Paulinho Demeneghi

ESTRUTURA DE IDEAIS EM ÁLGEBRAS DE STEINBERG

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To my parents.

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RESUMO

Dado uma ação ampla de um semigrupo inverso sobre um espaço topológico Hausdorff, localmente compacto e totalmente desconexo, estudamos a estrutura de ideais do produto cruzado associado. Através do desenvolvimento de uma teoria de ideais induzidos, provamos que todo ideal no produto cruzado pode ser obtido como intersecção de ideais induzidos a partir de álgebras de grupos de isotropia. Isto pode ser interpretado como uma versão algébrica da conjectura de Effros-Hahn. Finalmente, como uma aplicação de nosso resultado, estudamos a estrutura de ideais da álgebra de Steinberg associada a um grupoide amplo interpretando esta álgebra como um produto cruzado algébrico por um semigrupo inverso.

Palavras-chave: Semigrupo inverso, ação ampla, produto cruzado, álgebra de Steinberg, ideais, conjectura de Effros-Hahn.

RESUMO EXPANDIDO

Introdução

A conjectura de Effros-Hahn tem motivado muitos trabalhos relacionados ao estudo de ideais em produtos cruzados há aproximadamente 50 anos. Na sua forma original, a conjectura afirma que todo ideal primitivo no produto cruzado de uma C*-álgebra comutativa por um grupo localmente compacto é induzido por um ideal primitivo na C*-álgebra associada à algum grupo de isotropia.
Em [2], Sauvageot provou uma versão da conjectura para o caso de grupos discretos mediáveis e, desde então, a conjectura tem sido tratada em vários outros contextos. Gootman e Rosenberg, em [3], provaram uma versão para grupos localmente compactos agindo em C*-álgebras não necessariamente comutativas. Renault também introduziu uma versão da conjectura de Effros-Hahn em [4], no contexto de C*-álgebras associadas a grupoides. Além disso, os resultados de Renault foram refinados por Ionescu e Williams em [5].
Existem muitos outros trabalhos ao longo desses 50 anos que foram motivados pela conjectura de Effros-Hahn. Mas, por entender que os trabalhos citados já enfatizam suficientemente a importância da conjectura, citaremos apenas mais dois que são, fundamentalmente, os trabalhos que motivaram nosso estudo. No primeiro, Dokuchaev e Exel, em [6], introduziram uma versão da conjectura em um contexto totalmente diferente: o produto cruzado algébrico $\mathscr{L}_c(X) \rtimes G$, em que G é um grupo discreto agindo parcialmente em um espaço topológico localmente compacto, totalmente desconexo e Hausdorff. No segundo, Steinberg introduziu o que conhecemos hoje por álgebras de Steinberg e que podem ser consideradas como um viés algébrico das C*-álgebras de grupoides introduzidas por Renault. É bem conhecido que, dada uma ação parcial de um grupo discreto em um espaço topológico localmente compacto, totalmente desconexo e Hausdorff, o produto cruzado associado é isomorfo a álgebra de Steinberg associada ao grupoide de transformação da ação. Dessa forma, possivelmente, os resultados obtidos por Dokuchaev e Exel podem ser generalizados para álgebras de Steinberg.

Objetivos

Nesse momento, como recém comentado, a questão que surge é: os resultados de Dokuchaev e Exel em [6] podem ser generalizados para álgebras de Steinberg? Ou ainda mais especificamente: é possível obter uma versão da conjectura de Effros-Hahn para álgebras de Steinberg?

Nosso objetivo nesse trabalho é responder essas perguntas, apresentando uma versão da conjectura de Effros-Hahn para álgebras de Steinberg.

Metodologia

Analisando o trabalho de Dokuchaev e Exel, percebemos que uma ferramenta de desintegração e integração de representações foi fundamental para se obter a desejada versão da conjectura de Effros-Hahn e nossa esperança então foi obter uma ferramenta semelhante para o caso das álgebras de Steinberg. Cabe ressaltar que uma tal ferramenta de desintegração e integração de representações já apareceu no trabalho de Steinberg em [7], mas apenas para o caso em que o grupoide é Hausdorff. Como queríamos trabalhar em um contexto mais geral incluindo também grupoides não-Hausdorff, tentamos obter uma generalização do resultado de Steinberg. Contudo, a demonstração obtida por Steinberg baseava-se no fato que a interseção de dois conjuntos compactos é ainda um conjunto compacto, o que pode não acontecer no caso não-Hausdorff e, portanto, não conseguimos adaptar o argumento.
Como comentamos anteriormente, as álgebras de Steinberg podem ser interpretadas como um viés algébrico das C*-álgebras de grupoides introduzidas por Renault. E nesse sentido, Exel mostrou em [13] que, sob algumas condições, toda C*-álgebra associada a um grupoide étale é isomorfa a um produto cruzado por um semigrupo inverso. Nesse contexto surge a segunda tentativa de obter uma ferramenta de desintegração e integração de representações para álgebras de Steinberg. Se conseguíssemos obter uma versão algébrica do isomorfismo obtido por Exel e uma ferramenta de desintegração e integração para o produto cruzado algébrico por um semigrupo inverso, poderíamos transportar essa ferramenta para as álgebras de Steinberg através do isomorfismo. Novamente, apenas conseguimos obter o isomorfismo desejado para o caso em que o grupoide é Hausdorff.
Nesse momento a ideia que surgiu foi de mudar o objeto principal de estudo. Ao invés de focarmos em álgebras de Steinberg, poderíamos focar na versão algébrica do produto cruzado por um semigrupo inverso.

Resultados e Discussão

Ao concentrarmo-nos na versão algébrica do produto cruzado por um semigrupo inverso como objeto principal de estudo, fomos capazes de obter as desejados ferramentas de desintegração e integração de representações como queríamos. Além disso, nossas suspeitas foram confirmadas e essas ferramentas se mostraram fundamentais no nosso argumento

para obter, de fato, uma versão da conjectura de Effros-Hahn.
Um ponto muito interessante a ser ressaltado é que, nossos resultados se mostraram como uma ferramenta para obter o isomorfismo que inicialmente desejávamos obter, isto é, toda álgebra de Steinberg associado a um grupoide amplo é isomorfa a um produto cruzado algébrico por um semigrupo inverso e, portanto, a conjectura de Effros-Hahn pode ser transportada para álgebras de Steinberg através desse isomorfismo.

Considerações Finais
Dessa forma, respondemos a pergunta inicialmente feita, generalizando os resultados obtidos por Dokuchaev e Exel e obtendo uma versão da conjectura de Effros-Hahn para álgebas de Steinberg. Curiosamente, alguns resultados que esperávamos obter para usar como ferramentas acabaram aparecendo como consequência da teoria desenvolvida. Agora, novas perguntas podem ser feitas a partir do nosso trabalho, como por exemplo, se nosso resultado pode ser usado para obter condições suficientes no grupoide para garantir a simplicidade da álgebra de Steinberg associada.

Palavras-chave: Semigrupo inverso, ação ampla, produto cruzado, álgebra de Steinberg, ideais, conjectura de Effros-Hahn.

ABSTRACT

Given an ample action of an inverse semigroup on a locally compact, totally disconnected and Hausdorff topological space, we study the ideal structure of the crossed product algebra associated to it. By developing a theory of induced ideals, we manage to prove that every ideal in the crossed product algebra may be obtained as the intersection of ideals induced from isotropy group algebras. This can be interpreted as an algebraic version of the Effros-Hahn conjecture. Finally, as an application of our result, we study the ideal structure of a Steinberg algebra associated to an ample groupoid by interpreting it as an inverse semigroup crossed product algebra.

Keywords: Inverse semigroup, ample action, crossed product algebra, Steinberg algebra, ideals, Effros-Hahn conjecture.

CONTENTS

1 INTRODUCTION

There is a celebrated conjecture which has motivated most of the works in the study of ideals in crossed product C*-algebras since about fifty years ago, namely the Effros-Hahn conjecture [1]. The original conjecture states that every primitive ideal in the crossed product of a commutative C*-algebra by a locally compact group should be induced from a primitive ideal in the C*-algebra of some isotropy group.

It was proved by Sauvageout in [2] for the case of discrete amenable groups and, since then, it has been extended to various others contexts. Gootman and Rosenberg [3] have proved a version for locally compact groups acting on non-commutative C*-algebras. Renault has also introduced a version of the Effros-Hahn conjecture in [4] for groupoid C*-algebras, an entirely different setting. And Renault's results were later refined by Ionescu and Williams in [5].

We should also mention Dokuchaev and Exel work in [6] and Steinberg work in [7] and [8]. Dokuchaev and Exel have introduced the conjecture in an algebraic fashion, the algebraic partial crossed product $\mathscr{L}_c(X) \rtimes G$, where G is a discrete group partially acting on a locally compact and totally disconnected topological space X, and $\mathscr{L}_c(X)$ is the algebra consisting of all locally constant, compactly supported functions on X, taking values in a given field K. Steinberg introduced a notion of an algebra associated with an ample groupoid $\mathcal{G}$ over a given field K, known as Steinberg algebras nowadays. He obtained a remarkable number of results for these algebras and, among them, a theory of induction of modules from isotropy groups.

It is well known that the main object of study in [6], the algebra $\mathscr{L}_c(X) \rtimes G$, may also be described as the Steinberg algebra [7] for the transformation groupoid associated with the partial action of G on X. Hence, Steinberg results may be applied to $\mathscr{L}_c(X) \rtimes G$ as well.

The question that arises in this moment is: could Dokuchaev and Exel results be generalized for Steinberg algebras? The answer is affirmative and, in this paper we focus in showing this.

For that task, as our main object of interest, we first concentrate on crossed product algebras of the form $\mathscr{L}_c(X) \rtimes S$, where S is an inverse semigroup. In fact, in the last part of this paper, we show that every Steinberg algebra associated with an ample groupoid (not necessarily Hausdorff) over a given field K can be realized as an inverse semigroup crossed product of the form $\mathscr{L}_c(X) \rtimes S$. Similar results may be found in [9] and [10].

This paper is structured in four main parts. First, we introduce an algebraic notion of a Fell Bundle over an inverse semigroup, inspired by [11], and then build the cross-sectional algebra associated with it. Next, we show that an ample action of an inverse semigroup S over a locally compact and Hausdorff topological space X induces a Fell Bundle $\mathcal{B}^\theta$, referred as the semi-direct product bundle, and define the crossed product algebra $\mathscr{L}_c(X) \rtimes S$ as the cross-sectional algebra of the semi-direct product bundle. We then develop a theory linking representations of the crossed product algebra and covariant representations of the ample system (θ, S, X), obtaining results of integration and disintegration.

The second part is dedicated to present the theory of induction of ideals from isotropy groups algebras and to present the basics of the induction process. Based on Dokuchaev and Exel work, we study the relationship between the *input* ideal in the isotropy group algebra and its corresponding *output* induced ideal. It turns out that, for a point x in X, when inducing from the isotropy group G_x, not all ideals in KG_x play a relevant role. Those which do, we call *admissible*, inspired in Dokuchaev and Exel terminology. We show in (3.2.5) that, for every ideal $I \trianglelefteq KG_x$, there exists a unique admissible ideal $I' \subseteq I$, which induces the same ideal of $\mathscr{L}_c(X) \rtimes S$ as I does. Thus, the correspondence $I \mapsto \mathrm{Ind}_x(I)$ is seen to be an one-to-one mapping from the set of admissible ideals in KG_x to the set of ideals in $\mathscr{L}_c(X) \rtimes S$.

In the third part, we generalize Dokuchaev and Exel version of the Effros-Hahn conjecture ([6, Theorem 6.3]) for $\mathscr{L}_c(X) \rtimes S$, namely Theorem (4.2.6) which states that every ideal of $\mathscr{L}_c(X) \rtimes S$ is given as the intersection of ideals induced from isotropy groups. The method of the proof is inspired in [6] and does not rely on measure theoretical or analytical tools. The strategy adopted is as follows: given an ideal J of $\mathscr{L}_c(X) \rtimes S$, we first choose a representation π of $\mathscr{L}_c(X) \rtimes S$ whose null space coincides with J. Through the theory of integration and disintegration constructed before, we then build another representation, which we call the *discretization* of π, as done in [6], whose null space coincides with that of π, and hence also with J. The discretized representation is seen to decompose as a direct sum of sub-representations, which are finally shown to be equivalent to an induced representation, and hence the initially given ideal J is seen to coincide with the intersection of the null spaces of the various induced representations involved, each of which is then an induced ideal.

Finally, in the last part of this paper, as a consequence of Theorem (4.2.6) we show that every Steinberg algebra over a given field K is isomorphic to an inverse semigroup crossed product of the form $\mathscr{L}_c(X) \rtimes S$ and the induction theory introduced by Steinberg in [7] and [8] is compatible with our theory through the given isomorphism. So, our results can be all applied to Steinberg algebras.

2 INVERSE SEMIGROUP CROSSED PRODUCTS

In this chapter we explore inverse semigroup crossed product algebras and its universal property. Actually, this algebras have already came to light meanwhile this work was in progress in [9] and [10], for example. However, we prefer to build it in a slightly different fashion.

We introduce an intermediary step, namely, an algebraic notion of a Fell bundle over an inverse semigroup, based on Exel's paper [11]. With this in hand we then build the cross-sectional algebra associated with a Fell bundle and discuss an universal property with relation to representations. Finally, from an action of an inverse semigroup, when possible, we construct a Fell Bundle, named the semi-direct product bundle associated with the action. Turns out that the cross-sectional algebra of a semi-direct product bundle "coincides" with the crossed product algebra in the sense of [9] and [10].

From our context, we see that the crossed product algebra arising from actions of inverse semigroups on locally compact, totally disconnected and Hausdorff spaces inherits an universal property with relation to representations.

2.1 FELL BUNDLES OVER INVERSE SEMIGROUPS

We assume that the reader is familiar with the notion of an inverse semigroup and its basics notations: the semigroup is denoted by S, the involutive anti-homomorphism by $*$, and the set of all idempotent elements by $E(S)$.

Throughout this paper we fix a field K.

Most results in this chapter are still valid in the more general case obtained by replacing K by a commutative ring with identity. However, we prefer to maintain K as a field all long this paper since for the main results, this assumption is needed.

Definition 2.1.1. A *Fell Bundle* over an inverse semigroup S is a triple

$$\mathcal{B} = \left(\{B_s\}_{s\in S}, \{\mu_{s,t}\}_{s,t\in S}, \{j_{t,s}\}_{s,t\in S, s\leq t}\right)$$

such that, for each $s,t \in S$

(a) B_s is a K-vector space;

(b) $\mu_{s,t} : B_s \otimes B_t \to B_{st}$ is a K-linear map;

(c) $j_{t,s} : B_s \to B_t$ is a K-linear injective map for every $s \leq t$.

It is moreover required that for every $r, s, t \in S$,

(i) $\mu_{rs,t}\left(\mu_{r,s}(a \otimes b) \otimes c\right) = \mu_{r,st}\left(a \otimes \mu_{s,t}(b \otimes c)\right)$ for every $a \in B_r$, $b \in B_s$ and $c \in B_t$;

(ii) $\operatorname{span}\left\{\mu_{ss^*,s}\left(\mu_{s,s^*}(B_s \otimes B_{s^*}) \otimes B_s\right)\right\} = B_s$;

(iii) $j_{t,r} = j_{t,s} \circ j_{s,r}$ if $r \leq s \leq t$;

(iv) if $r \leq r'$ and $s \leq s'$, then the diagram

$$\begin{array}{ccc} B_r \otimes B_s & \xrightarrow{\mu_{r,s}} & B_{rs} \\ {\scriptstyle j_{r',r} \otimes j_{s',s}} \downarrow & & \downarrow {\scriptstyle j_{r's',rs}} \\ B_{r'} \otimes B_{s'} & \xrightarrow[\mu_{r',s'}]{} & B_{r's'} \end{array}$$

commutes.

If $s \leq t$, we shall use the map $j_{t,s}$ to identify B_s as a subspace of B_t. The last axiom then says that the multiplication operation is compatible with such an identification.

There are some immediate consequences of the definition.

Proposition 2.1.2.

(a) If $e \in E(S)$, then B_e is an associative K-algebra.

(b) For every $s \in S$, the map $j_{s,s}$ is the identity map on B_s.

(c) If $e, f \in E(S)$ and $e \leq f$, then $j_{f,e}(B_e)$ is a two-sided ideal in B_f.

Proof. The first item is obvious. For the second item, let $s \in S$ and notice that $j_{s,s}$ is an injective linear map from B_s to itself, which is idempotent by (2.1.1.iii). Therefore, $j_{s,s}$ must be the identity map on B_s, as stated. Finally, with respect to (c), let $a \in B_e$, $b \in B_f$ and notice that

$$\begin{aligned} j_{f,e}(a) \cdot b = \mu_{f,f}\Big(j_{f,e}(a) \otimes j_{f,f}(b)\Big) &\overset{(2.1.1.iv)}{=} j_{ff,ef}\Big(\mu_{e,f}(a \otimes b)\Big) \\ &= j_{f,e}\Big(\mu_{e,f}(a \otimes b)\Big) \in j_{f,e}(B_e), \end{aligned}$$

and similarly $b \cdot j_{f,e}(a) \in j_{f,e}(B_e)$. This shows that $j_{f,e}(B_e)$ is a two-sided ideal in B_f, as desired. ∎

Definition 2.1.3. A *pre-representation* of a Fell bundle $\mathcal{B} = \{B_s\}_{s\in S}$ in an algebra A is a family $\Pi = \{\pi_s\}_{s\in S}$ of linear maps

$$\pi_s : B_s \to A$$

such that, for all $s, t \in S$ and all $a \in B_s$ and $b \in B_t$, we have

(i) $\pi_{st}\left(\mu_{s,t}(a \otimes b)\right) = \pi_s(a)\pi_t(b)$.

Furthermore, Π is a representation if it satisfies

(ii) $\pi_t \circ j_{t,s} = \pi_s$, whenever $s \le t$.

In this context, if V is a K-vector space and $A = L(V)$, then we shall say that Π is a representation of $\mathcal{B}$ on V.

Definition 2.1.4. The *cross-sectional* algebra of $\mathcal{B}$, denoted by $\mathscr{Alg}(\mathcal{B})$, is the universal algebra generated by the disjoint union

$$\dot{\bigcup_{s\in S}} B_s,$$

subject to the relations stating that the natural maps

$$\pi^u_s : B_s \to \mathscr{Alg}(\mathcal{B})$$

form a representation of $\mathcal{B}$ in $\mathscr{Alg}(\mathcal{B})$.

The existence of $\mathscr{Alg}(\mathcal{B})$ is clear, as its uniqueness, up to isomorphism. For convenience, we spell out its universal property.

Proposition 2.1.5. *The cross-sectional algebra $\mathscr{Alg}(\mathcal{B})$ is an algebra and $\Pi^u = \{\pi^u_s\}_{s\in S}$ is a representation of $\mathcal{B}$ in $\mathscr{Alg}(\mathcal{B})$. Furthermore, given any representation $\Pi = \{\pi_s\}_{s\in S}$ of the Fell Bundle $\mathcal{B}$ in an algebra A, there exists a unique homomorphism $\Phi : \mathscr{Alg}(\mathcal{B}) \to A$ such that $\Phi \circ \pi^u_s = \pi_s$ for all $s \in S$.*

It will be useful to have a more concrete description of $\mathscr{Alg}(\mathcal{B})$ as follows. Let

$$\mathcal{L}(\mathcal{B}) = \bigoplus_{s\in S} B_s.$$

For each $s \in S$ and $b_s \in B_s$, we denote by $b\boldsymbol{\delta}_s$ the element of $\mathcal{L}(\mathcal{B})$ whose coordinates are equal to zero, except for the coordinate corresponding to s, which is equal to b. Then, it is clear that any element $b \in \mathcal{L}(\mathcal{B})$ can be represented uniquely in the form [1]

$$b = \sum_{s \in S} b_s \boldsymbol{\delta}_s.$$

Define a multiplication on $\mathcal{L}(\mathcal{B})$ such that

$$(b_s \boldsymbol{\delta}_s)(b_t \boldsymbol{\delta}_t) = \mu_{s,t}(b_s \otimes b_t) \boldsymbol{\delta}_{st}$$

for all $s, t \in S$, $b_s \in B_s$ and $b_t \in B_t$.

Then, with (2.1.1.i), we can prove that $\mathcal{L}(\mathcal{B})$ is an associative K-algebra.

Definition 2.1.6. Let $\Pi^0 = \{\pi_s^0\}_{s \in S}$ be the collection of maps such that, for each $s \in S$, $\pi_s^0 : B_s \to \mathcal{L}(\mathcal{B})$ is given by

$$\pi_s^0(b_s) = b_s \boldsymbol{\delta}_s.$$

In this fashion, Π_0 is a pre-representation of $\mathcal{B}$ in $\mathcal{L}(\mathcal{B})$ which is universal in the following sense.

Proposition 2.1.7. *Let A be an algebra. If $\Pi = \{\pi_s\}_{s \in S}$ is a pre-representation of $\mathcal{B}$ in A, then the map $\Phi : \mathcal{L}(\mathcal{B}) \to A$, given by*

$$\Phi \left(\sum_{s \in S} b_s \boldsymbol{\delta}_s \right) = \sum_{s \in S} \pi_s(b_s)$$

is a homomorphism. Conversely, given any homomorphism $\Phi : \mathcal{L}(\mathcal{B}) \to A$, consider for each $s \in S$, the map $\pi_s : B_s \to A$ given by

$$\pi_s = \Phi \circ \pi_s^0.$$

Then, $\Pi = \{\pi_s\}_{s \in S}$ is a pre-representation of $\mathcal{B}$ in A. Furthermore, the correspondences $\Pi \mapsto \Phi$ and $\Phi \mapsto \Pi$ are each other inverses, giving bijections between the set of all homomorphisms from $\mathcal{L}(\mathcal{B}) \to A$ and the set of all pre-representations of $\mathcal{B}$ in A.

Proposition 2.1.8. *Let $\mathcal{N}$ be the linear subspace of $\mathcal{L}(\mathcal{B})$ spanned by the set*

$$\{b_s \boldsymbol{\delta}_s - j_{t,s}(b_s) \boldsymbol{\delta}_t : s, t \in S, s \leq t, b_s \in B_s\}.$$

Then, $\mathcal{N}$ is a two-sided ideal of $\mathcal{L}(\mathcal{B})$.

[1] All sums considered in this paper are finite. Either because the summands are indexed on a finite set, or all but a finitely many summands are zero.

Proof. Given $r, s, t \in S$ such that $s \leq t$, let $b_s \in B_s$ and $b_r \in B_r$. Notice that, by (2.1.1.iv), we have $\mu_{t,r} \circ (j_{t,s} \otimes j_{r,r}) = j_{tr,sr} \circ \mu_{s,r}$ and so

$$\begin{aligned}(b_s\boldsymbol{\delta}_s - j_{t,s}(b_s)\boldsymbol{\delta}_t)\, b_r\boldsymbol{\delta}_r &= \mu_{s,r}(b_s \otimes b_r)\boldsymbol{\delta}_{sr} - \mu_{t,r}\left(j_{t,s}(b_s) \otimes b_r\right)\boldsymbol{\delta}_{tr} \\ &= \mu_{s,r}(b_s \otimes b_r)\boldsymbol{\delta}_{sr} - j_{tr,sr}\left(\mu_{s,r}(b_s \otimes b_r)\right)\boldsymbol{\delta}_{tr} \in \mathcal{N}.\end{aligned}$$

Therefore, we conclude that $\mathcal{N}$ is a right ideal and, similarly, we can show that $\mathcal{N}$ is a left ideal. □

Notice that, in the context of Proposition (2.1.7), Φ vanishes on $\mathcal{N}$ if and only if $\pi_t \circ j_{t,s} = \pi_s$, whenever $s \leq t$. Then, we immediately have the following proposition.

Proposition 2.1.9. *In the context of the correspondence $\Phi \leftrightarrow \Pi$ of (2.1.7), Φ vanishes on $\mathcal{N}$ if, and only if, Π is a representation.*

We now establish a very important representation of $\mathcal{B}$.

Corollary 2.1.10. *For each $s \in S$, let $\pi_s^+ = q \circ \pi_s^0$, where π_s^0 is like in (2.1.6) and $q : \mathcal{L}(\mathcal{B}) \to \mathcal{L}(\mathcal{B})/\mathcal{N}$ is the quotient map. Then, $\Pi^+ = \{\pi_s^+\}_{s\in S}$ is a representation of $\mathcal{B}$ in $\mathcal{L}(\mathcal{B})/\mathcal{N}$.*

Remark 2.1.11. We shall denote by $b\Delta_s$ the image of $b\boldsymbol{\delta}_s$ in $\mathcal{L}(\mathcal{B})/\mathcal{N}$ by the quotient map $q : \mathcal{L}(\mathcal{B}) \to \mathcal{L}(\mathcal{B})/\mathcal{N}$.

The importance of the representation Π^+ resides in the following result.

Proposition 2.1.12. *The algebra $\mathcal{L}(\mathcal{B})/\mathcal{N}$ possesses the universal property described in (2.1.5) with respect to the representation Π^+.*

Proof. Let $\Pi = \{\pi_s\}_{s\in S}$ be any representation of $\mathcal{B}$ in an algebra A and $\Psi : \mathcal{L}(\mathcal{B}) \to A$ be given as in (2.1.7) in terms of Π. By (2.1.9), Ψ vanishes at $\mathcal{N}$ and hence it factors through $\mathcal{L}(\mathcal{B})/\mathcal{N}$ giving a homomorphism $\Phi : \mathcal{L}(\mathcal{B})/\mathcal{N} \to A$ such that

$$\Phi(b_s\Delta_s) = \Phi(q(b_s\boldsymbol{\delta}_s)) = \pi_s(b_s)$$

whenever $b_s \in B_s$. Furthermore, notice that

$$\pi_s(b_s) = \Phi(q(b_s\boldsymbol{\delta}_s)) = \Phi(q(\pi_s^0(b_s))) = \Phi(\pi_s^+(b_s))$$

for every $s \in S$, as desired. It is also clear that such Φ must be unique. □

We then have an immediate corollary.

Corollary 2.1.13. *There exists an isomorphism $\Theta : \mathcal{L}(\mathcal{B})/\mathcal{N} \to \mathscr{A\!lg}(\mathcal{B})$, such that $\Theta \circ \pi_s^+ = \pi_s^u$, for every $s \in S$.*

We shall henceforth identify $\mathcal{L}(\mathcal{B})/\mathcal{N}$ and $\mathscr{A\!lg}(\mathcal{B})$, keeping in mind that this identification caries π_s^+ to π_s^u, for every $s \in S$.

Before we end this section, we introduce some important ingredients.

Definition 2.1.14. A representation $\Pi = \{\pi_s\}_{s\in S}$ of a Fell bundle $\mathcal{B}$ on a K-vector space V is *non-degenerate* if

$$\operatorname{span}\{\pi_s(b)\xi : s \in S, b \in B_s, \xi \in V\} = V.$$

Notice that, if $b \in B_{ss^*}$ and $c \in B_s$, then setting $a = \mu_{ss^*,s}(b \otimes c)$ we have

$$\pi_s(a) = \pi_{ss^*}(b)\pi_s(c).$$

Hence, by (2.1.1.ii), a representation of $\mathcal{B}$ on V is non-degenerate if and only if

$$\operatorname{span}\{\pi_e(b)\xi : e \in E(S), b \in B_e, \xi \in V\} = V.$$

Proposition 2.1.15. *In the context of Proposition (2.1.5), let $A = L(V)$ for some vector space V. Then, Π is non-degenerate if and only if Φ is non-degenerate.*

Proof. Suppose Π is non-degenerate and let $\xi \in V$ be such that $\xi = \pi_s(b_s)\eta$ for some $b_s \in B_s$ and $\eta \in V$. Then

$$\xi = \pi_s(b_s) = \Phi(\pi_s^u(b_s))\eta.$$

Since the vectors ξ of the above form spans V, Φ is non-degenerate. Conversely, suppose Φ is non-degenerate and let $\xi \in V$ be such that $\xi = \Phi(b)\eta$ where $b = q(\sum_{s\in S} b_s\boldsymbol{\delta}_s) \in \mathscr{A\!lg}(\mathcal{B})$ and $\eta \in V$. Then

$$\sum_{s\in S} \pi_s(b_s)\eta = \sum_{s\in S} \Phi(\pi_s^u(b_s))\eta = \Phi(b)\eta = \xi.$$

Since the vectors ξ of the above form spans V, Π is non-degenerate. $\square$

2.2 INVERSE SEMIGROUP ACTIONS AND ALGEBRAIC CROSSED PRODUCTS

The aim of this section is to construct a Fell bundle from an action of an inverse semigroup on an algebra. Unfortunately, this is not always possible, the problem in the construction will appear in the axioms (i) and (ii) of Definition (2.1.1), as we shall see.

Let X be any set, we denote by $\mathcal{I}(X)$ the inverse semigroup formed by all bijections between subsets of X, under the operation given by composition of functions in the largest domain in which the composition may be defined. We now present the definition of an action of an inverse semigroup on an algebra.

Definition 2.2.1. Let S be an inverse semigroup and let A be an algebra. An *action* of S on A is a semigroup homomorphism

$$\alpha : S \to \mathcal{I}(A)$$

such that

(i) for every $s \in S$, the domain (and hence also the range) of α_s is a two sided ideal of A and α_s is a homomorphism;

(ii) the linear span of the union of the domains of all the α_s coincides with A.

The triple (α, S, A) is called an (algebraic) *dynamical system.*

For every $e \in E(S)$, we denote by A_e the domain of α_e. Therefore, for each $s \in S$, we have that α_s is a homomorphism from A_{s^*s} to A_{ss^*}.

Throughout this section we fix an algebraic dynamical system (α, S, A)**, in order to describe the construction of the Fell bundle.**

We begin the construction defining, for each $s \in S$, the "fiber" $B_s = \{(a, s) \in A \times S : a \in A_{ss^*}\}$. To avoid excessive use of parentheses, we shall write $a\boldsymbol{\delta}_s$ to refer to (a, s) whenever $a \in A_{ss^*}$.

The linear structure of B_s is borrowed from A_{ss^*}, while the multiplication operation is defined on elementary tensors by

$$\begin{array}{rccc} \mu_{s,t} : & B_s \otimes B_t & \to & B_{st} \\ & a\boldsymbol{\delta}_s \otimes b\boldsymbol{\delta}_t & \mapsto & \alpha_s(\alpha_{s^*}(a)b)\boldsymbol{\delta}_{st}. \end{array}$$

We then define the inclusion maps naturally

$$\begin{array}{rccc} j_{t,s} : & B_s & \to & B_t \\ & a\boldsymbol{\delta}_s & \mapsto & a\boldsymbol{\delta}_t \end{array}$$

whenever $s, t \in S$ with $s \leq t$, which finally leads to a triple

$$\mathcal{B}^\alpha = \left(\{B_s\}_{s\in S}, \{\mu_{s,t}\}_{s,t\in S}, \{j_{t,s}\}_{s,t\in S, s\leq t} \right). \tag{2.2.2}$$

In order to the triple $\mathcal{B}^\alpha$ to be a Fell Bundle over S, we must worry about axioms (2.1.1.i-iv). Axioms (iii) and (iv) are easy to see, but as previous commented, axioms (i) and (ii) may not hold. To identify the origin of the problem with axiom (i), let $a\boldsymbol{\delta}_r \in B_r$, $b\boldsymbol{\delta}_s \in B_s$ and $c\boldsymbol{\delta}_t \in B_t$, for $r, s, t \in S$, and notice that, computing initially the left hand side of (2.1.1.i), we obtain:

$$\begin{aligned}\mu_{rs,t}\Big(\mu_{r,s}(a\boldsymbol{\delta}_r \otimes b\boldsymbol{\delta}_s) \otimes c\boldsymbol{\delta}_t\Big) &= \mu_{rs,t}\Big(\alpha_r\Big(\alpha_{r^*}(a)b\Big)\boldsymbol{\delta}_{rs} \otimes c\boldsymbol{\delta}_t\Big)\\ &= \alpha_{rs}\Big(\alpha_{s^*r^*}\Big(\alpha_r(\alpha_{r^*}(a)b)\Big)c\Big)\boldsymbol{\delta}_{rst}\\ &= \alpha_{rs}\Big(\alpha_{s^*}\Big(\alpha_{r^*}(a)b\Big)c\Big)\boldsymbol{\delta}_{rst}. \qquad (2.2.3)\end{aligned}$$

Additionally, computing the right hand side of (2.1.1.i), we have:

$$\begin{aligned}\mu_{r,st}\Big(a\boldsymbol{\delta}_r \otimes \mu_{s,t}(b\boldsymbol{\delta}_s \otimes c\boldsymbol{\delta}_t)\Big) &= \mu_{r,st}\Big(a\boldsymbol{\delta}_r \otimes \alpha_s\Big(\alpha_{s^*}(b)c\Big)\boldsymbol{\delta}_{st}\Big)\\ &= \alpha_r\Big(\alpha_{r^*}(a)\alpha_s\Big(\alpha_{s^*}(b)c\Big)\Big)\boldsymbol{\delta}_{rst}. \qquad (2.2.4)\end{aligned}$$

By these computations, wee see that (2.1.1.i) holds if and only if

$$\alpha_{rs}\Big(\alpha_{s^*}\Big(\alpha_{r^*}(a)b\Big)c\Big) = \alpha_r\Big(\alpha_{r^*}(a)\alpha_s\Big(\alpha_{s^*}(b)c\Big)\Big). \qquad (2.2.5)$$

Therefore, up to applying α_{r^*} in both sides of (2.2.5), we have proven:

Lemma 2.2.6. *A necessary and sufficient condition for the triple*

$$\mathcal{B}^\alpha = \Big(\{B_s\}_{s\in S}, \{\mu_{s,t}\}_{s,t\in S}, \{j_{t,s}\}_{s,t\in S, s\le t}\Big),$$

as defined in (2.2.2), to satisfy axiom (2.1.1.i) is that the equality

$$\alpha_s\Big(\alpha_{s^*}(\alpha_{r^*}(a)b)c\Big) = \alpha_{r^*}(a)\alpha_s\Big(\alpha_{s^*}(b)c\Big) \qquad (2.2.7)$$

holds for all $a \in A_{rr^*}$, $b \in A_{ss^*}$ and $c \in A_{tt^*}$, with $r, s, t \in S$.

We will know exploit sufficient conditions on the ideals A_{ss^*} in order to the triple $\mathcal{B}^\alpha$ to satisfy (2.1.1.i).

Proposition 2.2.8. *Given an action of an inverse semigroup S on an algebra A, a sufficient condition for the triple $\mathcal{B}^\alpha$ as defined in (2.2.2) to satisfy (2.1.1.i) is that, for each $s \in S$, the ideal A_{ss^*} is idempotent.*

Proof. Fix $s \in S$ and assume A_{ss^*} is idempotent. For $r, t \in S$, let $a \in A_{rr^*}$, $b_1, b_2 \in A_{ss^*}$, $c \in A_{tt^*}$ and $b = b_1 b_2$. Notice that

$$\alpha_s\Big(\alpha_{s^*}(\alpha_{r^*}(a)b)c\Big) = \alpha_s\Big(\alpha_{s^*}(\alpha_{r^*}(a)b_1b_2)c\Big) = \alpha_{r^*}(a)b_1\alpha_s\Big(\alpha_{s^*}(b_2)c\Big)$$
$$= \alpha_{r^*}(a)\alpha_s\Big(\alpha_{s^*}(b_1b_2)c\Big) = \alpha_{r^*}(a)\alpha_s\Big(\alpha_{s^*}(b)c\Big).$$

Since every element of A_{ss^*} is a sum of terms of the form $b_1 b_2$, we verify the equality (2.2.7) and, hence, by Lemma (2.2.6), we conclude (2.1.1.i). □

Finally, there is only one more axiom to worry about in our construction, namely (2.1.1.ii). As already mentioned, this may not hold as well.

Let $a, c \in B_s$ and $b \in B_{s^*}$ and notice, by the computation made in (2.2.3), that

$$\mu_{ss^*,s}\Big(\mu_{s,s^*}(a\boldsymbol{\delta}_s \otimes b\boldsymbol{\delta}_{s^*}) \otimes c\boldsymbol{\delta}_s\Big) = \alpha_{ss^*}\Big(\alpha_s\Big(\alpha_{s^*}(a)b\Big)c\Big)\boldsymbol{\delta}_s$$
$$= a\alpha_s(b)c\boldsymbol{\delta}_s.$$

So, we immediately have:

Lemma 2.2.9. *A necessary and sufficient condition for the triple*

$$\mathcal{B}^\alpha = \Big(\{B_s\}_{s\in S}, \{\mu_{s,t}\}_{s,t\in S}, \{j_{t,s}\}_{s,t\in S, s\le t}\Big),$$

as defined in (2.2.2), to satisfy axiom (2.1.1.ii) is that for each $s \in S$ the ideal A_{ss^*} satisfies

$$\operatorname{span} A_{ss^*}A_{ss^*}A_{ss^*} = A_{ss^*}.$$

Notice that, since $\operatorname{span} A_{ss^*}A_{ss^*}A_{ss^*} \subseteq \operatorname{span} A_{ss^*}A_{ss^*} \subseteq A_{ss^*}$, the equality in the Lemma above is equivalent to A_{ss^*} being idempotent. This, combined with Proposition (2.2.8), leads to the following result:

Theorem 2.2.10. *Given an action of an inverse semigroup S on an algebra A, the triple*

$$\mathcal{B}^\alpha = \Big(\{B_s\}_{s\in S}, \{\mu_{s,t}\}_{s,t\in S}, \{j_{t,s}\}_{s,t\in S, s\le t}\Big),$$

as defined in (2.2.2), is a Fell bundle over S if and only if, for each $s \in S$, the ideal A_{ss^*} is idempotent. In this case, it will be henceforth called the semi-direct product bundle *relative to the system (α, S, A).*

Definition 2.2.11. Let α be an action of an inverse semigroup S on an algebra A satisfying the equivalent conditions of Theorem (2.2.10). The *crossed product* algebra $A \rtimes_\alpha S$ is defined to be the cross-sectional algebra of the semi-direct product bundle $\mathcal{B}^\alpha$ associated with (α, S, A). We shall denote by $\mathcal{J}_\alpha$ the ideal defined in (2.1.8) and $q_{\mathcal{J}_\alpha}$ the quotient map from $\mathcal{L}(\mathcal{B}^\alpha)$ to $A \rtimes_\alpha S$.

Remark 2.2.12. We shall use the notation $a\boldsymbol{\delta}_s$ to denote an element of $\mathcal{L}(\mathcal{B}^\alpha)$ as well, instead of the awkward double notation $a\boldsymbol{\delta}_s\boldsymbol{\delta}_s$. The context should bring no confusion. Furthermore, according to Remark (2.1.11), we shall use the notation $a\Delta_s$ to denote the class of $a\boldsymbol{\delta}_s$ in $A \rtimes_\alpha S$.

Notice that, when it is possible to construct the semi-direct product bundle related to the action of an inverse semigroup on an algebra, the cross-sectional algebra of the semi-direct product bundle coincides with the definitions of crossed product algebras in [9] and [10]. Furthermore, notice still that the cross-sectional construction always leads to an associative K-algebra. Indeed, the problems faced in the construction of the semi-direct product bundle are related to the problems one would face to show that the crossed product algebra definitions in [9] and [10] are associative.

2.3 CROSSED PRODUCTS OF INTEREST

A special case of actions of inverse semigroups on algebras that leads to a Fell bundle will be investigated now and will be of high interest for us from now on.

Definition 2.3.1. Let S be an inverse semigroup and let X be a locally compact and Hausdorff topological space. An *action* of S on X is a semigroup homomorphism

$$\theta : S \to \mathcal{I}(X)$$

such that,

(i) for every $s \in S$, θ_s is continuous and its domain is open in X;

(ii) the union of the domains of all the θ_s coincides with X.

The triple (θ, S, X) is called a *(topological) dynamical system*. Furthermore, if X is totally disconnected and the domains are *clopen* (closed and open), we say that θ is an *ample action* and (θ, S, X) is an *ample dynamical system*.

The reader is invited to compare the definition of ample action above with the Definition 4.2 of ample action in [7]. Is his definition, Steinberg requires the domains to be compact-open instead of just clopen. As we shall see, there is no need to require the domains to be compact in order to obtain an ample groupoid of germs.

For every $e \in E(S)$, we denote by X_e the domain of θ_e. Therefore, for each $s \in S$, θ_s is a homeomorphism from X_{s^*s} to X_{ss^*}.

From now on, we fix a Hausdorff, locally compact, totally disconnected topological space X and an ample system (θ, S, X).

We will henceforth denote by $\mathscr{L}_c(X)$ the set of all locally constant, compactly supported, K-valued functions on X and denote by $\operatorname{supp}(f)$ the support of $f \in \mathscr{L}_c(X)$. With pointwise multiplication, $\mathscr{L}_c(X)$ is a commutative K-algebra, which is unital if and only if X is compact.

For each $s \in S$, we may also consider the K-algebra $\mathscr{L}_c(X_{s^*s})$, which we will identify with the set formed by all $f \in \mathscr{L}_c(X)$ vanishing on $X \setminus X_{s^*s}$. Under this identification $\mathscr{L}_c(X_{s^*s})$ becomes an ideal in $\mathscr{L}_c(X)$.

So, from the ample action of S on a locally compact Hausdorff space X in the sense of (2.3.1), it is easy to construct an action of S on $\mathscr{L}_c(X)$ in the sense of (2.2.1). Regarding the homeomorphism $\theta_s : X_{s^*s} \to X_{ss^*}$, we may define an isomorphism

$$\alpha_s : \mathscr{L}_c(X_{s^*s}) \to \mathscr{L}_c(X_{ss^*}),$$

by setting

$$\alpha_s(f) = f \circ \theta_{s^*}, \tag{2.3.2}$$

for all $f \in \mathscr{L}_c(X_{s^*s})$. This said, $\alpha : S \to \mathcal{I}(\mathscr{L}_c(X))$ is a semigroup homomorphism which is then easily seen to be an (algebraic) action of S on $\mathscr{L}_c(X)$.

Furthermore, it is clear that, for every $s \in S$, the ideal $\mathscr{L}_c(X_{ss^*})$ is idempotent. So, we may now construct the semi-direct product bundle associated with $(\alpha, S, \mathscr{L}_c(X))$, which we will denote by $\mathscr{B}^\theta$ in this case. This leads to the cross-sectional algebra $\mathscr{L}_c(X) \rtimes S$, which is our object of interest.

Notice that, since we are assuming that X_e is clopen for every $e \in E(S)$, its characteristic function 1_{X_e} is locally constant, but not necessarily compactly supported. However, for any $f \in \mathscr{L}_c(X)$, the product $f1_{X_e}$ is compactly supported and so, it lies in $\mathscr{L}_c(X_e)$. Then, we may globally define an endomorphism $\bar{\alpha}_s : \mathscr{L}_c(X) \to \mathscr{L}_c(X)$ given by

$$\bar{\alpha}_s(f) := \alpha_s(f1_{s^*s})$$

where 1_{ss^*} stands for $1_{X_{ss^*}}$.

In this context, notice that, for any $s, t \in S$, $f \in \mathscr{L}_c(X_{ss^*})$ and $g \in \mathscr{L}_c(X_{tt^*})$

$$\alpha_s(\alpha_{s^*}(f)g) = \alpha_s(\alpha_{s^*}(f)1_{s^*s}g) = \alpha_s(\alpha_{s^*}(f))\alpha_s(1_{s^*s}g) = f\bar{\alpha}_s(g).$$

Hence, we get a simpler formula for the product

$$f\Delta_s \cdot g\Delta_t = f\bar{\alpha}_s(g)\Delta_{st}.$$

Now we begin the preparations to obtain an universal property for $\mathscr{L}_c(X) \rtimes S$.

Definition 2.3.3. A *covariant representation* of the system (θ, S, X) on a K-vector space V is a pair (π, σ), where $\pi : \mathscr{L}_c(X) \to L(V)$ is a non-degenerate representation and $\sigma : S \to L(V)$ is a semigroup homomorphism such that:

(i) $\pi(\alpha_s(f)) = \sigma_s \pi(f) \sigma_{s^*}$ for $s \in S$ and $f \in \mathscr{L}_c(X_{s^*s})$;

(ii) $\operatorname{span}\{\pi(f)\xi : f \in \mathscr{L}_c(X_e), \xi \in V\} = \sigma_e(V)$ for every $e \in E(S)$.

Notice that, if the domains of all θ_s are compact, then condition (ii) of Definition (2.3.3) may be replaced equivalently by:

(ii') $\pi(1_{X_e}) = \sigma_e$ for $e \in E(S)$.

The next lemma is a very helpful tool

Lemma 2.3.4. *Let (π, σ) be a covariant representation of the system (θ, S, X) on a vector space V. If $f \in \mathscr{L}_c(X_e)$ for some $e \in E(S)$, then*

$$\sigma_e \pi(f) = \pi(f) = \pi(f)\sigma_e.$$

Proof. Let $\xi \in V$. By (2.3.3.ii), there exist $\eta \in V$ such that $\pi(f)\xi = \sigma_e(\eta)$. Notice that

$$\sigma_e \pi(f)\xi = \sigma_e(\sigma_e(\eta)) = \sigma_e(\eta) = \pi(f)\xi.$$

Hence, $\sigma_e \pi(f) = \pi(f)$ and the first equality is proved. For the second, observe that

$$\pi(f)\sigma_e = \sigma_e \pi(f) \sigma_e \overset{(2.3.3.i)}{=} \pi(\alpha_e(f)) = \pi(f),$$

proving the second equality. □

The next proposition shows that covariant representations of the system can be integrated to non-degenerate representations of the semi-direct product bundle associated and, hence, to a non-degenerate representation of the crossed product algebra.

Proposition 2.3.5. *Let (π, σ) be a covariant representation of the system (θ, S, X) on a vector space V. For each $s \in S$, consider the map $\pi_s : B_s \to L(V)$ given by $\pi_s(f\boldsymbol{\delta}_s) = \pi(f)\sigma_s$ for $f \in \mathscr{L}_c(X_{ss^*})$. Then, the collection $\Pi = \{\pi_s\}_{s\in S}$ is a non-degenerate representation of the semi-direct product bundle $\mathcal{B}^\theta$ on V.*

Proof. For each $s \in S$, the map π_s is clearly linear and, if $s, t \in S$, $f \in \mathscr{L}_c(X_{ss^*})$ and $g \in \mathscr{L}_c(X_{tt^*})$, we have

$$\begin{aligned}\pi_{st}\Big(\mu_{s,t}(f\boldsymbol{\delta}_s \otimes g\boldsymbol{\delta}_t)\Big) &= \pi_{st}\Big(\alpha_s(\alpha_{s^*}(f)g)\boldsymbol{\delta}_{st}\Big) = \pi\Big(\alpha_s(\alpha_{s^*}(f)g)\Big)\sigma_{st} \\ &\overset{(2.3.3.i)}{=} \sigma_s\pi(\alpha_{s^*}(f)g)\sigma_{s^*}\sigma_{st} \overset{(2.3.4)}{=} \sigma_s\pi(\alpha_{s^*}(f))\pi(g)\sigma_t \\ &\overset{(2.3.3.i)}{=} \sigma_s\sigma_{s^*}\pi(f)\sigma_s\pi(g)\sigma_t \overset{(2.3.4)}{=} \pi(f)\sigma_s\pi(g)\sigma_t = \pi_s(f\boldsymbol{\delta}_s)\pi_t(g\boldsymbol{\delta}_t).\end{aligned}$$

Furthermore, if $s \leq t$, then

$$\begin{aligned}\pi_t(j_{t,s}(f\boldsymbol{\delta}_s)) = \pi_t(f\boldsymbol{\delta}_t) = \pi(f)\sigma_t &\overset{(2.3.4)}{=} \pi(f)\sigma_{ss^*}\sigma_t \\ &= \pi(f)\sigma_s = \pi_s(f\boldsymbol{\delta}_s),\end{aligned}$$

concluding that Π is indeed a representation of $\mathcal{B}^\theta$ on V.

Finally, let $\xi \in V$ and write

$$\xi = \sum_{i=1}^{n} \pi(f_i)\xi_i,$$

by the non-degenerateness of π. Since $\mathscr{L}_c(X) = \sum_{e\in E(S)} \mathscr{L}_c(X_e)$, we may assume that each f_i lies in $\mathscr{L}_c(X_{e_i})$ for some $e_i \in E(S)$. Hence,

$$\xi = \sum_{i=1}^{n} \pi(f_i)\xi_i \overset{(2.3.4)}{=} \sum_{i=1}^{n} \pi(f_i)\sigma_{e_i}(\xi_i) = \sum_{i=1}^{n} \pi_{e_i}(f_i\boldsymbol{\delta}_{e_i})\xi_i,$$

concluding the proof. □

Combining this result with (2.1.5), we get:

Corollary 2.3.6. *Let (π, σ) be a covariant representation of the system (θ, S, X) on a vector space V. Then, there exists a non-degenerate representation $\pi \times \sigma : \mathscr{L}_c(X) \rtimes S \to L(V)$ such that $(\pi \times \sigma)(f\Delta_s) = \pi(f)\sigma_s$ for $f \in \mathscr{L}_c(X_{ss^*})$.*

Now we proceed the other way around. The goal is to prove that every non-degenerate representation Π of the semi-direct product bundle on a vector space V is given as above for a covariant representation (π, σ) of (θ, S, X). We begin with the following lemma.

Lemma 2.3.7. *Given a non-degenerate representation* $\Pi = \{\pi_s\}_{s\in S}$ *of the semi-direct product bundle* $\mathcal{B}^\theta$ *on a vector space* V*, there exists a non-degenerate representation* π *of* $\mathscr{L}_c(X)$ *on* V *such that*

$$\pi(f) = \pi_e(f\boldsymbol{\delta}_e)$$

for all $e \in E(S)$ *and all* $f \in \mathscr{L}_c(X_e)$.

Proof. Let $f \in \mathscr{L}_c(X)$. Since $\mathscr{L}_c(X) = \sum_{e\in E(S)} \mathscr{L}_c(X_e)$, we may write it as finite sum $f = \sum_{e\in E(S)} f_e$ where $f_e \in \mathscr{L}_c(X_e)$. We claim initially that $\sum_{e\in E(S)} \pi_e(f_e\boldsymbol{\delta}_e)$ vanishes when $f = 0$. In fact, since Π is non-degenerate, it is enough to prove that

$$\sum_{e\in E(S)} \pi_e(f_e\boldsymbol{\delta}_e)\pi_s(g\boldsymbol{\delta}_s) = 0$$

for all $s \in S$ and $g \in \mathscr{L}_c(X_{ss^*})$. Notice that

$$\begin{aligned}\sum_{e\in E(S)} \pi_e(f_e\boldsymbol{\delta}_e)\pi_s(g\boldsymbol{\delta}_s) &= \sum_{e\in E(S)} \pi_{es}(f_e g\boldsymbol{\delta}_{es}) \overset{es\leq s}{=} \sum_{e\in E(S)} \pi_s(f_e g\boldsymbol{\delta}_s)\\ &= \pi_s\Bigg(\sum_{e\in E(S)} f_e g\boldsymbol{\delta}_s\Bigg) = \pi_s(fg\boldsymbol{\delta}_s) = 0,\end{aligned}$$

proving the claim. Hence, the map $\pi : \mathscr{L}_c(X) \to L(V)$ defined by

$$\pi(f) = \sum_{e\in E(S)} \pi_e(f_e\boldsymbol{\delta}_e)$$

does not depend of the choice of the f_e's. Furthermore, notice that, for $e, e' \in E(S)$, $f \in \mathscr{L}_c(X_e)$ and $g \in \mathscr{L}_c(X_{e'})$, we have

$$\pi(f)\pi(g) = \pi_e(f\boldsymbol{\delta}_e)\pi_{e'}(g\boldsymbol{\delta}_{e'}) = \pi_{ee'}(fg\boldsymbol{\delta}_{ee'}) = \pi(fg).$$

By linearity, π is a representation of $\mathscr{L}_c(X)$ on V.

Finally, the non-degenerateness of π is a consequence of non-degenerateness of Π. In fact, let $s \in S$, $f \in \mathscr{L}_c(X_{ss^*})$ and choose $g, h \in \mathscr{L}_c(X_{ss^*})$ such that $f = gh$. Hence

$$\pi_s(f\boldsymbol{\delta}_s) = \pi_{ss^*}(g\boldsymbol{\delta}_{ss^*})\pi_s(h\boldsymbol{\delta}_s) = \pi(g)\pi_s(h\boldsymbol{\delta}_s),$$

concluding the argument. □

With this in hands, we proceed to the promised result.

Theorem 2.3.8. *Given a non-degenerate representation $\Pi = \{\pi_s\}_{s\in S}$ of the semi-direct product bundle $\mathcal{B}^\theta$ on a vector space V, there exists a covariant representation (π, σ) of the system (θ, S, X) on V such that*

$$\pi_s(f\boldsymbol{\delta}_s) = \pi(f)\sigma_s$$

for every $s \in S$ and $f \in \mathscr{L}_c(X_{ss^*})$.

Proof. Let π be the representation of $\mathscr{L}_c(X)$ on V given in the Lemma (2.3.7). Since π is non-degenerate, given any $\xi \in V$, we may write

$$\xi = \sum_{i=1}^{n} \pi(f_i)\xi_i,$$

where each $f_i \in \mathscr{L}_c(X)$ and $\xi_i \in V$. We then define, for each $s \in S$,

$$\sigma_s(\xi) = \sum_{i=1}^{n} \pi_s(\bar{\alpha}_s(f_i)\boldsymbol{\delta}_s)\xi_i.$$

To prove that σ_s is well defined, we must show that the right hand side of the equality above vanishes when $\xi = 0$. Hence, suppose $\xi = 0$ and let

$$C = \bigcup_{i=1}^{n} \mathrm{supp}(f_i) \cap X_{s^*s}.$$

So C is a compact open set and, for each $i = 1, \ldots, n$, we have $1_C f_i 1_{s^*s} = f_i 1_{s^*s}$. Therefore,

$$\begin{aligned}
\sum_{i=1}^{n} \pi_s(\bar{\alpha}_s(f_i)\boldsymbol{\delta}_s)\xi_i &= \sum_{i=1}^{n} \pi_s(\alpha_s(1_C f_i 1_{s^*s})\boldsymbol{\delta}_s)\xi_i \\
&= \sum_{i=1}^{n} \pi_s(1_{\theta_s(C)}\boldsymbol{\delta}_s)\pi_{s^*s}(f_i 1_{s^*s}\boldsymbol{\delta}_{s^*s})\xi_i \\
&= \sum_{i=1}^{n} \pi_s(1_{\theta_s(C)}\boldsymbol{\delta}_s)\pi(f_i 1_{s^*s})\xi_i \\
&= \sum_{i=1}^{n} \pi_s(1_{\theta_s(C)}\boldsymbol{\delta}_s)\pi(1_C f_i 1_{s^*s})\xi_i \\
&= \pi_s(1_{\theta_s(C)}\boldsymbol{\delta}_s)\pi(1_C 1_{s^*s}) \sum_{i=1}^{n} \pi(f_i)\xi_i \\
&= \pi_s(1_{\theta_s(C)}\boldsymbol{\delta}_s)\pi(1_C 1_{s^*s})\xi = 0,
\end{aligned}$$

concluding that σ_s is well defined. Furthermore, $\sigma : S \to L(V)$ given by $s \mapsto \sigma_s$ is a semigroup homomorphism. In fact, consider $s, t \in V$ and $\xi \in V$ a vector of the form $\xi = \pi(\varphi)\eta$ for some $\varphi \in \mathscr{L}_c(X)$ and $\eta \in V$. Additionally, consider $f, g \in \mathscr{L}_c(X_{tt^*})$ such that $fg = \bar{\alpha}_t(\varphi)$ and observe that

$$\begin{aligned}\sigma_s\sigma_t(\xi) &= \sigma_s\sigma_t\pi(\varphi)\eta = \sigma_s\pi_t(\bar{\alpha}_t(\varphi)\boldsymbol{\delta}_t)\eta = \sigma_s\pi_{tt^*}(f\boldsymbol{\delta}_{tt^*})\pi_t(g\boldsymbol{\delta}_t)\eta \\ &= \sigma_s\pi(f)\pi_t(g\boldsymbol{\delta}_t)\eta = \pi_s(\bar{\alpha}_s(f)\boldsymbol{\delta}_s)\pi_t(g\boldsymbol{\delta}_t)\eta \\ &= \pi_{st}(\bar{\alpha}_s(f)\bar{\alpha}_s(g)\boldsymbol{\delta}_{st})\eta = \pi_{st}(\bar{\alpha}_s(\bar{\alpha}_t(\varphi))\boldsymbol{\delta}_{st})\eta \\ &= \pi_{st}(\bar{\alpha}_{st}(\varphi)\boldsymbol{\delta}_{st})\eta = \sigma_{st}(\xi).\end{aligned}$$

Since the set of vectors of the form $\xi = \pi(\varphi)\eta$ spans V, we conclude that σ is a semigroup homomorphism. With the aim of proving (2.3.3.i), let $\xi = \pi(\varphi)\eta$ as above and notice that

$$\begin{aligned}\sigma_s\pi(f)\sigma_{s^*}(\xi) &= \sigma_s\pi(f)\sigma_{s^*}\pi(\varphi)\eta = \sigma_s\pi(f)\pi_{s^*}(\bar{\alpha}_{s^*}(\varphi)\boldsymbol{\delta}_{s^*})\eta \\ &= \pi_s(\alpha_s(f)\boldsymbol{\delta}_s)\pi_{s^*}(\bar{\alpha}_{s^*}(\varphi)\boldsymbol{\delta}_{s^*})\eta = \pi_{ss^*}(\alpha_s(f)\varphi\boldsymbol{\delta}_{ss^*})\eta \\ &= \pi(\alpha_s(f)\varphi)\eta = \pi(\alpha_s(f))\xi.\end{aligned}$$

For the proof of (2.3.3.ii), fix $e \in E(S)$ and let $f \in \mathscr{L}_c(X_e)$, $\xi \in V$ and notice that

$$\pi(f)\xi = \pi_e(f\boldsymbol{\delta}_e)\xi = \sigma_e(\pi(f)\xi)$$

and hence the essential space of $\pi(\mathscr{L}_c(X_e))$ is contained in the range of σ_e, that is $\operatorname{span}\{\pi(f)\xi : f \in \mathscr{L}_c(X_e), \xi \in V\} \subseteq \sigma_e(V)$. Conversely, let $\xi \in \sigma_e(V)$. Then, there exists $\eta \in V$ such that $\sigma_e(\eta) = \xi$. Since π is non-degenerate, there exists $f_i \in \mathscr{L}_c(X)$ and $\eta_i \in V$ such that $\eta = \sum_{i=1}^n \pi(f_i)\eta_i$. Therefore

$$\xi = \sigma_e(\eta) = \sum_{i=1}^n \pi_e(\bar{\alpha}_e(f_i)\boldsymbol{\delta}_e)\eta_i = \sum_{i=1}^n \pi(\bar{\alpha}_e(f_i))\eta_i,$$

from where we conclude that the range of σ_e is contained in the essential space of $\pi(\mathscr{L}_c(X_e))$.

Finally, we must prove that $\pi_s(f\boldsymbol{\delta}_s) = \pi(f)\sigma_s$ for every $s \in S$ and $f \in \mathscr{L}_c(X_{ss^*})$. For this task, let $\xi \in V$ and observe that

$$\pi_s(f\boldsymbol{\delta}_s)\xi = \sigma_s\pi(\alpha_{s^*}(f))\xi = \sigma_s\sigma_{s^*}\pi(f)\sigma_s(\xi) \overset{(2.3.4)}{=} \pi(f)\sigma_s(\xi),$$

concluding the proof. □

An immediate result about disintegration of representations of the crossed product algebra follows.

Corollary 2.3.9. *Given a non-degenerate representation Φ of the crossed product $\mathscr{L}_c(X) \rtimes S$ on a vector space V, there exists a unique covariant representation (π, σ) of the system (θ, S, X) on V such that*

$$\Phi(f\Delta_s) = \pi(f)\sigma_s$$

for every $s \in S$ and $f \in \mathscr{L}_c(X_{ss^*})$.

The next lemma is another helpful result.

Lemma 2.3.10. *Let (π, σ) be a covariant representation of the system (θ, S, X) on a vector space V. Then,*

$$\sigma_s\pi(f) = \pi(\bar{\alpha}_s(f))\sigma_s$$

for any $s \in S$ and $f \in \mathscr{L}_c(X)$.

Proof. In the presence of (2.3.5) and (2.3.8), we may assume that π and σ are given as in the proof of (2.3.8) for the non-degenerate representation of the semi-direct product bundle $\mathscr{B}^\theta$ given as in (2.3.5). Hence, let $\xi \in V$ be a vector of the form $\xi = \pi(\varphi)\eta$ for some $\varphi \in \mathscr{L}_c(X)$ and $\eta \in V$ and notice that

$$\begin{aligned}\sigma_s\pi(f)\xi &= \sigma_s\pi(f)\pi(\varphi)\eta = \sigma_s\pi(f\varphi)\eta = \pi_s(\bar{\alpha}_s(f\varphi)\boldsymbol{\delta}_s)\eta \\ &= \pi_{ss^*}(\bar{\alpha}_s(f)\boldsymbol{\delta}_{ss^*})\pi_s(\bar{\alpha}_s(\varphi)\boldsymbol{\delta}_s)\eta \\ &= \pi(\bar{\alpha}_s(f))\sigma_s(\pi(\varphi)\eta) = \pi(\bar{\alpha}_s(f))\sigma_s\xi\end{aligned}$$

Since the vectors of the above form spans V, by linearity we conclude the proof. ☐

Since we are discussing representations of $\mathscr{L}_c(X) \rtimes S$, we shall see now that, for any ideal J of $\mathscr{L}_c(X) \rtimes S$, there exists a non-degenerate representation whose kernel coincides with J. For that, we resort to Proposition (5.1) of [6].

Proposition 2.3.11. *Let A be a K-algebra possessing local units* [2]*. Then, for every ideal $J \trianglelefteq A$, there exists a vector space V and a non-degenerate representation*

$$\pi : A \to L(V),$$

such that $J = \ker(\pi)$.

[2]Recall that A is said to have local units if, for every a in A, there exists an idempotent $e \in A$, such that $ea = a = ae$.

To see that that above result applies to our situation, we give the following:

Proposition 2.3.12. *$\mathscr{L}_c(X) \rtimes S$ has local units.*

Proof. Let $b = \sum_{s\in F} f_s\Delta_s$, with $F \subseteq S$ finite. For each $s \in S$, let $C_s = \operatorname{supp}(f_s) \subseteq X_{ss^*}$. Then, C_s is a compact open set of X, as well as $\theta_{s^*}(C_s) \subseteq X_{s^*s}$. Therefore,

$$E := \bigcup_{s\in F} \Big(C_s \cup \theta_{s^*}(C_s) \Big),$$

is also compact open in X, since it is a finite union of compact open sets.

For each $\varepsilon \in \mathcal{P}(F)$, define

$$C_\varepsilon := \Big(\bigcap_{s\in\varepsilon} C_s \Big) \cap \Big(\bigcap_{s\in F\setminus\varepsilon} C_s^c \Big)$$

e

$$D_\varepsilon := \Big(\bigcap_{s\in\varepsilon} \theta_{s^*}(C_s) \Big) \cap \Big(\bigcap_{s\in F\setminus\varepsilon} \theta_{s^*}(C_s)^c \Big),$$

which are compact open sets as well, since they are closed sets contained in a compact set and X is Hausdorff.

Now, for every pair $(\varepsilon, \varsigma) \in \mathcal{P}(F) \times \mathcal{P}(F)$, such that $|\varepsilon|+|\varsigma|> 0$, define

$$E_{(\varepsilon,\varsigma)} := C_\varepsilon \cap D_\varsigma \quad \text{and} \quad e_{(\varepsilon,\varsigma)} := \Big(\prod_{s\in\varepsilon} ss^* \Big)\Big(\prod_{s\in\varsigma} s^*s \Big).$$

Since $E_{(\varepsilon,\varsigma)}$ is compact open, $\varphi_{(\varepsilon,\varsigma)} := 1_{E_{(\varepsilon,\varsigma)}}\Delta_{e_{(\varepsilon,\varsigma)}}$ is a well defined element of $\mathscr{L}_c(X) \rtimes S$. Moreover, $\big\{\varphi_{(\varepsilon,\varsigma)}\big\}_{|\varepsilon|+|\varsigma|>0}$ is a collection of orthogonal idempotent elements.

Hence,

$$\varphi = \sum_{|\varepsilon|+|\varsigma|>0} \varphi_{(\varepsilon,\varsigma)}$$

is an idempotent element of $\mathscr{L}_c(X) \rtimes S$ and notice that

$$\begin{aligned}
\varphi\Big(\sum_{s\in F} f_s\Delta_s\Big) &= \sum_{s\in F}\Big(\sum_{|\varepsilon|+|\varsigma|>0} \varphi_{(\varepsilon,\varsigma)}\Big) f_s\Delta_s\\
&= \sum_{s\in F}\sum_{|\varepsilon|+|\varsigma|>0} 1_{E_{(\varepsilon,\varsigma)}} f_s\Delta_{e_{(\varepsilon,\varsigma)}s}\\
&= \sum_{s\in F}\sum_{|\varepsilon|+|\varsigma|>0} 1_{E_{(\varepsilon,\varsigma)}} f_s\Delta_s\\
&= \sum_{s\in F} 1_E f_s\Delta_s\\
&= \sum_{s\in F} f_s\Delta_s
\end{aligned}$$

and

$$\begin{aligned}
\Big(\sum_{s\in F} f_s\Delta_s\Big)\varphi &= \sum_{s\in F}\sum_{|\varepsilon|+|\varsigma|>0} f_s\Delta_s\varphi_{(\varepsilon,\varsigma)}\\
&= \sum_{s\in F}\sum_{|\varepsilon|+|\varsigma|>0} f_s\bar{\alpha}_s(1_{E_{(\varepsilon,\varsigma)}})\Delta_{se_{(\varepsilon,\varsigma)}}\\
&= \sum_{s\in F}\sum_{|\varepsilon|+|\varsigma|>0} f_s\bar{\alpha}_s(1_{E_{(\varepsilon,\varsigma)}})\Delta_s\\
&= \sum_{s\in F} f_s\bar{\alpha}_s(1_E)\Delta_s\\
&= \sum_{s\in F} f_s 1_{\theta_s(E\cap X_{s^*s})}\Delta_s\\
&= \sum_{s\in F} f_s\Delta_s,
\end{aligned}$$

where the last equality holds because $C_s \subseteq \theta_s(E\cap X_{s^*s})$. $\square$

Joining this two results, we immediately have.

Corollary 2.3.13. *For every ideal $J \trianglelefteq \mathscr{L}_c(X)\rtimes S$, there exists a vector space V and a non-degenerate representation*

$$\pi : \mathscr{L}_c(X) \rtimes S \to L(V),$$

such that $J = \ker(\pi)$. In particular, $\mathscr{L}_c(X) \rtimes S$ has a faithful non-degenerate representation.

With this result in hand, we shall present two interesting consequences for $\mathscr{L}_c(X) \rtimes S$.

Proposition 2.3.14. *Theres is a monomorphism $\phi : \mathscr{L}_c(X) \to \mathscr{L}_c(X) \rtimes S$ such that*

$$\phi(f) = f\Delta_e,$$

whenever $e \in E(S)$ and $f \in \mathscr{L}_c(X_e)$.

Proof. Since $\mathscr{L}_c(X) = \sum_{e\in E(S)} \mathscr{L}_c(X_e)$, we may write any function $f \in \mathscr{L}_c(X)$ as a finite sum

$$f = \sum_{e\in E(S)} f_e,$$

with $f_e \in \mathscr{L}_c(X_e)$. By (2.3.13), $\mathscr{L}_c(X) \rtimes S$ has a faithful non-degenerate representation, which is the integrated form $\pi \times \sigma$ of some covariant representation (π, σ) for the system (θ, S, X), by (2.3.9).

Notice that,

$$(\pi \times \sigma)\Big(\sum_{e\in E(S)} f_e\Delta_e \Big) = \sum_{e\in E(S)} \pi(f_e)\sigma_e \overset{(2.3.4)}{=} \sum_{e\in E(S)} \pi(f_e) = \pi(f).$$

So, if $f = 0$, we must have $\sum_{e\in E(S)} f_e\Delta_e = 0$ by the faithfulness of $\pi \times \sigma$. Hence, the map $\phi : \mathscr{L}_c(X) \to \mathscr{L}_c(X) \rtimes S$ given by

$$\phi(f) = \sum_{e\in E(S)} f_e\Delta_e$$

is well defined and it is also a homomorphism.

Furthermore, ϕ is injective. Indeed, if $\phi(f) = 0$, then $\pi(f) = 0$. Let $x \in X$ and choose $e \in E(S)$ such that $x \in X_e$. Choose $\varphi \in \mathscr{L}_c(X_e)$ such that $\varphi(x) = 1$. Then,

$$0 = \pi(f)\pi(\varphi) = \pi(f\varphi) \overset{(2.3.4)}{=} \pi(f\varphi)\sigma_e = (\pi \times \sigma)(f\varphi\Delta_e)$$

from where we conclude that $f\varphi = 0$ and, hence, $f(x) = 0$. Since x is arbitrary, $f = 0$. □

Relying on this proposition, we may then identify $\mathscr{L}_c(X)$ as a subalgebra of $\mathscr{L}_c(X) \rtimes S$. Furthermore, with such an identification, keeping in mind the definition given in the proof of (2.3.7), we may interpret the map π of the covariant representation (π, σ) obtained by the disintegration of a non-degenerate representation ϕ of $\mathscr{L}_c(X) \rtimes S$ as its restriction to $\mathscr{L}_c(X)$.

The second consequence is an interesting characterization for the ideal $\mathcal{J}_\alpha$ defined in (2.2.11). Indeed, consider the ideal

$$\mathcal{I}_\alpha := \bigcap_{(\pi,\sigma)} \ker\left((\pi \times \sigma) \circ q_{\mathcal{J}_\alpha}\right) \trianglelefteq \mathcal{L}(\mathcal{B}^\theta), \tag{2.3.15}$$

in which (π, σ) ranges over covariant representations of the system (θ, S, X) and $\pi \times \sigma$ is the integrated form of (π, σ) for $\mathscr{L}_c(X) \rtimes S$.

Proposition 2.3.16. *The ideal $\mathcal{I}_\alpha$, as defined in* (2.3.15) *above, coincides with the ideal $\mathcal{J}_\alpha$.*

Proof. It is clear that $\mathcal{J}_\alpha \subseteq \mathcal{I}_\alpha$. It remains to prove the reverse inclusion. Indeed, by (2.3.13), $\mathscr{L}_c(X) \rtimes S$ has a faithful representation which is of the form $\pi \times \sigma$ for some covariant representation (π, σ) of the system (θ, S, X). Hence,

$$\mathcal{I}_\alpha \subseteq \ker\left((\pi \times \sigma) \circ q_{\mathcal{J}_\alpha}\right) = \mathcal{J}_\alpha,$$

concluding the proof. □

At first, there is no apparent reason for this result to be valid in the general case $A \rtimes_\alpha S$.

We end this section with a curious fact. Let (θ, S, X) be an ample dynamical system as usual all over this section. It is well known that one can always add a formal unit to a semigroup S, leading to the unitization

$$S^+ := S \sqcup \{1\}$$

of S. One may also extend θ to $\theta^+ : S^+ \to \mathcal{I}(X)$ in a natural way by defining θ_1 as the identity map on X. It is clear that (θ^+, S^+, X) is an ample action as well, which we shall call the unitization of (θ, S, X). We shall denote by α^+ the action of S^+ on $\mathscr{L}_c(X)$ induced by θ^+ in the sense of (2.3.2). In this context, we have:

Proposition 2.3.17. *Let (θ, S, X) be an ample dynamical system and (θ^+, S^+, X) its unitization. Then*

$$\mathscr{L}_c(X) \rtimes_\alpha S \simeq \mathscr{L}_c(X) \rtimes_{\alpha^+} S^+.$$

Proof. Let $\mathscr{B}^\theta$ be the semi-direct product bundle associated with (θ, S, X). For each $s \in S$, define $\pi_s : B_s \to \mathscr{L}_c(X) \rtimes_{\alpha^+} S^+$ by $\pi_s(f\boldsymbol{\delta}_s) = f\Delta_s$. It is clear that $\Pi = \{\pi_s\}_{s\in S}$ is a representation of $\mathscr{B}^\theta$ in $\mathscr{L}_c(X) \rtimes_{\alpha^+} S^+$. By, (2.1.5), there exists a homomorphism

$$\phi : \mathscr{L}_c(X) \rtimes_\alpha S \to \mathscr{L}_c(X) \rtimes_{\alpha^+} S^+$$

such that $\phi(f\Delta_s) = f\Delta_s$ for every $s \in S$.

Since every $f \in \mathscr{L}_c(X)$ may be written as a finite sum $f = \sum_{e\in E(S)} f_e$, we have

$$\phi\Bigg(\sum_{e\in E(S)} f_e\Delta_e\Bigg) = \sum_{e\in E(S)} f_e\Delta_e = \sum_{e\in E(S)} f_e\Delta_1 = f\Delta_1.$$

Thus proving that ϕ is onto $\mathscr{L}_c(X) \rtimes_{\alpha^+} S^+$.

By (2.3.13), $\mathscr{L}_c(X) \rtimes_\alpha S$ has a non-degenerate faithful representation which is the integrated form $\pi \times \sigma$ of some covariant representation (π, σ) of the system (θ, S, x), by (2.3.9). We may then extend σ to a semigroup homomorphism $\sigma^+ : S^+ \to L(V)$, by setting σ_1 as the identity map on V. It is then easy to see that (π, σ^+) is a covariant representation of the system (θ^+, S^+, X). By (2.3.6), there exists a representation $\pi \times \sigma^+$ of $\mathscr{L}_c(X) \rtimes_{\alpha^+} S^+$ on V such that $(\pi \times \sigma^+)(f\Delta_s) = \pi(f)\sigma_s^+$ for every $s \in S^+$.

We thus have the following commutative diagram

$$\begin{tikzcd}
\mathscr{L}_c(X) \rtimes_\alpha S \arrow[r, "\phi"] \arrow[d, "\pi\times\sigma"'] & \mathscr{L}_c(X) \rtimes_{\alpha^+} S^+ \arrow[dl, "\pi\times\sigma^+"] \\
L(V) &
\end{tikzcd}.$$

Finally, if $b \in \mathscr{L}_c(X) \rtimes_\alpha S$ lies in the kernel of ϕ, then it must also lie in the kernel of $\pi \times \sigma$. Hence, $b = 0$ and ϕ is injective. $\square$

3 INDUCTION PROCESS

In this chapter we follow the ideas introduced by Dokuchaev and Exel in [6]. They study the ideal structure of algebraic partial crossed products, in the context of a discrete group acting on a Hausdorff, locally compact, totally disconnected topological space. We shall study the ideal structure of the crossed product algebra in our context.

Throughout this chapter, we fix an ample dynamical system (θ, S, X).

3.1 INDUCTION PROCESS

For any point $x \in X$, as in the case of group actions, one can speak about its orbit

$$\mathrm{Orb}(x) := \{\theta_s(x) : x \in X_{s^*s}\}.$$

However, by trying to bring the concept of isotropy of a point from the case of group actions, one should come across the fact that the set

$$\tilde{G}_x := \{s \in S : x \in X_{s^*s},\, \theta_s(x) = x\}$$

does not need to have a group structure at all. In fact, $\tilde{G}_x$ as defined above is a $*$-subsemigroup of S. We shall define the isotropy group of x as the maximal group image [1] of $\tilde{G}_x$, but first we are going to introduce an auxiliary tool

$$\tilde{L}_x := \{s \in S : x \in X_{s^*s}\}.$$

We will introduce in $\tilde{L}_x$ an equivalence relation that identifies two elements $s, t \in \tilde{L}_x$ if, and only if, there is an idempotent element $e \in \tilde{L}_x$ such that $se = te$.

The motivation for this process comes from the interpretation of the well known concept of the isotropy group at a point in the unit space of a groupoid, for the case of the groupoid of germs for an action of an inverse semigroup on a space X, which we will explore later in this text. Thus, the class of an element $s \in \tilde{L}_x$ could be thought out as the germ of s at x, which also motivates the notation.

[1] Th maximal group homomorphic image of an inverse semigroup S is a group $G(S)$ satisfying the following property: if G is a group and $\psi : S \to G(S)$ is a surjective homomorphism, then ψ factors through $G(S)$. See Proposition 2.1.2 of [12] for further details.

Summarizing, we have:

$$\begin{array}{rcl} L_x & := & \{[s,x] : x \in X_{s^*s}\}, \\ G_x & := & \{[s,x] : x \in X_{s^*s} \text{ and } \theta_s(x) = x\}, \\ \mathrm{Orb}(x) & := & \{\theta_s(x) : x \in X_{s^*s}\}. \end{array} \tag{3.1.1}$$

Notice that, if s lies in $\tilde{L}_x$ and t lies in $\tilde{L}_{\theta_s(x)}$, then ts lies in $\tilde{L}_x$. Moreover, we have the following result.

Proposition 3.1.2. *Let $s \in \tilde{L}_x$ and $t \in \tilde{L}_y$, with $y = \theta_s(x)$. Then, ts lies in $\tilde{L}_x$ and the class of ts in L_x depends only on the classes of s and t in L_x and L_y, respectively.*

Proof. The first claim follows by the comment immediately before the proposition. For the second claim, notice initially that the fact that $\theta_s(x) = y$ does not depend on representatives. Indeed, let $s' \in \tilde{L}_x$ and $t' \in \tilde{L}_y$ be elements such that $[s', x] = [s, x]$ and $[t', y] = [t, y]$. Therefore there are idempotents $e \in \tilde{L}_x$ and $f \in \tilde{L}_y$ such that $se = s'e$ and $tf = t'f$. We then have that

$$\theta_s(x) = \theta_s(\theta_e(x)) = \theta_{se}(x) = \theta_{s'e}(x) = \theta_{s'}(\theta_e(x)) = \theta_{s'}(x).$$

It only remains to prove that $[ts, x] = [t's', x]$. For this task, let d be the idempotent given by $d = es^*fs$ and notice that d lies in $\tilde{L}_x$. Moreover,

$$\begin{aligned} t's'd &= t's'es^*fs = t'ses^*fs = t'ss^*fse = t'fss^*se \\ &= tfss^*se = tss^*fse = tses^*fs = tsd, \end{aligned}$$

concluding the argument. □

Therefore, as long as $y = \theta_s(x)$, we are allowed to operate the elements $[t, y]$ and $[s, x]$ to obtain $[ts, x]$. This provides a group structure on G_x such that $[s, x]^{-1} = [s^*, x]$ and whose identity element is $[e, x]$ for any idempotent element e in $\tilde{L}_x$.

Notice that, whenever there is an idempotent element $e \in \tilde{L}_x$ such that $se = te$ for a pair of elements in $\tilde{L}_x$, necessarily e lies in $\tilde{G}_x$, since θ_e is the identity map on its domain. Therefore, G_x coincides with the maximal group image of $\tilde{G}_x$ and, from now on, will be called the *isotropy group* of x.

Notice that $L_x G_x \subseteq L_x$ and the map

$$[s, x] \in L_x \mapsto \theta_s(x) \in \mathrm{Orb}(x)$$

is well defined (by the proof of (3.1.2)) and is onto. Moreover, two elements $[s, x]$ and $[t, x]$ in L_x satisfy $\theta_s(x) = \theta_t(x)$, if and only if, s^*t lies in $\tilde{G}_x$. Before we proceed, let us prove a technical result.

Lemma 3.1.3. *Let $s, t \in S$ such that $s \leq t$. If s lies in $\tilde{L}_x$, then t must also lie in $\tilde{L}_x$, $[s, x] = [t, x]$ and $\theta_s(x) = \theta_t(x)$. In particular, $\theta_t(X_{s^*s}) = X_{ss^*}$.*

Proof. Since $s \leq t$, there exists $e \in E(S)$ such that $s = te$. By hypothesis, $te = s \in \tilde{L}_x$, which means that

$$x \in X_{(te)^*(te)} = X_{t^*t} \cap X_e.$$

Hence, t and e lie in $\tilde{L}_x$. Moreover, $te = se$ and $\theta_s(x) = \theta_{te}(x) = \theta_t(\theta_e(x)) = \theta_t(x)$, as stated.

Finally, if $x \in X_{s^*s}$, then $s \in \tilde{L}_x$ and, by the previous part, $\theta_t(x) = \theta_s(x)$. This proves that $\theta_t(X_{s^*s}) = \theta_s(X_{s^*s}) = X_{ss^*}$. □

A central ingredient in the induction process is the vector space M_x with basis L_x. We shall denote a basis element of M_x by $\delta_{[s,x]}$ with $[s, x]$ in L_x. Since $L_xG_x \subseteq L_x$, M_x has a natural right KG_x-module structure.

Consider the bilinear form

$$\langle \cdot, \cdot \rangle : M_x \times M_x \to KG_x$$

such that

$$\langle \delta_{[s,x]}, \delta_{[t,x]} \rangle = \begin{cases} \delta_{[s^*t,x]}, & \text{if } s^*t \in \tilde{G}_x, \\ 0, & \text{otherwise.} \end{cases}$$

It is important to notice that this bilinear form is well defined, that is, does not depend on representatives. This said, we shall also express this form as

$$\langle \delta_{[s,x]}, \delta_{[t,x]} \rangle = [{\scriptstyle s^*t \in \tilde{G}_x}]\, \delta_{[s^*t,x]},$$

where the brackets indicate *boolean value* [2].

An important property of this form, which may be easily proved, is expressed by the identity

$$\langle m, na \rangle = \langle m, n \rangle a, \tag{3.1.4}$$

for all $m, n \in M_x$ and all $a \in KG_x$.

[2] We shall often use boolean value, even in a slightly abusive way. For example, in (3.1.6), we have the expression $[{\scriptstyle st \in \tilde{L}_x}]\, f\big(\theta_{st}(x)\big)\delta_{[st,x]}$. Indeed, if st does not lie in $\tilde{L}_x$, then it is not coherent to write $\theta_{st}(x)$. However, in this case, we mean that the expression equals $f\big(\theta_{st}(x)\big)\delta_{[st,x]}$ if the content in the brackets is true and the expression equals zero otherwise.

We say that $R_x \subseteq L_x$ is a *total system of representatives of left G_x-classes* if, for every $[s,x]$ in L_x, there exists precisely one element $[r,x]$ in R_x such that $\theta_r(x) = \theta_s(x)$. By the previous comment, this amounts to say that r^*s lies in $\tilde{G}_x$.

Proposition 3.1.5. *If $R_x \subseteq L_x$ is a total system of representatives of left G_x-classes, then, for all $m \in M_x$, we have*

$$m = \sum_{[r,x]\in R_x} \delta_{[r,x]}\langle \delta_{[r,x]}, m\rangle,$$

where the sum is always finite in the sense that there are only finitely many nonzero summands.

Proof. Assume initially that $m = \delta_{[s,x]}$ for some $[s,x] \in L_x$. So, there exists an unique element $[t,x]$ in R_x such that t^*s lies in $\tilde{G}_x$. Hence,

$$\begin{aligned}\sum_{[r,x]\in R_x} \delta_{[r,x]}\langle \delta_{[r,x]}, \delta_{[s,x]}\rangle &= \sum_{[r,x]\in R_x} \delta_{[r,x]} \left[r^*s\in\tilde{G}_x\right] \delta_{[r^*s,x]} \\ &= \delta_{[t,x]}\delta_{[t^*s,x]} \overset{(3.1.3)}{=} \delta_{[s,x]}.\end{aligned}$$

By writing m as a combination of elements of the form $m = \delta_{[s,x]}$ for $[s,x] \in L_x$, we may reach the general case. $\square$

We can now derive a very important fact about M_x. It is, there exists a left $\mathscr{L}_c(X) \rtimes S$-module structure compatible with its right KG_x-module structure.

Proposition 3.1.6. *There is a left $\mathscr{L}_c(X) \rtimes S$-module structure on M_x such that*

$$(f\Delta_s).\delta_{[t,x]} = \left[st\in\tilde{L}_x\right] f(\theta_{st}(x))\delta_{[st,x]},$$

for every $f \in \mathscr{L}_c(X_{ss^*})$ and every $t \in \tilde{L}_x$. Furthermore, with this structure M_x becomes a $\mathscr{L}_c(X) \rtimes S$-$KG_x$-bimodule.

Proof. We shall prove first that there is a well defined left $\mathcal{L}(\mathcal{B}^\theta)$-module structure on M_x, such that

$$(f\Delta_s).\delta_{[t,x]} = \left[st\in\tilde{L}_x\right] f(\theta_{st}(x))\delta_{[st,x]}.$$

Indeed, let t and t' in $\tilde{L}_x$ such that $[t,x] = [t',x]$. Hence, there exists $e \in \tilde{L}_x$ such that $te = t'e$.

Suppose $st \in \tilde{L}_x$. Since $e \in \tilde{L}_x$, we have:

$$\begin{aligned} x \in X_{(st)^*(st)} \cap X_e = X_{e(st)^*(st)e} &= X_{et^*s^*ste} \\ &= X_{et'^*s^*st'e} = X_{(st')^*(st')} \cap X_e. \end{aligned}$$

Hence, st' lies in $\tilde{L}_x$. Moreover, $[st, x] = [st', x]$, since $ste = st'e$ and

$$\theta_{st}(x) = \theta_{st}(\theta_e(x)) = \theta_{ste}(x) = \theta_{st'e}(x) = \theta_{st'}(\theta_e(x)) = \theta_{st'}(x).$$

Therefore, it is indeed a well defined action of $\mathcal{L}(\mathcal{B}^\theta)$ on M_x. We shall show now that the ideal $\mathcal{J}_\alpha$, as in (2.2.11), acts trivially on M_x. For the task, let $\delta_{[r,x]}$ in M_x and let $b = f\Delta_s - f\Delta_t$ be an element in $\mathcal{J}_\alpha$, with $s \leq t$. First, notice that $sr \leq tr$, since $s \leq t$.

If sr lies in $\tilde{L}_x$, by (3.1.3), we also have $tr \in \tilde{L}_x$, $[sr, x] = [tr, x]$ and $\theta_{sr}(x) = \theta_{tr}(x)$. In this case, $b \cdot \delta_{[r,x]} = 0$. The same thing happens if both sr and tr do not lie in $\tilde{L}_x$.

Suppose now that sr does not lie in $\tilde{L}_x$, but tr does. In this case, $\theta_{tr}(x)$ does not lie in X_{ss^*}, which contains the support of f. Indeed, otherwise, we should have $\theta_r(x) \in \theta_{t^*}(X_{ss^*}) = X_{s^*s}$, where the last equality comes from (3.1.3), since $s^* \leq t^*$. This gives $\theta_r(x) \in X_{s^*s} \cap X_{rr^*}$, from where we conclude that

$$x \in \theta_{r^*}(X_{s^*s} \cap X_{rr^*}) = X_{r^*s^*sr} = X_{(sr)^*(sr)},$$

which can not happen by assumption.

Hence, the action factors trough $\mathcal{J}_\alpha$, giving an action of $\mathscr{L}_c(X) \rtimes S$ on M_x, as desired. It is now standard to verify that M_x is a $\mathscr{L}_c(X) \rtimes S$-$KG_x$-bimodule. $\square$

We can now induce $\mathscr{L}_c(X) \rtimes S$-modules from KG_x-modules in the following way. Given any left KG_x-module V, the tensor product

$$M_x \otimes_{KG_x} V,$$

is a left $\mathscr{L}_c(X) \rtimes S$-module, henceforth denoted simply by $M_x \otimes V$.

Definition 3.1.7. The $\mathscr{L}_c(X) \rtimes S$-module $M_x \otimes V$ mentioned above is said to be the module *induced* by V and, will be denoted by $\mathrm{Ind}_x(V)$.

The next lemma is a technical result which will be an important tool to compute the annihilator of the induced module in terms of the annihilator of the original module V.

Lemma 3.1.8. *Let V be a left KG_x-module and let I be the annihilator of V in KG_x. Given $m \in M_x$, the following are equivalent:*

(i) $m \otimes v = 0$, *for all* $v \in V$;

(ii) $\langle n, m \rangle \in I$, *for all* $n \in M_x$.

Proof. Let $n \in M_x$ and consider the bilinear map

$$(m, v) \in M_x \times V \mapsto \langle n, m \rangle v \in V.$$

By (3.1.4), this is KG_x-balanced, so there is a well defined K-linear map $T_n : M_x \otimes V \to V$, such that

$$T_n(m \otimes v) = \langle n, m \rangle v,$$

So, if (i) is valid for $m \in M_x$, then

$$\langle n, m \rangle v = T_n(m \otimes v) = 0,$$

for all $n \in M_x$ and all $v \in V$. Hence, $\langle n, m \rangle$ lies in the annihilator of V for all $n \in M_x$, proving that (ii) is valid for m as well.

Conversely, let $m \in M_x$ and assume (ii) is valid for m. Let $R_x \subseteq L_x$ be a total system of representatives of left G_x-classes. Then, for every $v \in V$, we have

$$m \otimes v \overset{(3.1.5)}{=} \sum_{[r,x] \in R_x} \delta_{[r,x]} \langle \delta_{[r,x]}, m \rangle \otimes v = \sum_{[r,x] \in R_x} \delta_{[r,x]} \otimes \langle \delta_{[r,x]}, m \rangle v = 0,$$

proving that (i) is valid for m. □

We immediately obtain the following description for the annihilator of an induced module.

Corollary 3.1.9. *Let V be a left KG_x-module and let I be the annihilator of V in KG_x. Then,*

$$\{b \in \mathscr{L}_c(X) \rtimes S : \langle n, bm \rangle \in I, \forall n, m \in M_x\},$$

is the annihilator of $M_x \otimes V$ in $\mathscr{L}_c(X) \rtimes S$.

Notice that, if $I \trianglelefteq KG_x$, then KG_x/I is a left KG_x-module which is annihilated by I. Hence, every ideal of KG_x is the annihilator of a left KG_x-module. This motivates the following definition.

Definition 3.1.10. Given any ideal $I \trianglelefteq KG_x$, we define

$$\operatorname{Ind}_x(I) := \{b \in \mathscr{L}_c(X) \rtimes S : \langle n, bm \rangle \in I, \forall n, m \in M_x\},$$

and call it the *ideal induced by I*.

Notice that $\mathrm{Ind}_x(I)$ is a two-sided ideal. Moreover, notice that the annihilator of an induced $\mathscr{L}_c(X) \rtimes S$-module is the ideal induced from the annihilator of the original KG_x-module. For further reference, we reinterpret (3.1.9) from this new point of view.

Proposition 3.1.11. *Let V be a left KG_x-module and I the annihilator of V in KG_x. Then the annihilator of $M_x \otimes V$ coincides with the ideal induced by I.*

We now start to explore the induction process by introducing a clear fact about the behavior of the induction process under inclusion and intersection.

Proposition 3.1.12.

(i) If I_1 and I_2 are ideals of KG_x with $I_1 \subseteq I_2$, then

$$\mathrm{Ind}_x(I_1) \subseteq \mathrm{Ind}_x(I_2).$$

(ii) If $\{I_\lambda\}_{\lambda\in\Lambda}$ is a family of ideals of KG_x, then

$$\mathrm{Ind}_x\left(\bigcap_{\lambda\in\Lambda} I_\lambda\right) = \bigcap_{\lambda\in\Lambda} \mathrm{Ind}_x(I_\lambda).$$

Notice that the task of checking that $\langle n, bm\rangle \in I$ for all $n, m \in M_x$, as required by the above definition, may be simplified by considering $n = \delta_{[s,x]}$ and $m = \delta_{[t,x]}$, for $s, t \in \tilde{L}_x$, since these generate M_x. So, the next result is an important tool to use in this situation.

Proposition 3.1.13. *Given $b = \sum_{s\in S} f_s\Delta_s \in \mathscr{L}_c(X) \rtimes S$ and $k, l \in \tilde{L}_x$, we have that*

$$\langle \delta_{[k,x]}, b\delta_{[l,x]}\rangle = \sum_{s\in K_x} f_s(\theta_k(x))\delta_{[k^*sl,x]},$$

where K_x is the set of all elements $s \in S$ such that k^*sl lies in $\tilde{G}_x$.

Proof. By a simple computation, we obtain

$$\begin{aligned}
\langle \delta_{[k,x]}, b\delta_{[l,x]}\rangle &= \sum_{s\in S}\langle \delta_{[k,x]}, (f_s\Delta_s)\delta_{[l,x]}\rangle \\
&= \sum_{s\in S}[sl\in\tilde{L}_x]\, f_s(\theta_{sl}(x))\langle \delta_{[k,x]}, \delta_{[sl,x]}\rangle \\
&= \sum_{s\in K_x} [sl\in\tilde{L}_x]\, f_s(\theta_{sl}(x))\delta_{[k^*sl,x]} = \dots
\end{aligned}$$

Notice that, $k^*sl \in \tilde{G}_x$ means that x lies in the domains of θ_{k^*sl} and $\theta_{k^*sl}(x) = x$. By applying θ_k in both sides of the last equality, we obtain $\theta_{sl}(x) = \theta_k(x)$. Nevertheless, if the right side of the last equality is well defined, so is the left side, which amounts to say that sl lies in $\tilde{L}_x$. Hence, the above equals

$$\ldots = \sum_{s \in K_x} f_s(\theta_k(x))\delta_{[k^*sl,x]},$$

as desired. □

By combining this proposition with the comment that motivated it immediately before, we get a criteria for membership in $\mathrm{Ind}_x(I)$:

Proposition 3.1.14. *Given an ideal $I \trianglelefteq KG_x$ and $b = \sum_{s\in S} f_s\Delta_s \in \mathscr{L}_c(X) \rtimes S$, we have that $b \in \mathrm{Ind}_x(I)$, if and only if,*

$$\sum_{s \in K_x} f_s(\theta_k(x))\delta_{[k^*sl,x]} \in I,$$

for every $k, l \in \tilde{L}_x$.

We now proceed to introduce another fundamental concept in the induction process. For each $x \in X$, consider the map $\Gamma_x : \mathscr{L}_c(X) \rtimes S \to KG_x$ given by

$$\Gamma_x\left(\sum_{s\in S} f_s\Delta_s\right) = \sum_{s\in\tilde{G}_x} f_s(x)\delta_{[s,x]}. \tag{3.1.15}$$

We shall show next that it is indeed a well defined map.

Lemma 3.1.16. *For every $x \in X$, the map Γ_x introduced in (3.1.15) above is a well defined linear map.*

Proof. In fact, we shall show that the map Γ'_x, defined by

$$\sum_{s\in S} f_s\boldsymbol{\delta}_s \in \mathcal{L}(\mathcal{B}^\theta) \mapsto \sum_{s\in\tilde{G}_x} f_s(x)\delta_{[s,x]} \in KG_x$$

vanishes on $\mathscr{J}_\alpha$.

For the task, let $b = f\boldsymbol{\delta}_s - f\boldsymbol{\delta}_t$ lie in $\mathscr{J}_\alpha$, with $s \leq t$. Therefore, there are two possible scenarios for s:

- $s \in \tilde{G}_x$: In this case, by (3.1.3), t also lies in $\tilde{G}_x$ and $[s, x] = [t, x]$. Hence, b lies in the kernel of Γ'_x.

- $s \notin \tilde{G}_x$: In this case, again we have two distinct scenarios:

  If $s \in \tilde{L}_x$, by (3.1.3), $t \in \tilde{L}_x$ and $\theta_t(x) = \theta_s(x) \neq x$. Hence, again b lies in the kernel of Γ'_x.

  Otherwise, if both s and t does not lie in $\tilde{G}_x$, b lies in the kernel Γ'_x, and, if s does not lie in $\tilde{G}_x$ but t does lie, we must have $s^* \notin \tilde{L}_x$. Indeed, otherwise, since $s^* \leq t^*$, by (3.1.3), $x = \theta_{t^*}(x) = \theta_{s^*}(x) \in X_{s^*s}$, contradicting the initial assumption. But this means that $x \notin X_{ss^*}$ which contains the support of f, and so $f(x) = 0$. Hence, b lies in the kernel of Γ'_x.

Therefore, Γ'_x factors through $\mathcal{J}_\alpha$, giving the desired map Γ_x. $\square$

The next lemma suggests a close relation between the maps Γ_x and the induction process.

Lemma 3.1.17. *Let $k, l \in S$, $p \in \mathscr{L}_c(X_{k^*k})$, $q \in \mathscr{L}_c(X_{ll^*})$ and set*

$$u = p\Delta_{k^*} \text{ and } v = q\Delta_l.$$

Then, for every $b \in \mathscr{L}_c(X) \rtimes S$, one has that

$$\Gamma_x(ubv) = \begin{cases} p(x)q(\theta_l(x))\langle \delta_{[k,x]}, b\delta_{[l,x]}\rangle, & \text{if } k, l \in \tilde{L}_x, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Writing $b = \sum_{s\in S} f_s\Delta_s$, we have

$$\begin{aligned} ubv &= \sum_{s\in S} p\Delta_{k^*} \cdot f_s\Delta_s \cdot q\Delta_l \\ &= \sum_{s\in S} p\bar{\alpha}_{k^*}(f_s)\Delta_{k^*s} \cdot q\Delta_l \\ &= \sum_{s\in S} p\bar{\alpha}_{k^*}(f_s)\bar{\alpha}_{k^*s}(q)\Delta_{k^*sl}. \end{aligned}$$

Hence, by setting $K_x = \{s \in S : k^*sl \in \tilde{G}_x\}$, we obtain

$$\Gamma_x(ubv) = \sum_{s\in K_x} p(x)[k\in\tilde{L}_x]\, f_s(\theta_k(x))[s^*k\in\tilde{L}_x]\, q(\theta_{s^*k}(x))\delta_{[k^*sl,x]} = \dots$$

Notice that, if $s \in K_x$, then $\theta_{k^*sl}(x) = x$. By, applying θ_{s^*k} in both sides of the last equality, we obtain $\theta_l(x) = \theta_{s^*k}(x)$. Hence, the above equals

$$\begin{aligned} \dots &= [k,l\in\tilde{L}_x]\, p(x)q(\theta_l(x)) \sum_{s\in K_x} f_s(\theta_k(x))\delta_{[k^*sl,x]} \\ &= [k,l\in\tilde{L}_x]\, p(x)q(\theta_l(x))\langle \delta_{[k,x]}, b\delta_{[l,x]}\rangle, \end{aligned}$$

as desired. ☐

We now spell out an alternative definition of $\mathrm{Ind}_x(I)$ in terms of Γ_x.

Proposition 3.1.18. *If $I \trianglelefteq KG_x$ is an ideal, then*

$$\mathrm{Ind}_x(I) = \{b \in \mathscr{L}_c(X) \rtimes S : \Gamma_x(ubv) \in I \textit{ for all } u, v \in \mathscr{L}_c(X) \rtimes S\}.$$

Proof. Notice that, it is enough to prove that, for any $b \in \mathscr{L}_c(X) \rtimes S$, the following are equivalent:

(i) $\Gamma_x(ubv) \in I$ for all $u, v \in \mathscr{L}_c(X) \rtimes S$;

(ii) $\langle \delta_{[k,x]}, b\delta_{[l,x]} \rangle \in I$ for all $k, l \in \tilde{L}_x$.

(i) $\Rightarrow$ (ii): Let $k, l \in \tilde{L}_x$ and choose functions $p \in \mathscr{L}_c(X_{k^*k})$ and $q \in \mathscr{L}_c(X_{ll^*})$ such that $p(x) = 1$ and $q(\theta_l(x)) = 1$. Letting $u = p\Delta_{k^*}$ and $v = q\Delta_l$, by Lemma (3.1.17) we thus have

$$I \ni \Gamma_x(ubv) = p(x)q(\theta_l(x))\langle \delta_{[s,x]}, b\delta_{[l,x]} \rangle = \langle \delta_{[s,x]}, b\delta_{[l,x]} \rangle.$$

(ii) $\Rightarrow$ (i): Conversely, it is enough to prove (i) for $u = p\Delta_{k^*}$ and $v = q\Delta_l$, where k and l are arbitrary elements in S. By Lemma (3.1.17), we have

$$\Gamma_x(ubv) = p(x)q(\theta_l(x))\langle \delta_{[k,x]}, f\delta_{[l,x]} \rangle \in I,$$

if k and l lie in $\tilde{L}_x$, or

$$\Gamma_x(ubv) = 0 \in I,$$

otherwise, thus proving (i) in either case. ☐

3.2 ADMISSIBLE IDEALS

In this section we explore the relationship between induced ideals in $\mathscr{L}_c(X) \rtimes S$ and the ideals of the isotropy group algebra they came from. In this context, we introduce the concept of an admissible ideal. Roughly speaking, the admissible ideals are the ones which actually play a relevant role in the induction process.

For that task, again Γ_x will play a relevant role and we begin by expelling out an important behavior of Γ_x.

Proposition 3.2.1. *Let $t \in \tilde{G}_x$ and $\varphi \in \mathscr{L}_c(X_{tt^*})$. Then, setting $a = \varphi\Delta_t$, we have that*

$$\Gamma_x(ab) = \Gamma_x(a)\Gamma_x(b) \textit{ and } \Gamma_x(ba) = \Gamma_x(b)\Gamma_x(a),$$

for every $b \in \mathscr{L}_c(X) \rtimes S$.

Proof. Let $b = \sum_{s\in S} f_s\Delta_s$ and notice that, st lies in $\tilde{G}_x$ if and only if s lies in $\tilde{G}_x$, since we already have $t \in \tilde{G}_x$. Then, by a simple computation, we get

$$\begin{aligned}\Gamma_x(ba) &= \Gamma\left(\sum_{s\in S} f_s\bar{\alpha}_s(\varphi)\Delta_{st}\right) = \sum_{s\in\tilde{G}_x} f_s(x)\bar{\alpha}_s(\varphi)|_x\delta_{[st,x]}\\ &= \sum_{s\in\tilde{G}_x} f_s(x)\varphi(x)\delta_{[st,x]} = \left(\sum_{s\in\tilde{G}_x} f_s(x)\delta_{[s,x]}\right)\varphi(x)\delta_{[t,x]}\\ &= \Gamma_x(b)\Gamma_x(a).\end{aligned}$$

Similarly, we can show $\Gamma_x(ab) = \Gamma_x(a)\Gamma_x(b)$, concluding the proof. ☐

Proposition 3.2.2. *Let $J \trianglelefteq \mathscr{L}_c(X) \rtimes S$ be an ideal. Then, $\Gamma_x(J)$ is an ideal in KG_x.*

Proof. Let $a \in \Gamma_x(J)$ and $c = \delta_{[t,x]} \in KG_x$ for some $t \in \tilde{G}_x$. Then, there exists $b \in J$ such that $\Gamma_x(b) = a$. Notice that, by choosing $\varphi \in \mathscr{L}_c(X_{tt^*})$ such that $\varphi(x) = 1$, we have

$$ac = a\varphi(x)\delta_{[t,x]} = \Gamma_x(b)\Gamma_x(\varphi\Delta_t) \overset{(3.2.1)}{=} \Gamma_x(b\cdot\varphi\Delta_t) \in \Gamma_x(J).$$

By linearity, we deduce that $ac \in \Gamma_x(J)$ for arbitrary $c \in KG_x$ and similarly, we can show that $ca \in \Gamma_x(J)$. ☐

We then have the following proposition.

Proposition 3.2.3. *Let I be an ideal in KG_x, and put $I' = \Gamma_x(\mathrm{Ind}_x(I))$. Then*

(i) *I' is an ideal of KG_x;*

(ii) *$I' \subseteq I$;*

(iii) $\mathrm{Ind}_x(I') = \mathrm{Ind}_x(I)$.

Proof. (i) Follows from (3.2.2).

(ii) Given $a \in I'$, let $b \in \mathrm{Ind}_x(I)$ such that $\Gamma_x(b) = a$. Notice that, if we choose an idempotent $e \in \tilde{G}_x$ and $\varphi \in \mathscr{L}_c(X_e)$ such that $\varphi(x) = 1$ and then setting $d = \varphi\Delta_e$ we have

$$a = \delta_{[e,x]}a\delta_{[e,x]} = \Gamma_x(d)\Gamma_x(b)\Gamma_x(d) \overset{(3.2.1)}{=} \Gamma_x(dbd) \in I,$$

by (3.1.18).

(iii) The inclusion $\mathrm{Ind}_x(I') \subseteq \mathrm{Ind}_x(I)$ follows from (ii). On the other hand, if $b \in \mathrm{Ind}_x(I)$, then $ubv \in \mathrm{Ind}_x(I)$ for all $u, v \in \mathscr{L}_c(X) \rtimes S$. Hence,

$$\Gamma_x(ubv) \in \Gamma_x(\mathrm{Ind}_x(I)) = I'.$$

By (3.1.18), we conclude that $b \in \mathrm{Ind}_x(I')$, as wanted.

□

Notice that, if I is an ideal of KG_x, then I and $\Gamma_x(\mathrm{Ind}_x(I))$ induce the same ideal. This motivates the following definition, which intends to identify the ideals that play a relevant role in the induction process.

Definition 3.2.4. An ideal $I \trianglelefteq KG_x$ is said to be *admissible* if $\Gamma_x(\mathrm{Ind}_x(I)) = I$.

In this setting, we have.

Corollary 3.2.5. *For every ideal $I \trianglelefteq KG_x$, there exists an unique admissible ideal $I' \subseteq I$, such that* $\mathrm{Ind}_x(I') = \mathrm{Ind}_x(I)$.

Proof. Set $I' = \Gamma_x(\mathrm{Ind}_x(I))$. By (3.2.3.iii), we have $\mathrm{Ind}_x(I') = \mathrm{Ind}_x(I)$. Moreover,

$$\Gamma_x(\mathrm{Ind}_x(I')) = \Gamma_x(\mathrm{Ind}_x(I)) = I',$$

so I' is admissible. Finally, if I' and I'' are two admissible ideals inducing the same ideal of $\mathscr{L}_c(X) \rtimes S$, then

$$I' = \Gamma_x(\mathrm{Ind}_x(I')) = \Gamma_x(\mathrm{Ind}_x(I'')) = I''.$$

□

We already have two examples of induced ideals.

Proposition 3.2.6. *The trivial ideals of KG_x are admissible.*

Proof. Notice that

$$\{0\} \subseteq \Gamma_x(\mathrm{Ind}_x(\{0\})) \overset{(3.2.3.ii)}{\subseteq} \{0\}$$

so $\{0\}$ is admissible. On the other hand, we have

$$\Gamma_x(\mathrm{Ind}_x(KG_x)) = \Gamma_x(\mathscr{L}_c(X) \rtimes S) = KG_x,$$

so KG_x is admissible. □

Once we have studied the relationship of I and $\Gamma_x(\mathrm{Ind}_x(I))$, in the case I is an ideal of KG_x, we may ask ourselves about the relationship of J and $\mathrm{Ind}_x(\Gamma_x(J))$, in the case J is an ideal in $\mathscr{L}_c(X) \rtimes S$.

Proposition 3.2.7.

(i) If $J \trianglelefteq \mathscr{L}_c(X) \rtimes S$, then $J \subseteq \mathrm{Ind}_x(\Gamma_x(J))$.

(ii) If $I \trianglelefteq KG_x$, then $\mathrm{Ind}_x(I)$ is the largest among the ideals $J \trianglelefteq \mathscr{L}_c(X) \rtimes S$ satisfying $\Gamma_x(J) \subseteq I$.

Proof.

(i) If $b \in J$, then for every $u, v \in \mathscr{L}_c(X) \rtimes S$, $ubv \in J$ and so

$$\Gamma_x(ubv) \in \Gamma_x(J),$$

from where we deduce, by (3.1.18), that $b \in \mathrm{Ind}_x(\Gamma_x(J))$.

(ii) By (3.2.3.ii), $\Gamma_x(\mathrm{Ind}_x(I)) \subseteq I$. Moreover, if $J \trianglelefteq \mathscr{L}_c(X) \rtimes S$ satisfies $\Gamma_x(J) \subseteq I$, then

$$J \subseteq \mathrm{Ind}_x(\Gamma_x(J)) \subseteq \mathrm{Ind}_x(I).$$

□

Finally, Γ_x always leads to admissible ideals.

Proposition 3.2.8. *For any ideal $J \trianglelefteq \mathscr{L}_c(X) \rtimes S$, $\Gamma_x(J)$ is an admissible ideal of KG_x.*

Proof. By (3.2.3.i), $J \subseteq \mathrm{Ind}_x(\Gamma_x(J))$. Hence,

$$\Gamma_x(J) \subseteq \Gamma_x(\mathrm{Ind}_x(\Gamma_x(J))) \overset{(3.2.3.ii)}{\subseteq} \Gamma_x(J).$$

Therefore, $\Gamma_x(J)$ is admissible. □

4 EFFROS-HAHN LIKE THEOREM

In this chapter we shall prove that any ideal (always meaning two-sided ideal) of $\mathscr{L}_c(X) \rtimes S$ is the intersection of ideals induced from isotropy subgroups. Our methods will largely rely on representation theory, as we shall see.

4.1 REPRESENTATIONS

From this point on, we will fix an arbitrary ideal $J \trianglelefteq \mathscr{L}_c(X) \rtimes S$ which, in view of (2.3.11) and (2.3.12), we may assume it is the kernel of a likewise fixed non-degenerate representation

$$\pi : \mathscr{L}_c(X) \rtimes S \to L(V).$$

By (2.3.9), we may disintegrate it to a covariant representation (π_0, σ) of (θ, S, X) such that

$$\pi(f\Delta_s) = \pi_0(f)\sigma_s.$$

By the comment immediately after Proposition (2.3.14), we may identify $\mathscr{L}_c(X)$ as a subalgebra of $\mathscr{L}_c(X) \rtimes S$ and, in this fashion, we can also interpret π_0 as the restriction of π to $\mathscr{L}_c(X)$. Hence, by an abuse of notation, we shall also use π to denote π_0 from now on. The context should be enough to distinguish between the initial representation π of $\mathscr{L}_c(X) \rtimes S$ and the representation π of $\mathscr{L}_c(X)$ composing the covariant representation (π, σ) resulted from the disintegration of π.

Notice that the definition of induced ideals requires that a point of X to be chosen in advance, so we must begin to see our representation π from the point of view of a chosen point in X, a process which will eventually lead to a *discretization* of π.

For each $x \in X$, let

$$I_x = \{f \in \mathscr{L}_c(X) : f(x) = 0\}$$

which is clearly an ideal in $\mathscr{L}_c(X)$. Consequently,

$$Z_x := \operatorname{span}\{\pi(I_x)V\}$$

is invariant under $\mathscr{L}_c(X)$, so there is a well defined representation π_x of $\mathscr{L}_c(X)$ on $V_x := V/Z_x$ making the following diagram

$$\begin{array}{ccc} V & \xrightarrow{\pi(f)} & V \\ {\scriptstyle q_x}\downarrow & & \downarrow{\scriptstyle q_x} \\ V_x & \xrightarrow[\pi_x(f)]{} & V_x \end{array}$$

to commute for every $f \in \mathscr{L}_c(X)$.

The next proposition is an indication that the localization process is bearing fruits.

Proposition 4.1.1. *Let $x \in X$ and $f \in \mathscr{L}_c(X)$. Then, for every $\eta \in V_x$, we have*

$$\pi_x(f)\eta = f(x)\eta.$$

Proof. Since π is non-degenerate, it is enough to verify for $\eta = q_x(\pi(\varphi)\xi)$. Let C be a compact open set containing $\mathrm{supp}(\varphi) \cup \{x\}$ and notice that $1_C\varphi = \varphi$. Furthermore, $f - f(x)1_C$ lies in I_x and hence

$$f\varphi = (f - f(x)1_C + f(x)1_C)\varphi = (f - f(x)1_C)\varphi + f(x)\varphi \overset{mod\, I_x}{\equiv} f(x)\varphi.$$

We thus obtain

$$\pi(f\varphi)\xi \overset{mod\, Z_x}{\equiv} \pi(f(x)\varphi)\xi = f(x)\pi(\varphi)\xi.$$

Therefore,

$$\begin{aligned} \pi_x(f)\eta &= \pi_x(f)q_x(\pi(\varphi)\xi) = q_x(\pi(f)\pi(\varphi)\xi) = q_x(\pi(f\varphi)\xi) \\ &= q_x(f(x)\pi(\varphi)\xi) = f(x)q_x(\pi(\varphi)\xi) = f(x)\eta, \end{aligned}$$

concluding the proof. □

Combining the definition of π_x with the result above, we get the following useful formula

$$q_x(\pi(f)\xi) = \pi_x(f)q_x(\xi) = f(x)q_x(\xi) \tag{4.1.2}$$

for all $x \in X$, $f \in \mathscr{L}_c(X)$ and $\xi \in V$.

Now, we shall work with the homomorphism σ and, for a moment, let the maps π_x for aside.

Proposition 4.1.3. *If $x \in X_{s^*s}$, then $\sigma_s(Z_x) \subseteq Z_{\theta_s(x)}$. Furthermore, there exists a linear mapping*

$$\mu_s^x : V_x \to V_{\theta_s(x)},$$

such that, for all $\xi \in V$

$$\mu_s^x(q_x(\xi)) = q_{\theta_s(x)}(\sigma_s(\xi)).$$

Proof. Let $\xi \in Z_x$ be a vector of the form $\xi = \pi(\varphi)\eta$, where $\varphi \in I_x$ and $\eta \in V$. In such case,

$$\sigma_s(\xi) = \sigma_s\pi(\varphi)\eta \overset{(2.3.10)}{=} \pi(\bar{\alpha}_s(\varphi))\sigma_s\eta.$$

Noticing that

$$\bar{\alpha}_s(\varphi)|_{\theta_s(x)} = \varphi(\theta_{s^*}(\theta_s(x))) = \varphi(x) = 0,$$

we see that $\bar{\alpha}_s(\varphi)$ lies in $I_{\theta_s(x)}$. Hence, $\sigma_s(\xi)$ lies in $Z_{\theta_s(x)}$. The result then follows by linearity and the second part is an immediate consequence of the first. □

We shall explore some properties of the maps μ_s^x.

Proposition 4.1.4. *Let $e \in E(S)$ and $x \in X_e$, then μ_e^x is the identity map in V_x.*

Proof. Choose $\varphi \in \mathscr{L}_c(X_e)$ such that $\varphi(x) = 1$. Then,

$$q_x(\pi(\varphi)\eta) \overset{(4.1.2)}{=} \varphi(x)q_x(\eta) = q_x(\eta) \tag{4.1.5}$$

for all $\eta \in V$.

Let $\xi \in V$, then using (4.1.5) for $\eta = \sigma_e(\xi)$, we have

$$\begin{aligned}\mu_e^x(q_x(\xi)) = q_x(\sigma_e(\xi)) &\overset{(4.1.5)}{=} q_x(\pi(\varphi)\sigma_e\xi) \\ &\overset{(2.3.4)}{=} q_x(\pi(\varphi)\xi) \overset{(4.1.2)}{=} \varphi(x)q_x(\xi) = q_x(\xi),\end{aligned}$$

concluding the proof. □

The maps μ_s^x obey the following functorial property.

Proposition 4.1.6. *If $x \in X_{s^*t^*ts}$, then the composition*

$$V_x \xrightarrow{\mu_s^x} V_{\theta_s(x)} \xrightarrow{\mu_t^{\theta_s(x)}} V_{\theta_{ts}(x)}$$

coincides with μ_{ts}^x.

Proof. First notice that, since $x \in X_{s^*t^*ts}$, we must have $x \in X_{s^*s}$ and $\theta_s(x) \in X_{t^*t}$. Hence, for $\xi \in V$, we have

$$\begin{aligned}\mu_t^{\theta_s(x)}\Big(\mu_s^x(q_x(\xi))\Big) &= \mu_t^{\theta_s(x)}\Big(q_{\theta_s(x)}(\sigma_s(\xi))\Big) = q_{\theta_t(\theta_s(x))}\Big(\sigma_t\sigma_s(\xi)\Big)\\ &= q_{\theta_{ts}(x)}(\sigma_{ts}(\xi)) = \mu_{ts}^x(q_x(\xi)),\end{aligned}$$

proving the statement. □

Let us now consider the representation of $\mathscr{L}_c(X)$ on the cartesian product $\Pi_{x\in X}V_x$ given by

$$\Pi = \prod_{x\in X}\pi_x.$$

Thus, if $f \in \mathscr{L}_c(X)$, and $\eta = (\eta_x)_{x\in X} \in \prod_{x\in X} V_x$, we have

$$(\Pi(f)\eta)_x = \pi_x(f)\eta_x \overset{(4.1.1)}{=} f(x)\eta_x$$

for all $x \in X$.

Hence, $\Pi(f)$ is the block diagonal operator, acting on each V_x as scalar multiplication by $f(x)$.

Also, for each $s \in S$, consider the linear operator U_s on $\prod_{x\in X} V_x$, given by

$$U_s(\eta)_x = [x\in X_{ss^*}]\,\mu_s^{\theta_{s^*}(x)}(\eta_{\theta_{s^*}(x)})$$

for all $\eta = (\eta_x)_{x\in X} \in \prod_{x\in X} V_x$.

Proposition 4.1.7. *Identifying V_x as a subspace of $\prod_{x\in X} V_x$, in the natural way, we have:*

(i) if $x \notin X_{s^*s}$, then U_s vanishes on V_x;

(ii) if $x \in X_{s^*s}$, then U_s coincides with μ_s^x and hence maps V_x to $V_{\theta_s(x)}$;

(iii) if $x \in X_{s^*s}$, then U_s maps V_x bijectively onto $V_{\theta_s(x)}$;

(iv) if $x \in X_{s^*t^*ts}$, then the composition

$$V_x \xrightarrow{U_s} V_{\theta_s(x)} \xrightarrow{U_t} V_{\theta_{ts}(x)}$$

coincides with U_{ts} on V_x;

(v) U is a semigroup homomorphism.

Proof. Items (i) and (ii) are easy to see, while item (iv) follows immediately by (4.1.6). For item (iii), is is enough to notice that, by (iv), the restriction of U_{s^*} to $V_{\theta_s(x)}$ is the inverse of U_s restricted to V_x. Finally, in order to prove (v), notice that, $U_t \circ U_s = U_{ts}$ on V_x for every $x \in X_{s^*t^*ts}$ by item (iv), and, for $x \notin X_{s^*t^*ts}$, we must have $x \in X_{s^*s}$ or $\theta_s(x) \notin X_{t^*t}$. In either case, $U_t \circ U_s$ vanishes in V_x as U_{ts} does by item (i). □

Unfortunately, since we are considering the product of the V_x instead of the direct sum, the pair (Π, U) may not be a covariant representation of the system (θ, S, X). However, we can get around this problem with bare hands. The reason why we insist in maintain the product of the V_x will be clear later.

Proposition 4.1.8. *The pair (Π, U) can be integrated to a representation $\Pi \times U$ of $\mathscr{L}_c(X) \rtimes S$ on $\prod_{x\in X} V_x$ such that*

$$(\Pi \times U)(f\Delta_s) = \Pi(f)U_s.$$

Proof. We already know that U is a semigroup homomorphism and Π is a representation of $\mathscr{L}_c(X)$ on $\prod_{x\in X} V_x$, possibly degenerate. Furthermore, the pair (Π, U) may not satisfy condition (ii) of (2.3.3). However, it still satisfies condition (i). Indeed, for $s \in S$, $f \in \mathscr{L}_c(X_{s^*s})$ and $\eta \in \prod_{x\in X} V_x$, we have for all $x \in X$

$$\begin{aligned}
U_s\Pi(f)U_{s^*}(\eta)_x &= [x\in X_{ss^*}]\, \mu_s^{\theta_{s^*}(x)} \Big(\Pi(f)U_{s^*}(\eta)_{\theta_{s^*}(x)} \Big) \\
&= [x\in X_{ss^*}]\, \mu_s^{\theta_{s^*}(x)} \Big(f(\theta_{s^*}(x))U_{s^*}(\eta)_{\theta_{s^*}(x)} \Big) \\
&= [x\in X_{ss^*}]\, f(\theta_{s^*}(x))\mu_s^{\theta_{s^*}(x)}\mu_{s^*}^{\theta_s(x)}(\eta_x) \\
&\overset{(4.1.6)}{=} [x\in X_{ss^*}]\, \alpha_s(f)|_x \eta_x \\
&= \alpha_s(f)|_x \eta_x = \Big(\Pi(\alpha_s(f))\eta \Big)_x .
\end{aligned}$$

Now, let $e \in E(s)$, $f \in \mathscr{L}_c(X_e)$ and $\xi \in \prod_{x\in X} V_x$. Then, let η be the vector in $\prod_{x\in X} V_x$ such that $\eta_x = f(x)\xi_x$ and notice that

$$U_e(\eta)_x = [x\in X_e]\, \mu_e^x\Big(f(x)\xi_x\Big) = [x\in X_e]\, f(x)\xi_x = f(x)\xi_x = \Big(\Pi(f)\xi\Big)_x.$$

Hence, η is a vector such that $U_e(\eta) = \Pi(f)\xi$. Then, we can replace the use of condition (ii) of Definition (2.3.3) by this argument in the proof of Lemma (2.3.4) to obtain

$$U_e\Pi(f) = \Pi(f) = \Pi(f)U_e.$$

Replacing now the use of Lemma (2.3.4) by the equality above in the proof of (2.3.5) we obtain a representation of the semi-direct product bundle $\mathcal{B}^\theta$, possibly degenerate, which can be further integrated to a representation $\Pi \times U$ of $\mathscr{L}_c(X) \rtimes S$ on $\prod_{x\in X} V_x$ such that

$$(\Pi \times U)(f\Delta_s) = \Pi(f)U_s.$$

$\square$

Definition 4.1.9. The representation $\Pi \times U$ above will be referred as the *discretization* of the initially given representation π.

The reader may wonder why we have not considered the discretized representation acting on the direct sum of the V_x, instead of their product. The map which will be introduced in the next proposition is the main reason for that since it is an important tool to establish a relation between the null space of the original representation and the null space of its discretized form as we shall see.

Proposition 4.1.10. *The mapping*

$$Q : \xi \in V \mapsto (q_x(\xi))_{x\in X} \in \prod_{x\in X} V_x,$$

is injective and equivariant [1] *relative to the corresponding representations of $\mathscr{L}_c(X) \rtimes S$ on V and on $\prod_{x\in X} V_x$, respectively.*

[1] Recall that a linear map $T : E \to F$ between vector spaces E and F is equivariant relative to representations π_E and π_F of an algebra A on E and F, respectively, if it intertwines π_E and π_F, meaning that $T \circ \pi_E(a) = \pi_F(a) \circ T$ for every a in A.

Proof. Let $s \in S$, and $f \in \mathscr{L}_c(X_{ss^*})$. Then, for every ξ in V, and every $x \in X$, we have

$$\begin{aligned}((\Pi \times U)(f\Delta_s)Q(\xi))_x &= (\Pi(f)U_sQ(\xi))_x = f(x)(U_sQ(\xi))_x \\ &= f(x)[_{x\in X_{ss^*}}]\,\mu_s^{\theta_{s^*}(x)}(Q(\xi)_{\theta_{s^*}(x)}) \\ &= f(x)\mu_s^{\theta_{s^*}(x)}(q_{\theta_{s^*}(x)}(\xi)) \\ &= f(x)q_x(\sigma_s(\xi)) = q_x(\pi(f)\sigma_s(\xi)) \\ &= Q(\pi(f\Delta_s)\xi)_x.\end{aligned}$$

This proves that Q is covariant. In order to prove that Q is injective, suppose that $Q(\xi) = 0$, for some ξ in V. Since π is non-degenerate, there exists $f_i \in \mathscr{L}_c(X)$ and $\xi_i \in V$ such that

$$\xi = \sum_{i=1}^{n} \pi(f_i)\xi_i,$$

Define

$$D = \bigcup_{i=1}^{n} \operatorname{supp}(f_i),$$

so D is a compact open subset of X and we have

$$\xi = \sum_{i=1}^{n} \pi(1_D f_i)\xi_i = \pi(1_D)\sum_{i=1}^{n} \pi(f_i)\xi_i = \pi(1_D)\xi. \tag{4.1.11}$$

Now, for each x in X, we have $q_x(\xi) = 0$ by hypothesis. Thus, ξ lies in Z_x and so we may write

$$\xi = \sum_{i=1}^{n_x} \pi(f_i^{(x)})\xi_i^{(x)},$$

where $\xi_i^{(x)} \in V$ and $f_i^{(x)} \in I_x$. Since there are finitely many $f_i^{(x)}$, each of which locally constant, there exists a compact open neighborhood C_x of x where all of the $f_i^{(x)}$ vanish. Moreover,

$$\pi(1_{C_x})\xi = \sum_{i=1}^{n_x} \pi(1_{C_x} f_i^{(x)})\xi_i^{(x)} = 0. \tag{4.1.12}$$

Finally, $\{C_x\}_{x\in X}$ is an open cover of D, and hence we may find a finite set $\{x_1, \ldots, x_p\} \subseteq X$, such that $D \subseteq \bigcup_{i=1}^{p} C_{x_i}$. Putting

$$E_k = D \cap C_{x_k} \setminus \bigcup_{i=1}^{k-1} C_{x_i},$$

for $k = 1, \dots, p$, it is easy to see that the E_k are pairwise disjoint compact open sets, whose union coincides with D. Observing that $E_k \subseteq C_{x_k}$, we then have

$$\xi \overset{(4.1.11)}{=} \pi(1_D)\xi = \sum_{k=1}^{p} \pi(1_{E_k})\xi = \sum_{k=1}^{p} \pi(1_{E_k} 1_{C_{x_k}})\xi$$
$$= \sum_{k=1}^{p} \pi(1_{E_k})\pi(1_{C_{x_k}})\xi \overset{(4.1.12)}{=} 0.$$

This proves that Q is injective. □

Thus, we have an immediate consequence.

Corollary 4.1.13. *The null space of* $\Pi \times U$ *is contained in the null space of* π*.*

Proof. Let $b \in \ker(\Pi \times U)$. By (4.1.10), we have

$$0 = (\Pi \times U)(b)Q(\xi) = Q(\pi(b)\xi)$$

for all $\xi \in V$. Again by (4.1.10), Q is injective and, hence,

$$\pi(b)\xi = 0$$

for all $\xi \in V$, that is, $b \in \ker(\pi)$. □

From now on, we shall consider the subspace

$$\bigoplus_{x \in X} V_x \subseteq \prod_{x \in X} V_x,$$

consisting of the vectors with finitely many nonzero coordinates. It is easy to see that this subspace is invariant under $\Pi(f)$ for all $f \in \mathscr{L}_c(X)$, as well as under U_s for all $s \in S$. Consequently, it is also invariant under $\Pi \times U$.

Proposition 4.1.14. *The null space of the representation obtained by restricting* $\Pi \times U$ *to* $\bigoplus_{x \in X} V_x$ *coincides with the null space of* $\Pi \times U$ *itself.*

Proof. Suppose that $(\Pi \times U)(b)$ vanishes on $\bigoplus_{x \in X} V_x$ for some $b = \sum_{s \in S} f_s \Delta_s \in \mathscr{L}_c(X) \rtimes S$. Let $y \in X$ and $\eta = (\eta_x)_{x \in X} \in \prod_{x \in X} V_x$ and notice that

$$((\Pi \times U)(b)\eta)_y = \sum_{s \in S} (\Pi(f_s)U_s\eta)_y = \sum_{s \in S} f_s(y)[y \in X_{ss^*}]\, \mu_s^{\theta_{s^*}(y)}(\eta_{\theta_{s^*}(y)}).$$

Let $\eta' = (\eta'_x)_{x\in X}$ be the vector defined by either $\eta'_x = \eta_x$ if $x = \theta_{s^*}(y)$ for some $s \in S$ such that $y \in X_{ss^*}$ and $f_s \neq 0$, or $\eta'_x = 0$ otherwise. Then, it is clear that $\eta' \in \bigoplus_{x\in X} V_x$ and

$$((\Pi \times U)(b)\eta)_y = ((\Pi \times U)(b)\eta')_y = 0.$$

Since $y \in X$ and $\eta \in \prod_{x\in X} V_x$ are arbitrary, we deduce that $(\Pi\times U)(b) = 0$, concluding the argument. □

Regarding the space $\bigoplus_{x\in X} V_x$ where $\Pi \times U$ acts, we will identify each V_x as a subspace of $\bigoplus_{x\in X} V_x$, in the usual way. Thus, given $\xi \in V$, we shall think of $q_x(\xi)$ as the element of $\bigoplus_{x\in X} V_x$ whose coordinates all vanish, except for the x^{th} coordinate which takes on the value $q_x(\xi)$. In this fashion, notice that

$$\begin{aligned} \Pi(f)q_x(\xi) &= \pi_x(f)q_x(\xi) &&= q_x(\pi(f)\xi), \\ U_s(q_x(\xi)) &= [x\in X_{s^*s}]\, \mu_s^x(q_x(\xi)) &&= [x\in X_{s^*s}]\, q_{\theta_s(x)}(\sigma_s\xi), \end{aligned} \tag{4.1.15}$$

for all $f \in \mathscr{L}_c(X)$, $s \in S$, $x \in X$, and $\xi \in V$.

Since $\bigoplus_{x\in X} V_x$ is spanned by the union of the V_x, each of which is the range of the corresponding q_x, the formulas above determine the action of $\Pi(f)$ and U_s on the whole space $\bigoplus_{x\in X} V_x$. So, by combining them, we are able to give the following concrete description of the restriction of $\Pi \times U$ to $\bigoplus_{x\in X} V_x$.

Proposition 4.1.16. *Let $b = \sum_{s\in S} f_s\Delta_s$ in $\mathscr{L}_c(X) \rtimes S$. Then, for all $x \in X$ and $\xi \in V$, we have that*

$$(\Pi \times U)(b)q_x(\xi) = \sum_{s\in S} [x\in X_{s^*s}]\, q_{\theta_s(x)}(\pi(f_s)\sigma_s\xi).$$

Proof. By (4.1.15), the proof reduces to a direct computation:

$$\begin{aligned} (\Pi \times U)(b)q_x(\xi) &= \sum_{s\in S} \Pi(f_s)U_s(q_x(\xi)) = \sum_{s\in S} \Pi(f_s)[x\in X_{s^*s}]\, q_{\theta_s(x)}(\sigma_s\xi) \\ &= \sum_{s\in S} [x\in X_{s^*s}]\, q_{\theta_s(x)}(\pi(f_s)\sigma_s\xi). \end{aligned}$$

□

We are going to describe now the *matrix entries* of the operator $(\Pi\times U)(b)$ acting on $\bigoplus_{x\in X} V_x$. That is, for each x and y in X, we want an expression for the y^{th} component of the vector obtained by applying $(\Pi\times U)(b)$ to any given vector in V_x, say of the form $q_x(\xi)$, where $\xi \in V$.

It is clear that the desired expression is the y^{th} component of the expression given in (4.1.16), which is in turn given by the partial sum corresponding to the terms for which $\theta_s(x) = y$. So, we have

$$\begin{aligned} \big((\Pi \times U)(b)q_x(\xi)\big)_y &= \sum_{s\in S, \theta_s(x)=y} q_{\theta_s(x)}(\pi(f_s)\sigma_s\xi) \\ &= q_y\Bigg(\sum_{s\in S, \theta_s(x)=y} \pi(f_s)\sigma_s\xi\Bigg). \end{aligned} \tag{4.1.17}$$

We are going to prove now that the restriction of the discretized representation $\Pi \times U$ to $\bigoplus_{x\in X} V_x$ has the same null space as the original representation π has. But first, recall that in (4.1.13) and (4.1.14) we already proved the following relations among the null spaces:

$$\ker(\pi) \supseteq \ker(\Pi \times U) = \ker\big(\Pi \times U|_{\oplus_{x\in X} V_x}\big). \tag{4.1.18}$$

We are going to show now that equality in fact holds throughout.

Theorem 4.1.19. *The null space of the representation obtained by restricting $\Pi \times U$ to $\bigoplus_{x\in X} V_x$ coincides with the null space of π.*

Proof. It is important to emphasize that, since $(\Pi \times U)(b)$ is well defined on each V_x, then so is the right-hand-side in (4.1.17). Precisely, if ξ and ξ' are elements of V such that $q_x(\xi) = q_x(\xi')$, then

$$q_y\Bigg(\sum_{s\in S, \theta_s(x)=y} \pi(f_s)\sigma_s\xi\Bigg) = q_y\Bigg(\sum_{s\in S, \theta_s(x)=y} \pi(f_s)\sigma_s\xi'\Bigg). \tag{4.1.20}$$

By (4.1.18), if b is in the null space of π, it is enough to prove that $(\Pi \times U)(b)$ vanishes on $\bigoplus_{x\in X} V_x$, which amounts to prove that its matrix entries given by (4.1.17) vanish for all x and y in X.

Let $b = \sum_{s\in S} f_s\Delta_s$ and $\Lambda \subseteq S$ be the subset consisting of those s for which $f_s \neq 0$, and notice that Λ decomposes as the disjoint union of the following subsets:

$$\begin{aligned} \Lambda_1 &= \{s \in \Lambda : y \notin X_{ss^*}\}, \\ \Lambda_2 &= \{s \in \Lambda : y \in X_{ss^*},\ \theta_{s^*}(y) \neq x\}, \\ \Lambda_3 &= \{s \in \Lambda : y \in X_{ss^*},\ \theta_{s^*}(y) = x\}. \end{aligned}$$

From our hypothesis that $\pi(b) = 0$, we have that, for every $\eta \in V$,

$$0 = \pi(b)\eta = \sum_{s\in\Lambda} \pi(f_s\Delta_s)\eta = \sum_{s\in\Lambda} \pi(f_s)\sigma_s\eta. \tag{4.1.21}$$

Comparing this expression with the last part of (4.1.17), we are summing over all of Λ, while only the terms corresponding to Λ_3 are being considered there. In order to fix it, notice that x does not lie in the finite set $\{\theta_{s^*}(y) : s \in \Lambda_2\}$, so we may choose $\varphi \in \mathscr{L}_c(X)$ such that $\varphi(x) = 1$ and $\varphi(\theta_{s^*}(y)) = 0$ for all $s \in \Lambda_2$.

Let $\xi' := \pi(\varphi)\xi$ and notice that

$$q_y\Big(\pi(f_s)\sigma_s\xi'\Big) = q_y\Big(\pi(f_s)\sigma_s\pi(\varphi)\xi\Big) = q_y\Big(\pi(f_s\bar{\alpha}_s(\varphi))\sigma_s\xi\Big) \\ \overset{(4.1.2)}{=} f_s(y)\bar{\alpha}_s(\varphi)|_y\, q_y(\sigma_s\xi).$$

If $s \in \Lambda_1$, then the fact that f_s is supported on X_{ss^*} implies that $f_s(y) = 0$, so the above expression vanishes. Moreover, if $s \in \Lambda_2$, then

$$\bar{\alpha}_s(\varphi)|_y = \varphi(\theta_{s^*}(y)) = 0,$$

and the above expression vanishes again. From this we conclude that, for all $s \in \Lambda_1 \cup \Lambda_2$, we have

$$q_y\Big(\pi(f_s)\sigma_s\xi'\Big) = 0. \tag{4.1.22}$$

By noticing that

$$q_x\Big(\pi(\varphi)\xi\Big) \overset{(4.1.2)}{=} \varphi(x)q_x(\xi) = q_x(\xi),$$

and combining (4.1.22) with (4.1.21), we then have

$$\begin{aligned} 0 &= q_y\Bigg(\sum_{s\in\Lambda}\pi(f_s)\sigma_s\xi'\Bigg) \\ &= q_y\Bigg(\sum_{s\in\Lambda_1}\pi(f_s)\sigma_s\xi'\Bigg) + q_y\Bigg(\sum_{s\in\Lambda_2}\pi(f_s)\sigma_s\xi'\Bigg) + q_y\Bigg(\sum_{s\in\Lambda_3}\pi(f_s)\sigma_s\xi'\Bigg) \\ &= q_y\Bigg(\sum_{s\in\Lambda_3}\pi(f_s)\sigma_s\xi'\Bigg) \overset{(4.1.20)}{=} q_y\Bigg(\sum_{s\in\Lambda_3}\pi(f_s)\sigma_s\xi\Bigg) \\ &\overset{(4.1.17)}{=} \Big((\Pi\times U)(b)q_x(\xi)\Big)_y. \end{aligned}$$

This shows that $(\Pi \times U)(b)$ vanishes on $\bigoplus_{x\in X} V_x$, and hence the proof is concluded. $\square$

This result is fundamental for our study of ideals in $\mathscr{L}_c(X) \rtimes S$. The method we shall adopt will be to start with any ideal $J \trianglelefteq \mathscr{L}_c(X) \rtimes S$, and then use (2.3.11) and (2.3.12) to find a representation π, as above, such that $\ker(\pi) = J$. By (4.1.19) we may replace π by $\Pi \times U$ acting on $\bigoplus_{x \in X} V_x$, without affecting null spaces, and it will turn out that the latter decomposes as a direct sum of very straightforward sub-representations, which we will now describe.

Proposition 4.1.23. *Given any x in X, we have that*

$$\bigoplus_{y \in \mathrm{Orb}(x)} V_y$$

is invariant under $\Pi \times U$.

Proof. By (4.1.7.ii), for every $s \in S$, this space is invariant under U_s. It is also invariant under $\Pi(f)$, for every $f \in \mathscr{L}_c(X)$, since in fact each V_y has this property. The invariance under $\Pi \times U$ then follows. □

We shall now study the representation obtained by restricting $\Pi \times U$ to the invariant space mentioned above.

Definition 4.1.24. Given x in X, we shall denote the invariant subspace referred to in (4.1.23) by W_x, while the representation of $\mathscr{L}_c(X) \rtimes S$ obtained by restricting $\Pi \times U$ to W_x will be denoted by ρ_x.

If $R \subseteq X$ is a system of representatives for the orbit relation in X, namely, if R contains exactly one point of each orbit relative to the action of S on X, notice that

$$\bigoplus_{y \in X} V_y = \bigoplus_{x \in R} W_x,$$

while the restriction of $\Pi \times U$ to $\bigoplus_{y \in X} V_y$ is equivalent to $\bigoplus_{x \in R} \rho_x$.

Before we state the main result of this chapter we should recall that right after the proof of (2.3.12) we fixed an arbitrary ideal $J \trianglelefteq \mathscr{L}_c(X) \rtimes S$, which incidentally has been forgotten ever since.

Theorem 4.1.25. *Let J be an arbitrary ideal of $\mathscr{L}_c(X) \rtimes S$, and let π be a non-degenerate representation of $\mathscr{L}_c(X) \rtimes S$, such that $J = \ker(\pi)$. Considering the representations ρ_x constructed above, we have*

$$J = \bigcap_{x \in R} \ker(\rho_x),$$

where $R \subseteq X$ is any system of representatives for the orbit relation in X.

Proof. The null space of π coincides with the null space of the restriction of $\Pi \times U$ to $\bigoplus_{x\in X} V_x$ by (4.1.19). Since the latter representation is equivalent to the direct sum of the ρ_x, as seen above, the conclusion is evident. □

4.2 THE REPRESENTATIONS ρ_x

In this section we are going to maintain all standing hypothesis of the previous section, such as the ideal $J \trianglelefteq \mathscr{L}_c(X) \rtimes S$ and the representation $\pi : \mathscr{L}_c(X) \rtimes S \to L(V)$ fixed there.

The usefulness of Theorem (4.1.25) in describing J relies in our ability to describe the ideals $\ker(\rho_x)$ mentioned there. The good news is that the representations ρ_x are induced from representations of isotropy group algebras. The main goal of this chapter is to prove that this is indeed the case.

Initially, notice that, if $x \in X$ and $s, t \in \tilde{L}_x$ are such that $[s, x] = [t, x]$, then there exists $e \in E(S)$ such that $x \in X_e$ and $se = te$. Hence, for $\eta \in V_x$, we have

$$\begin{aligned} U_s(\eta) = \mu_s^x(\eta) &\overset{(4.1.4)}{=} \mu_s^x(\mu_e^x(\eta)) \overset{(4.1.6)}{=} \mu_{se}^x(\eta) \\ &= \mu_{te}^x(\eta) \overset{(4.1.6)}{=} \mu_t^x(\mu_e^x(\eta)) \overset{(4.1.4)}{=} \mu_t^x(\eta) = U_t(\eta). \end{aligned} \tag{4.2.1}$$

Our next result refers to the behavior of the operators U_s when $[s, x]$ lies in G_x.

Proposition 4.2.2. *Fixing x in X, let G_x be the isotropy group of x. Then, for each $[s, x]$ in G_x, we have that V_x is invariant under U_s. Moreover, the restriction of U_s to V_x is an invertible operator and the correspondence*

$$[s, x] \in G_x \mapsto U_s|_{V_x} \in GL(V_x)$$

is a group representation.

Proof. Its well definiteness follows from (4.2.1). The remaining statements are immediate consequence of (4.1.7). □

The representation of G_x on V_x referred to in the above Proposition may be integrated to a representation of KG_x, which in turn makes V_x into a left KG_x-module. Applying the machinery of Section 3, we may then form the induced module $M_x \otimes V_x$, as in (3.1.7), which we may also view as a representation of $\mathscr{L}_c(X) \rtimes S$ on $M_x \otimes V_x$.

Theorem 4.2.3. *For each x in X, we have that ρ_x is equivalent to the representation induced from the left KG_x-module V_x, as described above.*

Proof. Recalling from (4.1.24) that ρ_x acts on

$$W_x = \bigoplus_{y \in \mathrm{Orb}(x)} V_y,$$

and that M_x is a right KG_x-module, and viewing V_x as a left KG_x-module via the representation mentioned in (4.2.2), we claim that $T : M_x \times V_x \to W_x$ given by

$$T\Bigg(\sum_{[s,x] \in L_x} c_{[s,x]} \delta_{[s,x]}, \xi \Bigg) = \sum_{[s,x] \in L_x} c_{[s,x]} U_s(\xi)$$

is a well-defined, balanced, bilinear map.

Indeed, it is well defined by (4.2.1) and clearly bilinear. Moreover, for every $[s,x] \in L_x$, $[t,x] \in G_x$ and ξ in V_x, we have

$$\begin{aligned} T(\delta_{[s,x]}\delta_{[t,x]}, \xi) &= T(\delta_{[st,x]}, \xi) = U_{st}(\xi) \overset{(4.1.7)}{=} U_s(U_t(\xi)) \\ &= T(\delta_{[s,x]}, U_t(\xi)) = T(\delta_{[s,x]}, \delta_{[t,x]} \cdot \xi). \end{aligned}$$

Therefore, there exists a unique linear map $\tau : M_x \otimes V_x \to W_x$, such that $\tau(\delta_{[s,x]} \otimes \xi) = U_s(\xi)$. We shall next prove that τ is an isomorphism by exhibiting an inverse for it.

With this in mind, let $R_x \subseteq L_x$ be a total system of representatives for left G_x-classes. Thus, if y is in the orbit of x, there exists a unique $[r,x] \in R_x$ such that $\theta_r(x) = y$, so that U_{r^*} maps V_y onto V_x, by (4.1.7). We therefore let

$$\upsilon_y : V_y \to M_x \otimes V_x$$

be given by $\upsilon_y(\xi) = \delta_{[r,x]} \otimes U_{r^*}(\xi)$, for every ξ in V_y. Putting all of the υ_y together, let

$$\upsilon : W_x = \bigoplus_{y \in \mathrm{Orb}(x)} V_y \longrightarrow M_x \otimes V_x$$

be the unique linear map coinciding with υ_y on V_y, for every y in $\mathrm{Orb}(x)$.

We claim that υ is the inverse of τ. To see this, let $[s,x]$ be any element in L_x, and let ξ be picked in V_x arbitrarily. Let $[r,x] \in R_x$ be such that $\theta_s(x) = \theta_r(x)$. Then, r^*s lies in $\tilde{G}_x$ and $U_s(\xi) \in V_y$, where $y := \theta_s(x) = \theta_r(x)$. We then have

$$\begin{aligned}\upsilon\Big(\tau(\delta_{[s,x]} \otimes \xi)\Big) &= \upsilon(U_s(\xi)) = \delta_{[r,x]} \otimes U_{r^*}(U_s(\xi)) \\ &= \delta_{[r,x]} \otimes U_{r^*s}(\xi) = \delta_{[r,x]} \otimes \delta_{[r^*s,x]} \cdot \xi \\ &= \delta_{[r,x]}\delta_{[r^*s,x]} \otimes \xi = \delta_{[rr^*s,x]} \otimes \xi \\ &= \delta_{[s,x]} \otimes \xi.\end{aligned}$$

On the other hand, given any y in $\mathrm{Orb}(x)$ and $\xi \in V_y$, write $y = \theta_r(x)$, for $[r,x] \in R_x$, and notice that

$$\tau(\upsilon(\xi)) = \tau\Big(\delta_{[r,x]} \otimes U_{r^*}(\xi)\Big) = U_r(U_{r^*}(\xi)) = U_{rr^*}(\xi) = \xi.$$

Therefore τ is indeed an isomorphism between the K-vector spaces $M_x \otimes V_x$ and W_x. We will next prove that τ is equivariant for the respective actions of $\mathscr{L}_c(X) \rtimes S$, which amounts to say that it is linear as a map between left $\mathscr{L}_c(X) \rtimes S$-modules. For this, given $t \in S$, and $f \in \mathscr{L}_c(X_{tt^*})$, we must prove that

$$\tau\Big((f\Delta_t)\delta_{[s,x]} \otimes \xi\Big) = \rho_x(f\Delta_t)\Big(\tau(\delta_{[s,x]} \otimes \xi)\Big), \tag{4.2.4}$$

for all $[s,x] \in L_x$ and all $\xi \in V_x$.

Notice that, if $[s,x] \in L_x$ and $\xi \in V_x$, then the left-hand side of (4.2.4) equals

$$\begin{aligned}\tau\Big((f\Delta_t)\delta_{[s,x]} \otimes \xi\Big) &= [ts\in\tilde{L}_x]\, f(\theta_{ts}(x))\tau(\delta_{[ts,x]} \otimes \xi) \\ &= [ts\in\tilde{L}_x]\, f(\theta_{ts}(x))U_{ts}(\xi)\end{aligned}$$

while the right-hand side becomes

$$\rho_x(f\Delta_t)\Big(\tau(\delta_{[s,x]} \otimes \xi)\Big) = \Pi(f)U_tU_s(\xi) = \Pi(f)U_{ts}(\xi). \tag{4.2.5}$$

Since ξ lies in V_x, recall from (4.1.7) that U_{ts} vanishes on V_x, unless ts lies $\tilde{L}_x$, in which case U_{ts} maps V_x bijectively onto $V_{\theta_{ts}(x)}$. Hence, (4.2.5) becomes

$$\rho_x(f\Delta_t)\Big(\tau(\delta_{[s,x]} \otimes \xi)\Big) = \Pi(f)U_{ts}(\xi) = [ts\in\tilde{L}_x]\, f(\theta_{ts}(x))U_{ts}(\xi)$$

because, $\Pi(f)$ acts on $V_{\theta_{ts}(x)}$ by scalar multiplication by $f(\theta_{ts}(x))$, according to (4.1.1).

This proves (4.2.4), so τ is indeed covariant. □

Summarizing what we have done so far, the following is the main result of this work.

Theorem 4.2.6. *Let (θ, S, X) be an ample system and $\mathscr{L}_c(X) \rtimes S$ be the corresponding crossed product algebra over a field K. Then, every ideal $J \trianglelefteq \mathscr{L}_c(X) \rtimes S$ is the intersection of ideals induced from isotropy groups.*

Proof. Let $R \subseteq X$ be a system of representatives for the orbit relation on X. Using (4.1.25) we may write J as the intersection of the null spaces of the ρ_x, for x in R, while (4.2.3) tells us that ρ_x is equivalent to the representation induced from a representation of the isotropy group at x. The null space of ρ_x is therefore induced from an ideal in the group algebra of the said isotropy group by (3.1.11), whence the result. □

Next proposition goes in the way of describing explicitly a given ideal $J \trianglelefteq \mathscr{L}_c(X) \rtimes S$ as the intersection of induced ideals.

Proposition 4.2.7. *Under the assumptions of (4.2.6), choose a system R of representatives for the orbit relation on X. For each x in R, let G_x be the isotropy group at x, and let*

$$\Gamma_x : \mathscr{L}_c(X) \rtimes S \to KG_x$$

be as in (3.1.15). Then, given any ideal $J \trianglelefteq \mathscr{L}_c(X) \rtimes S$ we have that $\Gamma_x(J)$ is an admissible ideal of KG_x, and

$$J = \bigcap_{x \in R} \operatorname{Ind}_x(\Gamma_x(J)).$$

Proof. Let $I'_x := \Gamma_x(J)$. That each I'_x is an admissible ideal follows at once from (3.2.8). For each x in R, let I_x be the null space of the representation ρ_x referred to in the proof of (4.2.6), so that

$$J = \bigcap_{x \in R} \operatorname{Ind}_x(I_x).$$

Observe that for each $x \in R$, we have

$$I'_x = \Gamma_x(J) = \Gamma_x\Big(\bigcap_{y \in R} \operatorname{Ind}_y(I_y)\Big) \subseteq \Gamma_x\Big(\operatorname{Ind}_x(I_x)\Big) \overset{(3.2.3)}{\subseteq} I_x.$$

Consequently $\mathrm{Ind}_x(I'_x) \subseteq \mathrm{Ind}_x(I_x)$, whence

$$\bigcap_{x\in R} \mathrm{Ind}_x(I'_x) \subseteq \bigcap_{x\in R} \mathrm{Ind}_x(I_x) = J.$$

On the other hand, we have by (3.2.7) that $\mathrm{Ind}_x(I'_x)$ is the largest among the ideals of $\mathscr{L}_c(X) \rtimes S$ mapping into I'_x under Γ_x. Since $\Gamma_x(J) = I'_x$, by definition, we have that J is among such ideals, so $J \subseteq \mathrm{Ind}_x(I'_x)$, and then

$$J \subseteq \bigcap_{x\in R} \mathrm{Ind}_x(I'_x),$$

concluding the proof. $\square$

5 STEINBERG ALGEBRAS

In this chapter, we prove that every Steinberg algebra associated with an ample groupoid can be realized as an inverse semigroup crossed product algebra of the form $\mathscr{L}_c(X) \rtimes S$. For the task, we first show that the Steinberg algebra associated with the groupoid of germs of an ample dynamical system is isomorphic to the crossed product algebra as a consequence of the theory we have developed so far. Then, combining this with an Exel's result in [13], we get the promised result.

We assume the reader is familiar with the notion of topological groupoids and in particular with its basic notations: a groupoid is usually denoted by $\mathcal{G}$, its unit space by $\mathcal{G}^{(0)}$, and the set of composable pairs by $\mathcal{G}^{(2)}$. The source and range maps are denoted by d and r, respectively.

An *étale* groupoid is a topological groupoid $\mathcal{G}$, whose unit space $\mathcal{G}^{(0)}$ is locally compact and Hausdorff in the relative topology, and such that the range map $r : \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism [13].

A very important class of étale groupoids is that of *ample* groupoids [12]. An étale groupoid is called ample if the compact bisections form a basis for its topology, where a bisection is an open subset $U \subseteq \mathcal{G}$ such that the restrictions of d and r to U are injective.

If $\mathcal{G}$ is an ample groupoid, then the Steinberg algebra $A_K(\mathcal{G})$ is defined as the space of all K-valued functions on $\mathcal{G}$ spanned by functions $f : \mathcal{G} \to K$ such that:

- There is an open Hausdorff subspace V in $\mathcal{G}$ so that f vanishes outside V; and

- $f|_V$ is locally constant with compact support;

with pointwise sum and convolution product.

Note that if $\mathcal{G}$ is not Hausdorff, then $A_K(\mathcal{G})$ will contain discontinuous functions. The reader is referred to [14] and [7] for detailed treatment in the subject.

5.1 INDUCTION PROCESS FOR STEINBERG ALGEBRAS

From now on, we fix an ample groupoid $\mathcal{G}$ and its associated Steinberg algebra $A_K(\mathcal{G})$.

In [7], Steinberg also develops a theory of induction of modules from isotropy groups.

For a point $x \in \mathcal{G}^{(0)}$ he considers:

$$\begin{array}{rcl} \mathbf{L}_x & := & \{\gamma \in \mathcal{G} : d(\gamma) = x\}, \\ \mathbf{G}_x & := & \{\gamma \in \mathcal{G} : d(\gamma) = r(\gamma) = x\}, \\ \mathbf{Orb}(x) & := & \{r(\gamma) : \gamma \in \mathbf{L}_x\}. \end{array} \tag{5.1.1}$$

Moreover, he considers $\mathbf{M}_x$ as the free K-module with basis $\mathbf{L}_x$. Since $\mathbf{L}_x\mathbf{G}_x \subseteq \mathbf{L}_x$, there is a natural right $K\mathbf{G}_x$-module structure on $\mathbf{M}_x$. Moreover, $\mathbf{M}_x$ is $A_K(\mathcal{G})$-$K\mathbf{G}_x$-bimodule, where the left structure is such that

$$f \cdot \delta_\nu = \sum_{\gamma \in L} f(\gamma\nu^{-1})\delta_\gamma. \tag{5.1.2}$$

In this fashion, if $x \in \mathcal{G}^{(0)}$ and V is a left $K\mathbf{G}_x$-module, then the left $A_K(\mathcal{G})$-module induced by V is defined by

$$\mathrm{Ind}_x(V) := \mathbf{M}_x \otimes_{K\mathbf{G}_x} V.$$

We strongly encourage [7] for more details in the subject.

To introduce the notion of an induced ideal, we first talk about a map that will play a crucial role in the road to our ambitions. This is a version of Γ_x to the actual context. For each $x \in \mathcal{G}^{(0)}$, consider the map $\mathbf{\Gamma}_x : A_K(\mathcal{G}) \to K\mathbf{G}_x$ given by

$$\mathbf{\Gamma}_x(f) = \sum_{\gamma \in \mathbf{G}_x} f(\gamma)\delta_\gamma. \tag{5.1.3}$$

We then have the following proposition.

Proposition 5.1.4. *Let $x \in \mathcal{G}^{(0)}$ and let U be a compact open bisection such that $\mathbf{G}_x \cap U \neq \emptyset$. Then, for every $f \in A_K(\mathcal{G})$, we have*

$$\mathbf{\Gamma}_x(uf) = \mathbf{\Gamma}_x(u)\mathbf{\Gamma}_x(f) \quad \text{and} \quad \mathbf{\Gamma}_x(fu) = \mathbf{\Gamma}_x(f)\mathbf{\Gamma}_x(u),$$

where u stands for the characteristic function of U.

Proof. Notice that, since U is a bisection such that $\mathbf{G}_x \cap U \neq \emptyset$, there exists an unique element $\nu \in \mathbf{L}_x \cap U$. Hence, if $\gamma \in \mathbf{G}_x$, we have

$$fu(\gamma) = \sum_{\mu \in \mathbf{L}_x} f(\gamma\mu^{-1})u(\mu) = f(\gamma\nu^{-1})$$

and, therefore,

$$\mathbf{\Gamma}_x(fu) = \sum_{\gamma \in \mathbf{G}_x} (fu)(\gamma)\delta_\gamma = \sum_{\gamma \in \mathbf{G}_x} f(\gamma\nu^{-1})\delta_\gamma.$$

On the other hand,

$$\mathbf{\Gamma}_x(f)\mathbf{\Gamma}_x(u) = \left(\sum_{\gamma\in\mathbf{G}_x} f(\gamma)\delta_\gamma\right)\delta_\nu = \sum_{\gamma\in\mathbf{G}_x} f(\gamma)\delta_{\gamma\nu} = \sum_{\gamma\in\mathbf{G}_x} f(\gamma\nu^{-1})\delta_\gamma.$$

Similarly, we can show $\mathbf{\Gamma}_x(uf) = \mathbf{\Gamma}_x(u)\mathbf{\Gamma}_x(f)$, concluding the proof. □

Proposition 5.1.5. *Let $J \trianglelefteq A_K(\mathcal{G})$ be an ideal and $x \in \mathcal{G}^{(0)}$. Then, $\mathbf{\Gamma}_x(J)$ is an ideal in $K\mathbf{G}_x$.*

Proof. Let $a \in \mathbf{\Gamma}_x(J)$ and $b = \delta_\gamma \in K\mathbf{G}_x$ for some $\gamma \in \mathbf{G}_x$. Then, there exists $f \in J$ such that $\mathbf{\Gamma}_x(f) = a$. Notice that, by choosing a compact open bisection U containing γ, we have

$$ab = a\delta_\gamma = \mathbf{\Gamma}_x(f)\mathbf{\Gamma}_x(1_U) \overset{(5.1.4)}{=} \mathbf{\Gamma}_x(f1_U) \in \mathbf{\Gamma}_x(J).$$

By linearity, we deduce that $ab \in \mathbf{\Gamma}_x(J)$ for arbitrary $b \in K\mathbf{G}_x$ and similarly, we can show that $ba \in \mathbf{\Gamma}_x(J)$. □

The next definition should not be strange to the reader at this point. Indeed, we shall see that, if I is the annihilator of V in $K\mathbf{G}_x$, then $\mathrm{Ind}_x(I)$, as defined above, is the annihilator of $\mathbf{M}_x \otimes V$ in $A_K(\mathcal{G})$. Actually, we could verify it right now. However, since it shall become clear soon, we choose to spare the work.

Definition 5.1.6. Let $x \in \mathcal{G}^{(0)}$. Given any ideal $I \trianglelefteq K\mathbf{G}_x$, we define

$$\mathrm{Ind}_x(I) := \{f \in A_K(\mathcal{G}) : \mathbf{\Gamma}_x(ufv) \in I, \forall u, v \in A_K(\mathcal{G})\},$$

and call it the *ideal induced by I*.

5.2 UNIVERSAL PROPERTY FOR THE STEINBERG ALGEBRA ASSOCIATED WITH A GROUPOID OF GERMS

In section 4 of [13], Exel introduced the groupoid of germs associated with an action of an inverse semigroup on a locally compact Hausdorff topological space. For the convenience of the reader and to introduce some notations, we review briefly this theory. However, the interested reader is strongly encouraged to read [13] for more details in the subject.

For the moment, let (θ, S, X) be a topological dynamical system. The groupoid of germs, which we denote $S \ltimes_\theta X$ (or simply $S \ltimes X$ when the action is implicit in the context), as a set, is the quotient of the set

$$\{(s, x) \in S \times X : x \in X_{s^*s}\}$$

by the equivalence relation that identifies two pairs (s,x) and (t,y) if and only if $x=y$ and there exists an idempotent $e\in E(S)$ such that $x\in X_e$ and $se=te$. We denote by $[s,x]$ the equivalence class of (s,x) and call it the germ of s at x. The inversion is given by $[s,x]^{-1}=[s^*,\theta_s(x)]$ and the multiplication is given by defining $[s,x]\,.\,[t,y]$ if and only if $x=\theta_t(y)$, in which case the product is $[st,y]$.

A basis for the topology of $S\ltimes X$ is given by

$$\Theta(s,U)=\{[s,x]\in S\ltimes X : x\in U\}$$

where $s\in S$ and $U\subset X_{s^*s}$ is an open set. Furthermore, the map $x\in U\mapsto[s,x]\in\Theta(s,U)$ is a homeomorphism, where $\Theta(s,U)$ carries the topology induced from $S\ltimes X$.

The unit space is formed by elements $[e,x]$ with $e\in E(S)$ and $x\in X_e$, and the map

$$[e,x]\mapsto x \tag{5.2.1}$$

gives a homeomorphism between the unit space of $S\ltimes X$ and X. So, from now on, we identify the unit space with X and, with such an identification, we have that $d([s,x])=x$ and $r([s,x])=\theta_s(x)$ are the domain and range maps, respectively.

In this setting, $S\ltimes X$ is an étale groupoid and, for each $s\in S$ and each open subset U of X_{s^*s}, $\Theta(s,U)$ is a bisection. We will use the shorthand notation Θ_s for the bisection $\Theta(s,X_{s^*s})$.

Furthermore, if (θ,S,X) is an ample dynamical system, the unit space of $S\ltimes X$ is totally disconnected and, hence, the collection of all compact bisections forms a basis for the topology of $S\ltimes X$, according to Proposition (4.1) of [15]. This amounts to say that $S\ltimes X$ is an ample groupoid. Therefore, we can build the Steinberg algebra $A_K(S\ltimes X)$ associated with $S\ltimes X$.

It worths to mention that the groupoid of germs does not need be Hausdorff. The interested reader is referred to [16] for a characterization of Hausdorffness for the groupoid of germs.

From now on, we fix an ample dynamical system (θ,S,X), as well as the Steinberg algebra $A_K(S\ltimes X)$ associated with the groupoid of germs $S\ltimes X$.

We will denote by d_s and r_s the restrictions of the source and range maps to Θ_s, respectively. The maps d_s and r_s are homeomorphisms onto their images X_{s^*s} and X_{ss^*}, respectively. Notice that, if $\varphi\in\mathscr{L}_c(X_{ss^*})$, then the composition $\varphi\circ r_s$ is a compactly supported locally constant function on Θ_s. So, we shall also see $\varphi\circ r_s$ as a function in $A_K(S\ltimes X)$ by extending them to be zero outside Θ_s, which we shall denote by $\varphi\Delta_s$.

For each $s \in S$ and each $f \in \mathscr{L}_c(X_{s^*s})$ we will denote by $\alpha_s(f)$ the element of $\mathscr{L}_c(X_{ss^*})$ given by

$$\alpha_s(f)\big|_x = f(\theta_{s^*}(x)) \text{ for all } x \in X.$$

In this context, we have the result.

Proposition 5.2.2. *Given $f \in \mathscr{L}_c(X_{ss^*})$ and $g \in \mathscr{L}_c(X_{tt^*})$, we have*

$$f\varDelta_s * g\varDelta_t = \alpha_s(\alpha_{s^*}(f)g)\varDelta_{st}.$$

Proof. Let's prove initially that $\Theta_s\Theta_t = \Theta_{st}$. Indeed, it is clear that $\Theta_s\Theta_t \subseteq \Theta_{st}$. Conversely, let $[st, y] \in \Theta_{st}$ and notice that we must have

$$y \in X_{(st)^*st} = X_{t^*s^*st} = \theta_{t^*}(X_{s^*s} \cap X_{tt^*}).$$

This means that there exists $x \in X_{s^*s} \cap X_{tt^*}$ such that $y = \theta_{t^*}(x)$. In particular, $x \in X_{s^*s}$ and $y \in X_{tt^*}$, which implies that $[s, x] \in \Theta_s$ and $[t, y] \in \Theta_t$. Then, since $x = \theta_t(y)$, we have

$$[st, y] = [s, x]\,[t, y] \in \Theta_s\Theta_t,$$

concluding the initial assumption.

Since $f \in \mathscr{L}_c(X_{ss^*})$ and $g \in \mathscr{L}_c(X_{tt^*})$, we have

$$\operatorname{supp}(f\varDelta_s * g\varDelta_t) \subseteq \Theta_s\Theta_t = \Theta_{st}.$$

Notice now that

$$\begin{aligned}\Big(f\varDelta_s * g\varDelta_t\Big)([st, y]) &= (f\varDelta_s)([s, x])(g\varDelta_t)([t, y]) = f(\theta_s(x))g(\theta_t(y)) \\ &= f(\theta_{st}(y))g(\theta_t(y)) = \Big(\alpha_s(\alpha_{s^*}(f)g)\varDelta_{st}\Big)([st, y]).\end{aligned}$$

□

This proposition is important to establish the isomorphism between the crossed product algebra and the Steinberg algebra associated with the groupoid of germs, which is our aim now.

Before we proceed, for the sake of understanding, let's take a pause to interpret the objects of last section in the context of the groupoid of germs $S \ltimes X$.

Laying the groundwork, from the point of view of a groupoid of germs, having in mind the identification done in (5.2.1), notice that the sets defined in (5.1.1) can be interpreted as

$$\begin{array}{rcl} \mathbf{L}_x & := & \{[s,x] : x \in X_{s^*s}\} \\ \mathbf{G}_x & := & \{[s,x] : x \in X_{s^*s} \text{ and } \theta_s(x) = x\} \\ \mathbf{Orb}(x) & := & \{\theta_s(x) : x \in X_{s^*s}\} \end{array} \tag{5.2.3}$$

In relation to the definitions in (3.1.1), notice that the map

$$[s,x] \in \mathbf{L}_x \mapsto [s,x] \in L_x$$

is a bijection which restricts to an isomorphism between $\mathbf{G}_x$ and G_x. Hence, the isotropy in each sense coincide.

From now on, we identify these objects as well as the isotropy group algebra and, hence, we abolish the bold notation in (5.2.3).

We now have the tools to the establish an isomorphism between the inverse semigroup crossed product algebra associated with an ample system (θ, S, X) and the Steinberg algebra associated with the respective groupoid of germs $S \ltimes X$. The reader may compare with Theorem 5.4 of [17].

Theorem 5.2.4. *Let (θ, S, X) be an ample dynamical system, α the action of S on $\mathscr{L}_c(X)$ given by (2.3.2) and $S \ltimes X$ the associated groupoid of germs. Then $\mathscr{L}_c(X) \rtimes_\alpha S$ is isomorphic to $A_K(S \ltimes X)$.*

Proof. Let $\mathcal{B}^\theta$ be the semi-direct product bundle associated with (θ, S, X), as in (2.2.2). Consider, for each $s \in S$, the map

$$\pi_s : f\boldsymbol{\delta}_s \in B_s \mapsto f\mathit{\Delta}_s \in A_K(S \ltimes X).$$

Then, by (5.2.2), $\{\pi_s\}_{s\in S}$ is a pre-representation of the semi-direct product bundle $\mathcal{B}^\theta$ in $A_K(S \ltimes X)$. Furthermore, if $s \leq t$ in S and f lies in $\mathscr{L}_c(X_{ss^*})$, then it is easy to see that $f\mathit{\Delta}_t$ vanishes outside Θ_s and, then, coincides with $f\mathit{\Delta}_s$. This amounts to say that $\{\pi_s\}_{s\in S}$ is a representation of $\mathcal{B}^\theta$ in $A_K(S \ltimes X)$.

By Proposition (2.1.5), there exists an epimorphism $\Phi : \mathscr{L}_c(X) \rtimes_\alpha S \to A_K(S \ltimes X)$ such that $\Phi(f\Delta_s) = f\mathit{\Delta}_s$.

To show that Φ is injective, we claim first that the diagram

$$\begin{tikzcd} \mathscr{L}_c(X) \rtimes S \arrow[r, "\Phi"] \arrow[dr, "\Gamma_x"'] & A_K(S \ltimes X) \arrow[d, "\mathbf{\Gamma}_x"] \\ & KG_x \end{tikzcd} \tag{5.2.5}$$

is commutative for every $x \in X$. Indeed, let $f \in \mathscr{L}_c(X_{ss^*})$ and notice that, since $f\Delta_s$ is a function supported on Θ_s and the latter is a bisection, there exists at most one element in $\Theta_s \cap G_x$. Actually, if $x \in X_{s^*s}$ and $\theta_s(x) = x$ we must have $\Theta_s \cap G_x = \{[s, x]\}$ and, otherwise, we must have $\Theta_s \cap G_x = \emptyset$. Therefore,

$$\begin{aligned}(\mathbf{\Gamma}_x \circ \Phi)(f\Delta_s) = \mathbf{\Gamma}_x(f\Delta_s) &= [x \in X_{s^*s}, \theta_s(x) = x]f(x)\delta_{[s,x]} \\ &= [s \in \tilde{G}_x]f(x)\delta_{[s,x]} = \Gamma_x(f\Delta_s).\end{aligned}$$

This concludes the diagram commutativity.

Let $J \trianglelefteq \mathscr{L}_c(X) \rtimes S$ be the kernel of Φ. Then, by the commutativity of the diagram we have

$$\Gamma_x(J) = (\mathbf{\Gamma}_x \circ \Phi)(J) = \mathbf{\Gamma}_x(0) = 0,$$

for every $x \in X$. Hence, by (4.2.7)

$$J = \bigcap_{x\in X} \mathrm{Ind}_x\left(\Gamma_x(J)\right) = \bigcap_{x\in X} \mathrm{Ind}_x\left(0\right) = \bigcap_{x\in X} \mathrm{Ind}_x\left(\Gamma_x(0)\right) = 0,$$

concluding the proof. □

There are some immediate consequences of this result. First, note that $A_K(S \ltimes X)$ inherits the universal property of $\mathscr{L}_c(X) \rtimes S$, which we spell out. The reader is invited to compare with Theorem 4.27 of [7]. On the one hand, Steinberg demands the groupoid of germs $S \ltimes X$ to be Hausdorff, on the other hand, the assumption that K is a field is relaxed by considering algebras over a commutative ring with identity.

Proposition 5.2.6. *Let (θ, S, X) be an ample dynamical system and $S \ltimes X$ its associated groupoid of germs. Then, for any covariant representation (π, σ) of (θ, S, X), there exists a non-degenerate representation $\pi \times \sigma$ of $A_K(S \ltimes X)$ such that*

$$(\pi \times \sigma)(f\Delta_s) = \pi(f)\sigma_s$$

Furthermore, the mapping

$$(\pi, \sigma) \mapsto \pi \times \sigma$$

gives a bijection between covariant representations of (θ, S, X) and non-degenerate representations of $A_K(S \ltimes X)$.

Proof. Just join (5.2.4), (2.3.9) and (2.3.6). □

Furthermore, by the commutativity of diagram (5.2.5), we see that Φ maps $\mathrm{Ind}_x(I)$ in the sense of Definition (3.1.10) onto $\mathrm{Ind}_x(I)$ in the sense of Definition (5.1.6).

Moreover, Φ is compatible with the actions of $\mathscr{L}_c(X) \rtimes S$ and $A_K(S \ltimes X)$ on M_x in the sense that, the left $A_K(\mathcal{G})$-module structure on M_x induced by Φ from (3.1.6) is exactly the same structure as defined in (5.1.2). This immediately gives the following two propositions.

Proposition 5.2.7. *Let $x \in \mathcal{G}^{(0)}$. If I is the annihilator of V in KG_x, then* $\mathrm{Ind}_x(I)$ *is the annihilator of $M_x \otimes V$ in $A_K(S \ltimes X)$.*

This proposition makes sense to Definition (5.1.6), while the next one translates the main result for inverse semigroup crossed product algebras to Steinberg algebras (associated with groupoid of germs). But first, let us bring the definition of admissible ideal from (3.2.4) for the current context.

Definition 5.2.8. An ideal $I \trianglelefteq KG_x$ is said to be *admissible* if $\Gamma_x(\mathrm{Ind}_x(I)) = I$.

Then, we finally have the desired result.

Proposition 5.2.9. *Let (θ, S, X) be an ample system and $S \ltimes X$ the associated groupoid of germs. Choose a system R of representatives for the orbit relation on X. For each x in R, let G_x be the isotropy group at x, and let*

$$\Gamma_x : A_K(S \ltimes X) \to KG_x$$

be as in (5.1.3). Then, given any ideal $J \trianglelefteq A_K(S \ltimes X)$ we have that $\Gamma_x(J)$ is an admissible ideal of KG_x, and

$$J = \bigcap_{x \in R} \mathrm{Ind}_x(\Gamma_x(J)).$$

5.3 STEINBERG ALGEBRAS AS CROSSED PRODUCTS

In section 5 of [13], Exel presents an example of an inverse semigroup action which is intrinsic to every étale groupoid. We therefore fix an étale groupoid $\mathcal{G}$ from now on and denote by $\mathcal{S}(\mathcal{G})$ the set of all bisections in $\mathcal{G}$. It is well known that $\mathcal{S}(\mathcal{G})$ is an inverse semigroup under the operations

$$UV = \left\{uv : u \in U, v \in V, (u,v) \in \mathcal{G}^{(2)}\right\} \text{ and } U^* = \left\{u^{-1} : u \in U\right\},$$

The idempotent semilattice of $\mathcal{S}(\mathcal{G})$ consist precisely of the open subsets of $\mathcal{G}^{(0)}$.

Moreover, for any bisection U, its source $d(U)$ and range $r(U)$ are open subsets of $\mathcal{G}^{(0)}$ and the maps $d_U : U \to d(U)$ and $r_U : U \to d(U)$, obtained by restricting d and r, respectively, are homeomorphisms. Hence, we can define a topological action $\theta : \mathcal{S}(\mathcal{G}) \to \mathcal{G}^{(0)}$ such that, for each $U \in \mathcal{S}(\mathcal{G})$,

$$\begin{array}{rccc} \theta_U : & d(U) & \longrightarrow & r(U) \\ & x & \longmapsto & r_U\Big(d_U^{-1}(x)\Big). \end{array} \tag{5.3.1}$$

Notice that $\theta_U(x) = y$, if and only if there exists some $u \in U$ such that $d(u) = x$ and $r(u) = y$.

Additionally, given any *-subsemigroup $S \subseteq \mathcal{S}(\mathcal{G})$, we may restrict θ to S, thus obtaining a semigroup homomorphism

$$\theta|_S : S \to \mathcal{I}(X)$$

which is an action of S on X, provided (2.3.1.2.3.1) can be verified. The next result gives sufficient conditions for the groupoid of germs for such an action to be equal to $\mathcal{G}$.

Proposition 5.3.2. *Let $\mathcal{G}$ be an étale groupoid and let S be a *-subsemigroup of $\mathcal{S}(\mathcal{G})$ such that*

(i) $\mathcal{G} = \bigcup_{U \in S} U$, *and*

(ii) for every $U, V \in S$, and every $u \in U \cap V$, there exists $W \in S$, such that $u \in W \subseteq U \cap V$.

Then $\theta|_S$ is an action of S on $X = \mathcal{G}^{(0)}$, and the groupoid of germs for $\theta|_S$ is isomorphic to $\mathcal{G}$.

Proof. Proposition (5.4) of [13]. □

An interesting consequence of the above proposition is that, if $\mathcal{G}$ is an ample groupoid, then the set of compact bisections of $\mathcal{G}$ are in the hypotheses of (5.3.2). Hence, the groupoid of germs obtained by the restriction of the action (θ, S, X) above to the *-subsemigroup of compact bisections is isomorphic to the original groupoid $\mathcal{G}$. Moreover, this restriction forms an ample action (with domains compact).

We could even add the unit space to the *-subsemigroup of compact bisections that it would still satisfy the hypotheses of (5.3.2). In this situation, the restriction of the action would still form an ample action and the semigroup involved would have a unit.

We shall denote by S^a the *-subsemigroup of $\mathcal{S}(\mathcal{G})$ formed by the compact bisections of $\mathcal{G}$. The importance of the comments above arises from the next proposition.

Proposition 5.3.3. *Let $\mathcal{G}$ be an ample groupoid and let S be a *-subsemigroup of $\mathcal{S}(\mathcal{G})$ satisfying the hypotheses of (5.3.2) and such that the restriction θ to S of the action of $\mathcal{S}(\mathcal{G})$ on $\mathcal{G}^{(0)}$ given by (5.3.1) is ample. If α is the induced action of S on $\mathscr{L}_c(\mathcal{G}^{(0)})$, as in (2.3.2), then*

$$A_K(\mathcal{G}) \simeq \mathscr{L}_c(\mathcal{G}^{(0)}) \rtimes_\alpha S.$$

Proof. Let $S \ltimes \mathcal{G}^{(0)}$ be the groupoid of germs for the given action of S on $\mathcal{G}^{(0)}$. Applying (5.2.4), we conclude that

$$A_K(S \ltimes \mathcal{G}^{(0)}) \simeq \mathscr{L}_c(\mathcal{G}^{(0)}) \rtimes_\alpha S.$$

By (5.3.2), $S \ltimes \mathcal{G}^{(0)} \simeq \mathcal{G}$. Hence,

$$A_K(\mathcal{G}) \simeq \mathscr{L}_c(\mathcal{G}^{(0)}) \rtimes_\alpha S,$$

as desired. □

The reader is invited to compare this result with Corollary 5.6 of [17] and Theorem 5.2 of [9]. Summarizing, every Steinberg algebra associated to an ample groupoid can be viewed as an inverse semigroup crossed product algebra. In particular, by the comments immediately before Proposition (5.3.3), one may always choose S to be S^a or even S^a added by the unit space if it is desired to deal only with inverse semigroups with a unit.

6 CONCLUSION

The question that inspired this work is: can a version of the Effors-Hahn conjecture be obtained for Steinberg algebras?

Trying to answer this question, we initially analyzed the work done by Dokuchaev and Exel in [6]. It became clear in their paper that, in order to obtain a version of the Effros-Hahn conjecture for the case of the algebraic crossed product formed by a partial action of a discrete group on a totally disconnected, locally compact and Hausdorff topological space, the key was a tool of disintegration and integration of representations.

Hence, we started trying to obtain a somewhat result of disintegration and integration of representations for Steinberg algebras. In [7], Steinberg obtained such a result for Hausdorff groupoids. But, since we were not disposed to give up the non-Hausdorff case, we hoped to obtain a generalized version of Steinberg result. However, we were not able to adapt the proof of Steinberg to include non-Hausdorff case since the proof relies in the fact that, in the Hausdorff setting, the intersection of two compacts sets is a compact set.

Meanwhile, another idea came to light. Since Steinberg algebras can be thought as algebraic versions of groupoid C*-algebras, inspired on the work of Exel in [13], we have tried to define an algebraic version of inverse semigroup crossed products and obtain an isomorphism between a given Steinberg algebra and a suitable inverse semigroup crossed product algebra as done by Exel in the C*-algebras setting, under some extra hypothesis. We hoped then to obtain a tool of disintegration and integration of representations for these new kind of crossed product algebras and then transport this tool for Steinberg algebras through the isomorphism. It is worth pointing out that, at that time, there was no notion of inverse semigroup crossed product in the literature, but when this work was in progress it appeared in papers like [9]. Anyway, we were just able to obtain the desired isomorphism in the Hausdorff setting.

At this point, we realized that would be a nice idea to change the main object of interest. So, we started focusing on inverse semigroup crossed products as our main object of study, instead of Steinberg algebras.

It turns out that we were able the obtain the desired tools of disintegration and integration for this algebras as can be seen in Corollaries

(2.3.9) and (2.3.6), respectively. And, as expected, these tools guided to the desired version of the Effros-Hahn conjecture for inverse semigroup crossed products as can be seen in Proposition (4.2.7).

The interesting fact is that these results showed themselves as a tool to obtain the initial isomorphism we hoped to obtain with no success so far, as stated in (5.2.4) and (5.3.3). It is worth saying that this isomorphism includes even the non Hausdorff case, but the Steinberg algebras are considered only over fields.

Therefore, we were able then to transport our version of the Effros-Hahn conjecture to the case of Steinberg algebras through the isomorphism, as our initial goal. This is the content of Proposition (5.2.9) and answers the initial question.

Furthermore, this answer stimulates new works in the tentative of answering another interesting questions as:

- Should every primitive ideal in a Steinberg algebra be induced by a primitive ideal in some isotropy group algebra?
- Could we use our results to obtain sufficient conditions on the groupoid to ensure simplicity in the Steinberg algebra associated with it?

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