UNIVERSIDADE FEDERAL DE SANTA CATARINA
CENTRO DE CIÊNCIAS FÍSICAS E MATEMÁTICAS
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA PURA E APLICADA

Víctor Luis Espinoza

On the genericity of singularities in Lorentzian geometry

Florianópolis
2024

Victor Luis Espinoza

On the genericity of singularities in Lorentzian geometry

Tese submetida ao Programa de Matemática Pura e Aplicada da Universidade Federal de Santa Catarina para a obtenção do título de Doutor em Geometria e Topologia.
Orientador: Prof. Ivan Pontual Costa e Silva, Dr.

Florianópolis
2024

Ficha catalográfica gerada por meio de sistema automatizado gerenciado pela BU/UFSC.
Dados inseridos pelo próprio autor.

Espinoza, Victor Luis
On the genericity of singularities in Lorentzian geometry / Victor Luis Espinoza ; orientador, Ivan Pontual Costa e Silva, 2024.
81 p.

Tese (doutorado) - Universidade Federal de Santa Catarina, Centro de Ciências Físicas e Matemáticas, Programa de Pós-Graduação em Matemática Pura e Aplicada, Florianópolis, 2024.

Inclui referências.

1. Matemática Pura e Aplicada. 2. geometria Lorentziana. 3. incompletude geodésica. 4. genericidade. I. Costa e Silva, Ivan Pontual. II. Universidade Federal de Santa Catarina. Programa de Pós-Graduação em Matemática Pura e Aplicada. III. Título.

Victor Luis Espinoza

On the genericity of singularities in Lorentzian geometry

O presente trabalho em nível de doutorado foi avaliado e aprovado por banca examinadora composta pelos seguintes membros:

Prof. Ivan Pontual Costa e Silva, Dr.
Universidade Federal de Santa Catarina

Prof. Jónatan Herrera Fernandez, Dr.
Universidad de Córdoba

Prof. José Luis Flores Dorado, Dr.
Universidad de Málaga

Prof. Francisco Carlos Caramello Junior, Dr.
Universidade Federal de Santa Catarina

Profa. Marianna Ravara Vago, Dra. [**suplente**]
Universidade Federal de Santa Catarina

Certificamos que esta é a **versão original e final** do trabalho de conclusão que foi julgado adequado para obtenção do título de doutor em geometria e topologia.

ICP-Edu
Documento assinado digitalmente
Douglas Soares Goncalves
Data: 10/05/2024 08:22:31-0300
CPF: ***.566.318-**
Verifique as assinaturas em https://v.ufsc.br

Coordenação do Programa de Pós-Graduação

ICP-Edu
Documento assinado digitalmente
Ivan Pontual Costa e Silva
Data: 10/05/2024 16:56:42-0300
CPF: ***.450.424-**
Verifique as assinaturas em https://v.ufsc.br

Prof. Ivan Pontual Costa e Silva, Dr.
Orientador

Florianópolis, 2024.

Agradecimentos

Esta tese é resultado de um trabalho que efetivamente durou quatro anos, passando por um longo ano de pandemia onde diversos obstáculos tiveram que ser contornados.

Primeiramente, agradeço aos meus pais, Fernando Luis Espinoza e Maristela Cardoso, pelo apoio durante todo meu percurso acadêmico.

Agradeço muito a meu orientador, Prof. Dr. Ivan Pontual Costa e Silva, que me guiou durante todo este trajeto, com muitas ideias e *insights*, e que depois muitas reuniões, cálculos e demonstrações, eventualmente culminaram na presente tese. Seu vasto conhecimento tanto em matemática quanto em física são uma fonte de inspiração, e espero continuar colaborando com ele no futuro.

Agradeço também os demais colegas dos departamentos de matemática e física por todas as boas lembranças que carrego destes onze em que passei nestes departamentos. Em especial, agradeço meu amigo e colega de graduação Paulo Henrique Dos Santos por todas as conversas iluminadoras nos mais diversos e aleatórios assuntos, indo de política internacional, passando por medicina, até *animes*, agradeço minha querida amiga Maria de Lourdes Deglmann por todo o apoio, conversas e fofocas em nossas diversos encontros presenciais e vituais ao longo destes mais de dez anos de amizade, e por fim também agradeço a meu amigo Gabriel Michels, por diversos papos sobre matemática, geometria, mercado financeiro, e por recentemente me acompanhar ganhando e perdendo (mais frequentemente perdendo) dinheiro no mercado de criptomoedas.

O presente trabalho foi realizado com apoio da Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES) – Código de Financiamento 001.

Resumo

O objetivo desta tese é obter condições em espaços-tempo para que o fenômeno de incompletude geodésica causal é genérico no sentido topológico (ou seja, é válido em um conjunto residual).

Em nosso primeiro conjunto de novos resultados, baseamos nossas técnicas topológicas naquelas desenvolvidas por Lerner (1973) e obtemos um teorema de genericidade da incompletude geodésica em uma classe de métricas de espaço-tempo na topologia de Whitney forte (sob condições abertas adequadas) para uma variedade não compacta fixada contendo subvariedades fracamente presas de codimensão dois. Com algumas restrições extras para a curvatura em tais classes de espaços-tempos, obtemos um resultado análogo no caso de codimensão maior que dois.

Para nosso último conjunto de novos resultados, exploramos uma situação semelhante, agora para conjuntos de dados iniciais contendo MOTS, sob condições adequadas. Para este caso, estruturas de variedades com dimensão infinita já estabelecidas na literatura podem ser adotadas. Com métodos de análise funcional baseados no trabalho de Biliotti, Javaloyes, and Piccione (2009), obtemos a genericidade da incompletude luminosa para este caso.

Palavras chave: geometria Lorentziana, incompletude geodésica, genericidade.

Abstract

The objective of this thesis is to obtain conditions on spacetimes for when the phenomena of nonspacelike geodesic incompleteness is generic in the topological sense (i.e. is valid in a residual set).

In our first set of new results, we basis our topological techniques from the ones developed by Lerner (1973) and obtain a genericity of geodesic incompleteness theorem in a class of spacetime metrics with strong Whitney topology (under suitable open conditions) for a fixed noncompact manifold containing codimension two weakly trapped submanifolds. With some extra restrictions for the curvature on such classes of spacetimes we can give an analogous higher codimensional result.

For our last set of new results we explore a similar situation, now for initial data sets containing MOTS, under suitable conditions. For this case known infinite dimensional manifold structures can adopted. With functional-analytical methods based on the work of Biliotti, Javaloyes, and Piccione (2009) we obtain genericity of null incompleteness for this case as well.

Keywords: Lorentzian geometry, geodesic incompleteness, genericity.

Resumo Expandido

Introdução

A importância dos teoremas de singularidade (Beem, Ehrlich, and Easley (1999), Hawking (1966), Hawking and Penrose (1970) and Penrose (1965)) em teorias geométricas da gravidade não podem ser subestimadas. Como é bem conhecido, todos os teoremas deste tipo estabelecem a existência de geodésicas causais inextensíveis incompletas (as chamadas "singularidades") em espaços-tempos sob suposições geométricas fisicamente motivadas. Quase tão importante é garantir que as conclusões e/ou suposições nesses teoremas são estáveis sob "pequenas perturbações" da métrica no espaço-tempo se tais devem ser fisicamente relevantes, uma vez que atingir a precisão absoluta para medição de campos físicos tais como o gravitacional é impossível em princípio.

O trabalho seminal de Lerner (1973) introduziu um método natural para discutir questões de estabilidade de forma rigorosa, utilizando topologias fortes de Whitney C^s no espaço das métricas Lorentzianas em uma dada variedade. Lerner também apresentou a forma de adequar estas topologias na relatividade matemática, analisando a estabilidade de uma série de propriedades causais e de curvatura usadas nos teoremas de singularidade (veja também Beem, Ehrlich, and Easley (1999), cap. 7, para uma discussão detalhada e mais resultados e referências sobre o assunto).

Embora o interior dos buracos negros seja o lugar principal onde se espera que as singularidades ocorram, é bem conhecido que as noções *matemáticas* de buracos negros e singularidades são logicamente independentes. Roger Penrose propôs a chamada *conjectura da censura cósmica* para preencher essa lacuna, afirmando aproximadamente que os buracos negros deveriam *genericamente* (em um sentido adequado) surgir quando existem singularidades (veja, por exemplo, Wald (1984), pgs. 299-308, para uma discussão didática e referências originais). Embora a conjectura não tenha sido provada ainda, uma maneira de abordá-la é considerando a existência de singularidades na presença de superfícies marginalmente exteriormente aprisionadas (MOTS), que são especialmente úteis para modelar horizontes de buracos negros em um conjunto de dados iniciais. Pertinente aos nossos propósitos aqui, um "teorema de singularidade genérico" pode ser dado neste caso (Chruściel and Galloway (2014), prop. 1.1; veja também Silva (2012) para resultados relacionados). Esses resultados dependem de uma variante da condição genérica no tensor de curvatura (cf. Beem, Ehrlich, and Easley (1999), seção 2.5), uma suposição já usada no teorema de singularidade clássico de Penrose-Hawking (Hawking and Penrose (1970)). Embora a condição genérica parecesse ser uma restrição de curvatura um tanto forçada, seu caráter "verdadeiramente genérico" foi analisado em espaços tangentes por Beem and Harris (1993), e globalmente - também usando topologias de Whitney - na dissertação de mestrado mais recente de Larsson (2014).

Em qualquer caso, uma perspectiva conceitual mais clara dos teoremas de singularidade em Chruściel and Galloway (2014) and Silva (2012) é se eles são vistos como manifestações da densidade/genericidade de uma *classe inteira* de espaços-tempos singulares próximos (em relação a uma geometria/topologia adequada) de um espaço-tempo contendo uma subvariedade fechada fracamenta aprisionada tal como uma MOTS. Para um exemplo de tal caso, o trabalho de Chruściel and Galloway (2014) (ver teo. 1.2) mostra que, além de certos "casos excepcionais", conjuntos de dados iniciais que satisfazem a condição de energia dominante (DEC) e contêm uma MOTS Σ podem ser arbitrariamente aproximados (na topologia C^∞) por dados iniciais que também satisfazem DEC e para os quais Σ se torna uma superfície aprisionada *externa*, cujo no desenvolvimento de Cauchy a existência de uma geodésica causal inextensível incompleta pode ser diretamente provada (se a variedade subjacente for adicionalmente não compacta).

Objetivos

A Proposta desta tese é estudar condições para topológicas e geométricas em espaços-tempo para estudar a genericidade (no sentido topológico, ou seja, válido em um conjunto residual) da incompletude geodésica. Abordamos estas questões em contextoa distintos mas que se complementam: primeiro no conjunto de métricas, e em segundo no formalismo de dados iniciais. Em cada situação precisamos lidar com condições topologicas e geometricas adequeadas para que técnicas já bem estabelecidas na literatura possam ser adaptadas em tais contextos.

Metodologia

A metologia utilizada neste trabalho de matemática pura é a pesquisa bibliográfica por meio de artigos, teses e livros relevantes e ja estabelicidos na literatura, e reuinões do autor com orientador e outros pesquisadores para discutir, propor e checar a validade dos resultados obtidos.

Resultados e Discussão

O primeiro conjunto de problemas abordados são inspirados no trabalho de Lerner (1973), onde, induzindo topologias de Whtiney C^s forte no espaço de métricas Lorentzianas sobre uma variedade não compacta, obtem-se propriedades de estabilidade (i.e. propriedades abertas) de estruturas relevantes em tais topologias.

Introduzimos noção auxiliar para genericidade topológica que chamamos de *prevalência*, que irá captar a noção de o que é ou não "grande" dentro de conjuntos fechados em sua topologia de subespaço, uma vez que tratar de genericidade diretamente em conjuntos que não são abertos traz alguns fenomenos indesejáveis (exemplo 2.1.5).

Para obter resultados relevantes, nos restringimos ao conjunto de métricas temporalmente orientadas em uma direção pré-fixada X, que são causalmente estaveis, e para as quais uma subvariedade Σ compacta fixada é espacial (talé aberto na topologia de interesse). A partir disso, consideramos o conjunto de métricas para as quais Σ é fracamente futuro-aprisionada, e $\mathcal{FA}$ seu fecho (métricas para as quais Σ é *fracamente futuro-aprisionada*). Obtemos um resultado de prevalência da incompletude causal no caso de Σ ter codimensão 2 (teorema 3.1.7), e com considerações extras no tensor de Riemann das métricas, obtemos o análogo para Σ de codimensão maior (teorema 3.2.6).

Em uma segunda abordagem, exploramos o problema por meio de *dados iniciais de vácuo*. Com algumas restrições geométricas, tal espaço possui uma estrutura de variedade de dimensão infinita (Chruściel and Delay (2004), Bartnik (2005)), possibilitando uma aplicação do clássico teorema de Sard-Smale. A técnica de obter um conjunto genérico é baseada numa adaptação das técnicas em Biliotti, Javaloyes, and Piccione (2009), para a situação do funcional de expansão escalar θ_+, onde a pré-imagem por zero é exatamente o conjunto de dados iniciais possuindo MOTS. Com uma sequência de aplicações de tais métodos e com algumas restrições técnicas, e um teorema de singularidade em presença de MOTS desenvolvido aqui (teorema 1.6.3) obtemos um resultado de genericidade na presença de MOTS para dados iniciais (teoremas 4.3.1 e 4.3.4).

Considerações Finais

Neste trabalho conseguimos obter essencialmente dois resultados para genericidade de singularidades em situanções distintas mas complementares. Este problema é de interesse tanto téorico quando aplicado para a relatividade geral, e com respostas pouco satisftórias na literatura até o momento. Apesar de respostas positivas, algumas restrições fortes precisaram ser assumidas para utilizar resultados técnicos disponíveis na literatura, mas não está claro se tais restrições são realmente necessárias, e isto o que abre possibilidade para novos estudos com o objetivo de enfraquecer tais hipóteses. Também, as técnicas envolvendo topologias de Whitney são um tanto gerais e aparentam ter outras aplicações na geometria Lorentziana, que motivam projetos de investigação futuros.

Palavras chave: geometria Lorentziana, incompletude geodésica, genericidade.

Contents

Introduction

The importance of the singularity theorems (Beem, Ehrlich, and Easley (1999), Hawking (1966), Hawking and Penrose (1970) and Penrose (1965)) in geometric theories of gravity cannot be overstated. As is well known, all theorems of this kind establish the existence of incomplete inextendible causal geodesics (the so called "singularities") in spacetimes under physically motivated geometric assumptions. Almost as important is to ensure that the conclusions and/or assumptions in these theorems are stable under "small perturbations" of the spacetime metric if they are to be truly physically relevant, since attaining infinite precision for the values of physical fields such as the gravitational one is untenable on principle.

The seminal work by Lerner (1973) introduced a natural framework to discuss stability issues rigorously and in full nonlinear generality, under strong Whitney C^s topologies on the space of Lorentzian metrics on a given manifold. Lerner also presented a cogent case for the special suitability of these topologies in mathematical relativity, analyzing and establishing the stability of a number of causal and curvature properties used in the original singularity theorems (see also Beem, Ehrlich, and Easley (1999), Ch. 7, for a detailed discussion and further results and references on the subject).

Although the interior of black holes are prime places where singularities are expected to occur, it is well-known that the *mathematical* notions of black holes and singularities are logically independent. This was already recognized by R. Penrose himself, who proposed the so-called *cosmic censorship conjecture* to bridge this gap, roughly stating that black holes should *generically* (in a suitable sense) arise when singularities exist (see, e.g., Wald (1984), pgs. 299-308, for a didactic discussion and original references). Although the full conjecture remains elusive, one way to approach it is by considering the existence of singularities in the presence of marginally outer trapped surfaces (MOTS), which are especially useful to model black holes horizons in initial data sets. Pertinent to our purposes here, a "generic singularity theorem" can be given in this case (Chruściel and Galloway (2014), prop. 1.1; see also Silva (2012) for related results). These results rely on a variant of the generic condition on the curvature tensor (cf. Beem, Ehrlich, and Easley (1999), section 2.5), an assumption already used in the Penrose-Hawking classic singularity theorem (Hawking and Penrose (1970)). Although the generic condition at first seemed to be a somewhat contrived curvature constraint, its "truly generic character" has been analyzed at tangent spaces by Beem and Harris (1993), and globally - also using Whitney topologies - in the more recent master's thesis of Larsson (2014).

In any case, an arguably more transparent conceptual perspective of the singularity theorems in Chruściel and Galloway (2014) and Silva (2012) is if they are viewed as manifestations of the density/genericity of a *whole class* of singular spacetimes near (with respect to a suitable geometry/topology) a spacetime containing a closed weakly trapped submanifold such as a MOTS. For an example of such case, the work of Chruściel and Galloway (2014) (see thm. 1.2) shows that apart from certain "exceptional cases", initial data sets satisfying the dominant energy condition (DEC) and containing a MOTS Σ can be arbitrarily approximated (in the C^∞-topology) by initial data sets also satisfying DEC for which Σ becomes an *outer* trapped surface, in whose Cauchy development the existence of an incomplete inextendible causal geodesic can be directly proven (if the underlying manifold is in addition noncompact).

The objective of this thesis is, in the spirit of Chruściel and Galloway (2014), to develop techniques as to obtain conditions where geodesic incompleteness is generic for suitable topologies on the set metrics over a given manifold with certain topological and geometrical properties that will allow us to use some well established (and variations of) topological and functional-analytical methods.

The original part of the work has been divided into two main sets. The first set of genericity results presented here have been heavily influenced by the ideas in Lerner (1973) and in particular we work with the strong (or fine) Whitney topologies throughout, using comparatively more well-known topological notions, and applying these to a broader problem: that of analyzing the genericity of causal geodesic incompleteness in spacetimes containing the so-called *weakly trapped* submanifolds, a class which includes MOTS. The second group of genericity results discussed here uses rather different techniques, and are now directly influenced by the arguments in Chruściel and Galloway (2014). Our goal here is obtain, in similar fashion, a "genericity of incompleteness near MOTS" type of result. Dealing with with the so-called *initial data sets* rather than working direct with spacetimes has the advantage that in this context there are infinite dimensional (Banach/Hilbert) manifold structures for the set of initial data (under some reasonable restrictions, cf. Bartnik (2005) and Chruściel and Delay (2003)), and here the basic technical tool to obtain genericity is the Sard-Smale theorem (Smale (1965)).

Of course, there are advantages and disadvantages of each approach. Among the perks of the first approach we might count: (i) the proofs are less technical since they rely on already standard topological techniques, (ii) our curvature assumption in the codimension 2 case (strong energy condition) is strictly weaker than the dominant energy condition, (iii) we handle weakly trapped submanifolds, a larger class than just MOTS, (iv) we include a result for higher codimension with little extra cost, and last but not least (v) we weaken the causality requirements on spacetime. The latter point is relevant especially in physical applications, because unless strong cosmic censorship applies, any incomplete causal geodesic one predicts in the (globally hyperbolic) maximal Cauchy development of a given initial data set might still be complete in an isometric extension of lower causality, as the case of data induced in a suitable smooth partial Cauchy hypersurfaces in anti-de Sitter spacetime (which is in particular stably causal but not globally hyperbolic) illustrates.

On the flip side, however, the initial data approach, while technically harder, is more convenient if one wishes to focus one's attention only on MOTS, which are natural models, in this context, for black hole horizons. Initial data sets also have broader applicability in PDE analysis of the Einstein fields equations of general relativity, not least in numerical methods (cf.

Cook (2000)). From a physical perspective, again, one might argue that one can hardly expect to glean information of spacetimes as a whole. Rather, all one can expect is to make predictions from current data, and initial data sets are a natural model for such a situation. While we can only guarantee genericity of causal incompleteness for the Cauchy development of initial data sets, and in particular only for globally hyperbolic spacetimes, such already cover a vast amount of interesting and relevant cases.

This thesis is organized as follows. In part I we recall some basic notions of geometry and topology forms the basic language of the work, mostly to establish notation and terminology. In chapter 1 we review semi-Riemannian and spacetime geometry. Most of the contents in this part is quite standard, but we also introduce a more specialized singularity theorem (due to Silva (2012)) on the presence of MOTS (theorem 1.6.3).

Chapter 2 is dedicated to review what one understands by *genericity* in the context of this work, as well as to introduce a weaker notion of "topological largeness" which we call *prevalence*, and also review stability and genericity properties for the space of Lorentzian metrics with strong Whitney topologies (the main results here are due to Lerner (1973)).

With the basics established, we go to the first part of our main results. In chapter 3 we introduce a suitable context were our causal structures are well defined. Our first prevalence/generic on codimension 2 is theorem 3.1.7, and under some extra curvature restrictions, we give a higher codimensional analogous (theorem 3.2.6).

Finally, chapter 4 is dedicated to the initial data/MOTS case. We start with an abstract, Banach manifold genericity method (which has been especially adapted from Biliotti, Javaloyes, and Piccione (2009)). "This abstract approach has the enormous advantage of flexibility: we can choose among a number of variants of Banach/Hilbert manifold structures on initial data sets and set of embeddings extant in the literature, subject only to relatively mild technical restrictions. It also bypasses many of the tremendously technical details behind each of these structures. With a functional-analytical method and a concrete separable Hilbert manifold structure, we obtain the genericity result (theorem 4.3.1 and theorem 4.3.4) as a straightforward consequence of the abstract machinery.

Part I

Preliminaries

1 Review of Lorentzian Geometry

In this chapter we give a e review some of the main aspects of Lorentzian geometry that will be relevant and recurrent throughout this thesis, mainly to introduce notation and terminology, and refer the reader to standard textbooks for semi-Riemannian and Lorentzian geometry, eg. O'Neill (1983), Beem, Ehrlich, and Easley (1999) and John M. Lee (2018) for other topics that won't be mentioned in this review (see also Costa e Silva (n.d.) and Espinoza (2020), also for a discussion on initial data and MOTS see Hafemann (2023) for a comprehensive review on the subject).

1.1 Semi-Riemmanian and Lorentzian Manifolds

Let M be an n-dimensional smooth manifold, which will usually be denoted by M^n. A symmetric smooth [1] $(0, 2)$-tensor is said to be a *semi-Riemannian metric of index* $\nu \in \mathbb{N}$ $(0 \leq \nu \leq n)$ if at each point $p \in M$, the symmetric bilinear form $g_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate bilinear form of index ν. The pair (M, g) is said to be a *semi-Riemannian manifold (of class C^r)*. The relevant cases for this thesis are when $\nu = 0$, called a *Riemannian manifold*, and $\nu = 1$ and $n \geq 2$, called a *Lorentzian manifold*.

Example 1.1.1. The simplest example of a semi-Riemannian manifold is given on $M = \mathbb{R}^n$, Consider the usual Cartesian coordinates $(x^1, \ldots, x^n)$, define a metric η_ν by the line element

$$ds_\nu^2 = -\sum_{i=1}^{\nu}(dx^i)^2 + \sum_{j=\nu+1}^{n}(dx^j)^2.$$

This metric is called the *semi-Euclidean* metric of index ν. $\mathbb{R}^n$ with this metric is denoted by $\mathbb{R}^n_\nu$. For $\nu = 1$, we refer to $\mathbb{R}^n_1$ as the *Minkowski spacetime*. ◄

For (M, g) a semi-Riemannian manifold with index $0 < \nu < n$, we can partition the tangent vectors in mutually disjoint classes. We say $v \in T_pM$ is

(i) *timelike* if $g_p(v, v) < 0$,

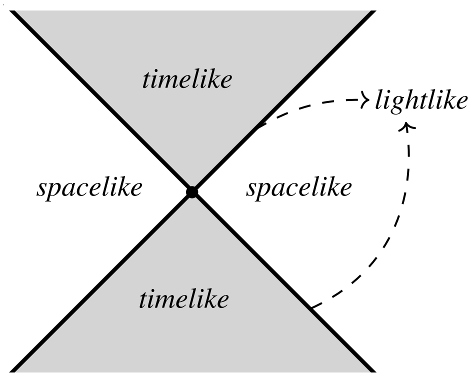
(ii) *lightlike* (ou *null*) if $g_p(v, v) = 0$ and $v \neq 0$,

(iii) *spacelike* if either $g_p(v, v) > 0$ or $v = 0$.

This is the *causal character* of tangent vectors. We also say that $v \in T_pM$ is a *causal vector* if v is either timelike or null. The set of timelike (resp. causal) vectors is called the *time cone* (resp. *causal cone*).

[1] While standard textbooks on the subject describe semi-Riemannian theory for smooth objects, here finite differentiability will be relevant at some points.

Figure 1.1: Cones in Minkowski spacetime.



The causal character is also meaningful for other objects. A vector field X is *timelike* (resp. *lightlike*, *spacelike* or *causal*) if for each $p \in M$ we have X_p *timelike* (resp. *lightlike*, *spacelike* or *causal*). Similarly, a differentiable curve $\gamma : I \to M$ has a causal character if $\gamma'(t)$ has the same causal character for all $t \in I$.

1.1.1 Connection and Curvature

One of the main properties of semi-Riemannian manifolds is the existence of a canonical connection associated with its metric. We denote by $\mathfrak{X}(M)$ the set of smooth vector fields over M (smooth sections of TM), and define an *affine connection* over the tangent bundle TM to be an application $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ satisfying

(i) $\nabla_{X_1 + fX_2} Y = \nabla_{X_1} Y + f \nabla_{X_2} Y$, for $X_1, X_2 \in \mathfrak{X}(M), Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$.

(ii) $\nabla_X (Y_1 + cY_2) = \nabla_X Y_1 + c \nabla_X Y_2$, for $X \in \mathfrak{X}(M), Y_1, Y_2 \in \mathfrak{X}(M)$ and $c \in \mathbb{R}$.

(iii) $\nabla_X (fY) = (Xf)Y + f \nabla_X Y$, for $X \in \mathfrak{X}(M), Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$.

Such a connection is said to be *symmetric* or *torsion free* if $\nabla_X Y - \nabla_Y X = [X, Y]$ (the Lie bracket between X and Y) for all $X, Y \in \mathfrak{X}(M)$.

The *fundamental theorem of semi-Riemmanian geometry* is the following:

Theorem 1.1.2 (Costa e Silva (n.d.), thm. 3.2.1). *For semi-Riemannian manifold (M, g) there exists a unique affine connection $\nabla^g \equiv \nabla$ that is torsion-free and* compatible with the metric g*, meaning that given $X, Y, Z \in \mathfrak{X}(M)$, ∇ satisfies*

$$Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).$$

Such connection is called the Levi-Civita connection of (M, g) *and can be characterized by the* Koszul formula*:*

$$\begin{aligned} 2g(\nabla_X Y, Z) = {} & X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) \\ & - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]), \end{aligned}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. ■

Therefore, computations involving a connection ∇ on a semi-Riemannian manifold (M, g) will be implicitly understood to be with respect to the Levi-Civita connection.

The connection acting on local coordinate vector field defines the *Christoffel symbols* Γ_{ij}^k :

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k.$$

For a symmetric connection, the Christoffel symbols are symmetric on the covariant indices $(\Gamma_{ij}^k = \Gamma_{ji}^k)$.

A connection gives rise to the *curvature tensor* $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X, Y, Z \in \mathfrak{X}(M).$$

In coordinates, $R^i_{\ jkl}\partial_i = R\left(\partial_k, \partial_l\right)\partial_j$, where

$$R^i_{\ jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^m_{jl}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{ml}.$$

The *Riemann tensor* on the semi-Riemannian manifold (M, g) is the $(0, 4)$ tensor metrically equivalent to the curvature tensor, here denoted by $Riem$. Its coordinate symbol is $R_{ijkl} = g_{ir} R^r_{\ jkl}$.

The *Ricci curvature tensor* on (M, g) is a symmetric $(0, 2)$-tensor Ric given by the contraction of the curvature tensor. In coordinates it will be denoted by $R_{ij} = \sum_{k=1}^n R^k_{\ ikj}$.

Lastly, the *scalar curvature* $Scal \in C^\infty(M)$ is the trace of the Ricci curvature, $Scal = g^{ij} R_{ij}$.

Remark 1.1.3. Since we will be interested in varying the metric over a manifold, these curvature tensors will be a function of the metric, and we will sometimes use the descriptive notation R_g or $R(g)$, Ric_g or $Ric(g)$ and so on to emphasize this dependence on the metric of such objects.

1.1.2 Connections Over Maps

Let M and N be smooth manifolds and let $F : N \to M$ be a smooth map. A *vector field over* F is a map $V : N \to TM$ for which $F = \pi_M \circ V$ holds, where $\pi_M : TM \to M$ is the standard projection. For any smooth map $F : N \to M$, we define $\mathfrak{X}(F)$ to be the set of smooth vector fields over F, which is a $C^\infty(N)$-module with respect to pointwise operations. For instance, if F is a smooth map, then given $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$, the maps $X \circ F$ and $dF \circ Y$ are smooth vector fields over F where here (and hereafter) dF denotes the derivative of F.

At this point, the definition of connection can be extended to vector fields over maps as follows: Let $F : N \to M$ be a smooth map. A *connection on* F is a map

$$D : (X, V) \in \mathfrak{X}(N) \times \mathfrak{X}(F) \mapsto D_X V \in \mathfrak{X}(F),$$

such that

1. D is $\mathbb{R}$-bilinear;

2. $D_{fX}Y = fD_XY$, for $X \in \mathfrak{X}(N), V \in \mathfrak{X}(F), f \in C^\infty(N)$;

3. $D_X(fY) = (Xf)Y + fD_XY$ for $X \in \mathfrak{X}(N), V \in \mathfrak{X}(F), f \in C^\infty(N)$.

We can also define a curvature tensor for the connection D. The *curvature tensor of a connection D on the map $F : N \to M$* is given by

$$R^D(X,Y)V := D_XD_YV - D_YD_XV - D_{[X,Y]}V$$

for all $X, Y \in \mathfrak{X}(N)$ and $V \in \mathfrak{X}(F)$. This curvature tensor is $C^\infty(N)$-trilinear. The following results regarding the so-called *induced connection* will be constantly employed in our calculations.

Theorem 1.1.4 (Costa e Silva (n.d.)). *Let ∇ be a connection on the manifold M, let N be any smooth manifold and $F : N \to M$ be a smooth map. Then, there exists a unique connection D^∇ on F such that*

$$D_X^\nabla(V \circ F)(p) = \nabla_{dF_p(X_p)}V(F(p)), \quad \forall p \in N, \forall X \in \mathfrak{X}(N), \forall V \in \mathfrak{X}(M).$$

D^∇ is called the induced connection *on F.*

Proposition 1.1.5 (Costa e Silva (n.d.)). *Let ∇ be a connection on the manifold M, let N be any manifold and let $F : N \to M$ be a smooth map. Finally, let $D = D^\nabla$ be the induced connection on F. Then*

$$R_p^D(x,y)v = R_{F(p)}^\nabla(dF_p(x), dF_p(y))v, \tag{1.1}$$

for any $p \in N$ and $\forall x, y \in T_pN, \forall v \in T_{F(p)}M$. Moreover, if ∇ is symmetric, then

$$D_X(dF \circ Y) - D_Y(dF \circ X) = dF \circ [X,Y].$$

When computations involving connections over maps appear in the context of semi-Riemannian manifolds, we will always assume that D is the induced connection from the Levi-Civita connection.

1.1.3 Geometry of Immersions

It will be important to introduce not only some terminology and notation for the geometry of semi-Riemannian submanifolds, but also more generally for semi-Riemannian immersions.

Let (M, g) be a semi-Riemannian manifold. Given a smooth map $F : N \to M$, the pullback F^*g defines a symmetric $(0,2)$-type smooth tensor field on N, but this is *not* necessarily a semi-Riemannian metric. A necessary condition to ensure it is a metric is that F is an immersion:

Proposition 1.1.6 (Costa e Silva (n.d.), prop. 4.1.1). *Let $F : N \longrightarrow M$ be a smooth map into the semi-Riemannian manifold (M, g). If F^*g is a semi-Riemannian metric on N, then F is an immersion.* ■

Given a semi-Riemannian manifold (M, g) we then define a *semi-Riemannian immersion* to be a smooth immersion $F : N \longrightarrow M$ for which (N, F^*g) is also a semi-Riemannian manifold. A *semi-Riemannian embedding* is an embedding that is also a semi-Riemannian immersion. In this case, F^*g is called the *induced metric*. When $N \subseteq M$ is a submanifold, we will say that N itself is a *semi-Riemannian submanifold* if the inclusion $i : N \hookrightarrow M$ is a semi-Riemannian embedding.

Moreover, any semi-Riemannian immersion $F : N \longrightarrow M$ for a *Lorentzian* (M, g) induces a metric F^*g which is either

1) Riemannian, in which case we say that F is *spacelike*;

2) Lorentzian, in which case we say that F is *timelike*;

If $N \subseteq M$ is a semi-Riemannian submanifold of the Lorentzian manifold (M, g), then it is said to be *spacelike* [resp. *timelike*] if the inclusion $i : N \hookrightarrow M$ is spacelike [resp. timelike].

The following notation for the scalar product over semi-Riemmanian immersions will be frequent: for $F : N \longrightarrow M$ a semi-Riemannian immersion, given any $V, W \in \mathfrak{X}(F)$, we define at each $p \in N$

$$\langle\langle V, W \rangle\rangle_p = g_{F(p)}(V_p, W_p). \tag{1.2}$$

The is a natural orthogonal decomposition of vectors over a semi-Riemannian immersion: given any $p \in N$, and any $v \in T_{F(p)}M$, there exist unique $v^\top \in dF_p(T_pN)$ and $v^\perp \in dF_p(T_pN)^\perp$, called the *tangent* and *normal* parts of v, respectively, such that

$$v = v^\top + v^\perp. \tag{1.3}$$

More generally, given a smooth vector field $V \in \mathfrak{X}(F)$ on F, we have a *pointwise* decomposition in tangent and normal parts. A smooth vector field $V \in \mathfrak{X}(F)$ is said to be *tangent* [resp, *normal*] if $V_p \in dF_p(T_pN)$ [resp. $V_p \in dF_p(T_pN)^\perp$] for all $p \in N$. We denote the subset of smooth tangent [resp. normal] vector fields over F by

$$\mathfrak{X}^\top(F)[resp. \mathfrak{X}^\perp(F)].$$

The normal-tangent decomposition of the connection over F gives rise to a very important geometric object.

Theorem 1.1.7 (Costa e Silva (n.d.), thm. 4.1.8). *For any $X, Y \in \mathfrak{X}(N)$*

$$D_X(dF \circ Y) = dF(\nabla^N_X Y) + II(X, Y), \tag{1.4}$$

where ∇^N denotes the Levi-Civita connection of F^*g in N, and

$$II(X,Y) := (D_X(dF \circ Y))^{\perp}.$$

Moreover, the map $II : \mathfrak{X}(N) \times \mathfrak{X}(N) \to \mathfrak{X}^{\perp}(F)$ thus defined is $C^{\infty}(M)$-bilinear and symmetric, and is called the second fundamental form tensor *or* shape tensor *(of F).* ■

From the shape tensor we define $H \in \mathfrak{X}^{\perp}(F)$ the *mean curvature vector* of the semi-Riemann immersion F as the trace of the shape tensor with respect to the metric F^*g. At each $p \in N$ this vector can be computed as

$$H_p = \sum_{i=1}^{\dim N} \varepsilon_i II(e_i, e_i),$$

where $\{e_1, \ldots, e_{\dim N}\}$ is a F^*g-orthonormal basis at T_pN. One can then easily check that the definition is actually independent of the choice of the orthonormal basis.

Remark 1.1.8. As in remark 1.1.3, the mean curvature vector is also dependent on the metric and F, so when an emphasis on the metric and the map is needed the notations H_g^F or $H^F(g)$ will be used.

1.2 Spacetimes and Causality

We shall restrict our analysis almost exclusively to a class of Lorentzian manifolds where a notion of "past and future" can be given in a precise manner.

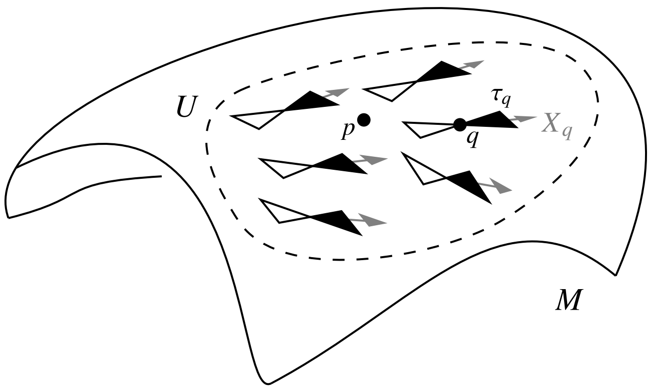
Let (M, g) be a Lorentzian manifold. At each tangent space we choose a time cone. Similarly to the usual definition of orientability of manifolds, the notion of time orientatility is related to a continuous choice of such time cones along M. More precisely, let τ be a function such that at each point $p \in M$ chooses a time cone $\tau(p)$ in T_pM.

Definition 1.2.1. *We say that τ is* smooth *if at each $p \in M$ there exists a smooth vector field X defined on a open neighborhood of U of p such that $X_q \in \tau(q)$ for all $q \in U$ (figure 1.2). Such a smooth function τ is called a* time-orientation *on (M, g). If (m, g) admits a time orientation, we say that it is* time orientable. *The act of choosing a specific time orientation then* time orients (M, g).

Proposition 1.2.2. *If (M, g) is a connected, time orientable Lorentzian manifold, there exists exactly two time orientations for (M, g).* ■

In practice, the following lemma gives a more useful characterization of time orientability that we shall adopt.

Figure 1.2: Time orientability.



Lemma 1.2.3. *A Lorentzian manifold (M, g) is time orientable if and only if there exists a globally defined smooth timelike vector field $X \in \mathfrak{X}(M)$.* ■

If there is such vector field X as in lemma 1.2.3, the time orientation defined by X is called the *time orientation induced by X.*

A time orientation now gives us the notion of past and future directions. In a spacetime (M, g) with $X \in \mathfrak{X}(M)$ inducing its time orientation, a causal vector $v \in TM$ is said to be *future-directed* if $g(v, X) < 0$, and *past-directed* if $g(v, X) > 0$. Similarly a vector field $V \in \mathfrak{X}(M)$ is *future-directed* if V_p is future-directed for all $p \in M$, and a differentiable curve $\gamma : I \longrightarrow M$ is *future-directed* if $\gamma'(t)$ is future-directed for all $t \in I$, with the notion of *past-directed* being analogous.

Remark 1.2.4. The spacetime condition is not as restrictive as it might seem, since for any connected Lorentzian manifold (M, g) it is possible to find a double covering $\pi : \widetilde{M} \longrightarrow M$ that is also a connected Lorentzian manifold and is time orientable, so it is a spacetime locally isometric to (M, g) (see the discussion in Beem, Ehrlich, and Easley (1999) after definition 3.1).

With spacetimes having a precise notion of future and past directions, we can define the *causal relations*: given p, q points in a spacetime (M, g),

- q is in the *chronological future* of p if there is a timelike future-directed curve starting at p and ending at q.
- q is in the *causal future* of p if there is a causal future-directed curve starting at p and ending at q.

The notion of *chronological* and *causal past* points are defined analogously.

The chronological [resp. causal] future set of a point p is the set of all points in the chronological [resp. causal] future of p, denoted by $I^+(p)$ [resp. $J^+(p)$]. By time duality, we the denote $I^-(p), J^-(p)$ for the past case, and we also have a analogous notion for past and future of a set $A \subseteq M$. Concretely, $I^+(A) = \cup_{p \in A} I^+(p)$, and analogously for $J^+(A)$.

We define the so-called *causality conditions* on a spacetime that will be relevant in this work[2].

Definition 1.2.5. *Let (M, g) be a spacetime. We say that (M, g) is*

(1) chronological *if no point in M is an element of its own chronological future* (there are no timelike loops).

(2) causal *if no point in M is an element of its own causal future* (there are no causal loops).

Lastly, a subset $S \subseteq M$ is said to be a Cauchy hypersurface *if every timelike inextendible curve intersects S exactly once, and a spacetime that has a Cauchy hypersurface is said to be* globally hyperbolic[3].

1.3 The Generic Property for Spacetimes

The generic propriety is a restriction for causal vectors that is needed as a hypothesis for some singularity theorems. Seemingly technical and algebraic in nature, it has a more physical intepretation that we will point out shortly. But the generic property has also another, precise topological meaning that will be discussed in section 2.2.1 and that will be its most appropriate connotation for us here.

Definition 1.3.1. *On a Lorentzian manifold (M, g), let $v \in TM$ be any vector, and denote R_{ijkl} the components of the $(0, 4)$ Riemann curvature tensor Riem with respect to some local basis. We say that v satisfies the* generic condition *if*

$$v^k v^l v_{[i} R_{j]kl[m} v_{r]} \neq 0, \tag{1.5}$$

for some *combination of free indices. Such a vector v satisfying the generic condition is said to be a* generic vector.

Similarly, we say that the generic condition *holds for an inextendible geodesic $\gamma : I \longrightarrow M$ if for some time $t_0 \in I$ we have that $\gamma'(t_0)$ is a generic vector in the sense of (1.5). In this sense, we say that* generic condition *holds for the Lorentzian manifold (M, g) if the generic condition holds on each inextendible causal geodesic.*

The condition (1.5) for a vector v can be expressed in the following invariant way: v is generic if and only if

$$(v^\flat \otimes v^\flat) \owedge Riem(\cdot, v, \cdot, v) \neq 0,$$

[2] There are other causality conditions that we shall have no use for here and are therefore omitted.

[3] It can then be shown (cf. O'Neill (1983), ch. 14) that a Cauchy hypersurface S is a closed subset of M, that it is indeed a C^0 hypersurface, and that S is *achronal*, i.e., there are no two of its points that can be connected via a timelike curve segment. Also, in a globally hyoerbolic space it is also possible to choose a *smooth* and spacelike Cauchy hypersurface S (Bernal and Sánchez (2003)).

where $v^\flat$ is the 1-form metrically equivalent to v and $\owedge$ denotes the Kulkarni-Nomizu product of two (0,2)-tensors (John M. Lee (2018), p. 213).

The generic condition for timelike vectors can be interpreted more geometrically to be equivalent to the condition that the *tidal force operator* $R_v = R(\,\cdot\,, v)v$ for $v \in TM$ timelike (Beem, Ehrlich, and Easley (1999), prop. 2.7) is not the zero operator. A similar, slightly more technical characterization exists for lightlike vectors (Beem, Ehrlich, and Easley (1999), sect. 2.5).

Geodesics of interest for the singularity theorems are causal. Physically speaking, a timelike geodesic fails to be generic if a corresponding observer along the geodesic fails to measure a tidal force effect. A similar analysis holds for lightlike vectors and geodesics, with the usual vector space quotient analysis to deal with the degeneracy of certain associated vector spaces (Beem, Ehrlich, and Easley (1999), prop. 2.11).

1.4 Trapped Surfaces and Submanifolds

We now discuss the notion of trapped and marginally outer trapped submanifolds and surfaces; these special surfaces play many roles in Lorentzian geometry and general relativity.

Definition 1.4.1 (Trapped Submanifolds). *Let (M^{m+k}, g) be a spacetime, and let Σ be an m-dimensional manifold. A map $\psi : \Sigma \to M$ is said to be a* future trapped immersion *if ψ is a spacelike immersion and the mean curvature vector field H of ψ is past-directed and timelike. An immersed submanifold $\Sigma \subset M$ is a* future-trapped submanifold *if the inclusion $\Sigma \hookrightarrow M$ is a trapped immersion.*

A particular case of interest is when $k = 2$ and the spacelike immersion $\psi : \Sigma \to M$ has trivial normal bundle in the spacetime (M, g) (when $\Sigma \subseteq M$ is a codimension 2 submanifold we say that Σ is a *surface*). In this case, one can see that there exists two linearly independent and globally defined null future-directed vector fields $\ell_+, \ell_- \in \mathfrak{X}^\perp(\psi)$, that can be normalized such that $\langle\langle \ell_+, \ell_- \rangle\rangle = -1$. In this situation we can define $(0,2)$-tensors over Σ called the *null second fundamental forms* $\mathcal{X}_\pm$ associated with $\ell_\pm$ as

$$\mathcal{X}_\pm(X, Y) = \langle\langle D_X \ell_\pm, d\psi \circ Y \rangle\rangle, \quad \text{for } X, Y \in \mathfrak{X}(\Sigma).$$

The *null mean curvatures* (or *null expansion scalars*) $\theta_\pm \in C^\infty(\Sigma)$ are defined as

$$\theta_\pm = \operatorname{tr}_{\psi^* g} \mathcal{X}_\pm = \operatorname{div}_{\psi^* g} \ell_\pm$$

where $\psi^* g$ is the induced Riemannian metric in Σ. With some computations we obtain a more friendly formula for the null expansion scalars:

$$\theta_\pm = -\langle\langle H, \ell_\pm \rangle\rangle.$$

These quantities are all dependent on the choice of the null vector fields $\ell_\pm$, but their signs remain the same when multiplying by a positive smooth function, and the signs of the null expansion scalars are the meaningful quantitiy here. For the case of a codimension two immersion ψ, we see that the mean curvature vector H of ψ has the form

$$H = \theta_- \ell_+ + \theta_+ \ell_-.$$

Also, $\langle\langle H, H\rangle\rangle = -2\theta_+\theta_-$. If ψ is future trapped, then H is past-directed and timelike, and this implies both $\theta_\pm < 0$. Conversely, if both $\theta_\pm$ are negative over Σ then clearly H is timelike past-directed, so ψ is future-trapped, establishing the equivalence between the negative sign of the null expansion scalars and the future-trapped characteristic of the immersion.

Other signs for $\theta_\pm$ are also meaningful,and here a very important case is when $\theta_+ = 0$. A surface (immersion) satisfying this condition is called a *marginally outer trapped surface* (*immersion*) (for surfaces this is abbreviated to *MOTS*). For this thesis MOTS will be more relevant to the initial data formulation, discussed in the next section.

Remark 1.4.2. Although we have developed the submanifold geometry in the broader class of immersions, for simplicity we shall focus on embeddings from now on. However, most results can be adapted for immersions as well (but not all, especially those in chapter 4).

1.5 Einstein Field Equations and the Initial Data Formulation

The theory of general relativity first envisioned by Einstein (1915), provides a geometric description of the phenomenon of gravity. From the the general relativistic point of view, a region of the universe of interest can be described via a 4-dimensional spacetime (M, g), and gravitational phenomena are a manifestation of its underlying geometry, rather than by forces as in classical mechanics.

The dynamics of general relativity is then described by the *Einstein Field Equations* (abbreviated to *EFE*). In coordinates, these are a system of second-order nonlinear partial differential equations that associate the geometry of spacetime with the distribution of matter and energy in the universe. These equations are significantly hard to solve for a number of reasons, not only purely technical, but also conceptual. Among the latter is the fact that a distribution of matter only makes sense in the context of a background spacetime and therefore the distribution of matter and the geometry of spacetime must be solved simultaneously.

In the language of Lorentzian geometry, the universe is modeled as spacetime (M, g), where g represents the gravitational field. The distribution of matter and energy is represented by a symmetric $(0, 2)$-tensor field T, known as the *stress-energy tensor*, rather than a mass density function in classical mechanics. Additionally, the model also incorporates a constant called *cosmological constant* $\Lambda \in \mathbb{R}$ that represents a form of energy that is inherent to space itself. The relationship between the spacetime (M, g), the tensor T and the constant Λ is established via the

EFE:

Definition 1.5.1 (Einstein field equations). *Let (M, g) be a spacetime and T be a symmetric $(0, 2)$-tensor field on M. We say that the spacetime (M, g) is a* solution to the Einstein field equations *with* stress-energy tensor T *and* cosmological constant Λ *if g satisfies*

$$Ric - \frac{1}{2}Scal \cdot g + \Lambda \cdot g = T, \tag{1.6}$$

where $Ric, Scal$ are the Ricci and scalar curvature associated to g. Alternatively, the equations can be written as $G = T - \Lambda g$, where G is the Einstein tensor *of g defined by*

$$G := Ric - \frac{1}{2}Scal \cdot g.$$

The simplest version of the EFE is the *vacuum spacetime* case, which is obtained when T is identically zero and $\Lambda = 0$. In this scenario, the Minkowski space satisfies the EFE and it is considered as a fundamental model of empty spacetime. The vacuum case also includes other non-trivial solutions of interest, such as the Schwarzschild spacetime. In particular, a spacetime (M, g) is a *vacuum solution of the EFE* if and only if it is *Ricci-flat*, i.e, $Ric = 0$. Whenever we refer to the EFE in this thesis, we shall almost always consider only the vacuum case. In particular, we set $\Lambda = 0$ for the rest of the discussion.

1.5.1 Constraint Equations and Initial Data

It is natural from the PDE point of view to formulate an initial value problem for the EFE, specifying an initial metric and stress-energy tensor at a given starting time to then evolve the EFE forward on time. The notion of "time" in general relativity is rather delicate, and unlike usual PDE theory, there isn't a fixed notion of time with respect to which one could define evolution equations in the usual sense, as the idea of time for a given observer only has meaning after we know the spacetime in question, which is actually a variable to be obtained only after we solve the EFE. All these questions gave rise to the *initial data formulation* of general relativity, which we briefly discuss now.

For (M^{n+1}, g) a spacetime with G its associated Einstein tensor, given $u \in TM$ a future-directed unit timelike vector, the *energy density* associated with u is defined by $\rho_u = G(u, u)$, and we also define the *energy momentum current* as the one-form over the vector space $\{u\}^\perp$ given by $J_u(\cdot) = -G(u, \cdot)$.

Consider now $\mathcal{S} \subseteq M$ an embbeded spacelike hypersuface in M with induced Riemannian metric h, and let $\mathcal{K}$ denote its scalar second fundamental form with respect to $U \in \mathfrak{X}^\perp(\mathcal{S})$ which is the unique future-directed unit timelike vector field normal to $\mathcal{S}$. The following equations, called the *constraint equations*, hold on $\mathcal{S}$ (Wald (1984), ch. 10):

$$\mathrm{Scal}_h - |\mathcal{K}|_h^2 + (\mathrm{tr}_h\mathcal{K})^2 = 2\rho_U, \quad \text{(Hamiltonian Constraint Equation)}$$
$$\mathrm{div}_h\,\mathcal{K} - d(\mathrm{tr}_h\mathcal{K}) = J_U. \quad \text{(Momentum Constraint Equation)}$$

Although the previous equations are induced on a spacelike hypersurface in a previously defined solution of the EFE, and thus give a *necessary* condition on such a solution, a surprising fact is that these conditions are actually *sufficient* to characterize the underlying spacetime in a precise sense we now discuss. This allows one to start with *abstractly defined initial data* and obtain a solution of the EFE from that: this summarizes the idea behind the so-called *initial data formulation* of general relativity.

Definition 1.5.2 (Initial Data Set). *An* initial data set *is a triple* $(\mathcal{S}^n, h, \mathcal{K})$ *where* $(\mathcal{S}^n, h)$ *is a Riemannian manifold and* $\mathcal{K}$ *is a symmetric* $(0,2)$*-tensor field on* $\mathcal{S}$*. Given an initial data* $(\mathcal{S}, h, \mathcal{K})$*, we define a function* $\rho \in C^\infty(\mathcal{S})$ *and a one-form* $J \in \Omega^1(\mathcal{S})$*, called respectively the* energy density *and* energy-momentum current *associated with the data by*

$$\begin{aligned} \rho &= \frac{1}{2}\left[\mathrm{Scal}_h - |\mathcal{K}|_h^2 + (\mathrm{tr}_h\,\mathcal{K})^2\right], \\ J &= \mathrm{div}_h\,\mathcal{K} - d(\mathrm{tr}_h\,\mathcal{K}). \end{aligned} \tag{1.7}$$

The seminal work of Choquet-Bruhat and Geroch (1969) shows that for *vacuum initial data*, i.e., those for which $\rho = 0$ and $J = 0$, and $(\mathcal{S}, h, \mathcal{K})$ is one such initial data, the manifold $\mathcal{S}$ can be viewed as suitably embedded spacelike hypersurfaces in a uniquely defined maximal globally hyperbolic vacuum spacetime. We summarize this result in the following theorem:

Theorem 1.5.3. *Let* $(\mathcal{S}^n, h)$ *be a Riemannian manifold and let* $\mathcal{K}$ *be a smooth symmetric* $(0,2)$*-tensor field on* $\mathcal{S}$*. Suppose that eqs. (1.7) are satisfied for vacuum conditions with* $\rho = 0$ *and* $J = 0$*. Then there exists an* $(n+1)$*-dimensional vacuum spacetime* (M^{n+1}, g) *such that* $(\mathcal{S}^n, h)$ *isometrically embeds into* (M, g) *as a Cauchy hypersurface with second fundamental form* $\mathcal{K}$*. Furthermore, there is a unique (up to isometry)* maximal *such solution in the sense that any other solution satisfying these conditions can be isometrically embedded therein.* ■

1.5.2 MOTS and Initial Data

For our applications we are interested in MOTS and its null expansion scalar from an initial data set point of view. Let $\Sigma^{n-1} \subseteq \mathcal{S}^n$ be an embedded submanifold[4], and recall that Σ is *two-sided* if it has a trivial normal bundle in $\mathcal{S}$.

Definition 1.5.4 (Null expansion and MOTS - Initial data version). *Let* $(\mathcal{S}, h, \mathcal{K})$ *be an initial data set and* $\Sigma \subseteq \mathcal{S}$ *a two-sided embedded submanifold, with one of the two normal vectors*

[4]For simplicity we work with an embbeded submanifold for the last of the chapter. Slight modifications to the formulas exhibited here are needed for general immersions, and we return to them on appendix B.

$\boldsymbol{\nu} \in \mathfrak{X}^{\perp}(\Sigma)$ *fixed, and referred to as the* outward-pointing unit normal *vector field on* Σ. *The* *outward* null expansion θ_+ *[resp. inward* null expansion θ_-*] of* Σ *is defined as*

$$\theta_{\pm} = \operatorname{tr}_{\Sigma} \mathcal{K} \pm H_{\boldsymbol{\nu}}, \tag{1.8}$$

where $H_{\boldsymbol{\nu}}$ *is the mean curvature scalar of* Σ *with respect to the normal* $\boldsymbol{\nu}$ *and the partial trace is in respect to the induced metric. For the sign of* θ_+ *we then define* Σ *to be*

- *outer trapped if* $\theta_+ < 0$,
- *weakly outer trapped if* $\theta_+ \leq 0$,
- *marginally outer trapped if* $\theta_+ = 0$.

MOTS are important in general relativity because they provide a "quasi-local" analgue of the fully global notion of *(black hole) event horizon*, and as such can be adapted to numerical studies, for example Cook (2000). In purely mathematical terms, MOTS are important because they provide spacetime/initial data analogues of minimal surfaces (cf. Dan A. Lee (2019), section 7.5).

1.5.3 MOTS Stability Operator

As observed above, MOTS can be viewed as spacetime analogues of minimal surfaces. More concretely, since $\theta_+ = 0$, if $\mathcal{K}$ is traceless (for example if $\mathcal{K} = 0$, called a *symmetric data set*) then by eq. (1.8) Σ has vanishing mean curvature, so the MOTS is a minimal surface. However, while minimal surfaces can be described via a variational formulation, there is no known analogue for a MOTS.

A powerful tool for studying minimal surfaces is the notion of *stability*, which comes from the sign of the second variation of the volume measure; this concept of stability can be generalized to the setting of MOTS through the linearization of the null expansion θ_+. Such notion was introduced by Andersson, Mars, and Simon (2008). We give a brief account of more technical aspects of the MOTS stability operator in appendix B. For now, the relevant definition is the following:

Definition 1.5.5 (MOTS Stability Operator - Initial Data). *Let* Σ^{n-1} *be a closed MOTS (compact without boundary) within an initial data* $(\mathcal{S}^n, h, \mathcal{K})$. *We define the* MOTS stability operator $L : C^{\infty}(\Sigma) \to C^{\infty}(\Sigma)$ *to be*

$$L(\psi) = -\Delta\psi + 2\langle X, \operatorname{grad}\psi\rangle + (Q + \operatorname{div} X - \|X\|^2)\psi, \quad \forall \psi \in C^{\infty}(\Sigma), \tag{1.9}$$

where

$$Q = \frac{1}{2}\mathrm{Scal}_{\Sigma} - [J(\boldsymbol{\nu}) + \rho] - \frac{1}{2}|\mathcal{K}_{\boldsymbol{\nu}} + \mathcal{K}|^2. \tag{1.10}$$

All the geometric objects her are defined on Σ, $\boldsymbol{\nu}$ is the outward pointing unit normal vector field on Σ, $\mathcal{K}_{\boldsymbol{\nu}}$ is the scalar second fundamental form of Σ with respect to the induced metric from $(\mathcal{S}, h)$ on the direction $\boldsymbol{\nu}$, X is the vector field dual to the one-form $\mathcal{K}(\boldsymbol{\nu}, \cdot)$ on Σ. Finally, ρ and J are defined as in definition 1.5.2.

In the case of time-symmetric initial data ($\mathcal{K} = 0$), the operator L reduces to the self-adjoint, classic stability (or Jacobi) operator of the minimal surface theory, which consists of the second variation of the volume. Although the operator L is not self-adjoint in general, the operator possesses crucial spectral properties as stated below.

Proposition 1.5.6 (Andersson, Mars, and Simon (2008), Galloway (2018)). *Let Σ be a closed MOTS (compact without boundary) within an initial data set $(\mathcal{S}^n, h, \mathcal{K})$. The following statements hold for the MOTS stability operator L.*

1. *There is a real eigenvalue $\lambda_1 = \lambda_1(L)$, called the* principal eigenvalue *of L, such that for any other eigenvalue μ, $Re(\mu) \geq \lambda_1$. The associated eigenfunction $\phi \in C^\infty(\Sigma)$, $L\phi = \lambda_1\phi$, is unique up to a multiplicative constant, and can be chosen to be strictly positive.*

2. *$\lambda_1 \geq 0$ (resp., $\lambda_1 > 0$) if only and if there exist some $\psi \in C^\infty(\Sigma), \psi > 0$, such that $L(\psi) \geq 0$ (resp., $L(\psi) > 0$).* ■

1.6 Singularity Theorems

For later reference we end this chapter by recalling the statements of the classic singularity theorem by Hawking and Penrose, and also present a less frequent singularity theorem with the presence of MOTS. . Both types of singularity theorems will be relevant in chapters 3 and 4.

The following is the celebrated theorem of Hawking and Penrose (1970).

Theorem 1.6.1 (Hawking-Penrose). *Let (M, g) be a spacetime with dimension $n \geq 3$ such that*

(i) *(M, g) is chronological;*

(ii) *every inextendible causal geodesic has a pair of conjugated points;*

(iii) *there is a future (or past) trapped set $A \subseteq M$.*

Then (M, g)has at least one inextendible, incomplete causal geodesic. ■

The following corollary for the Hawking and Penrose theorem is the statement needed in chapter 3.

Corollary 1.6.2 (Beem, Ehrlich, and Easley (1999), thm. 12.47). *Let (M, g) be a chronological spacetime of dimension $n \geq 3$ that satisfies the generic condition (definition 1.3.1) and the* timelike convergence condition *(i.e. $Ric(v, v) \geq 0$ for all $v \in TM$ timelike). Then (M, g) is nonspacelike incomplete if any of the following conditions are satisfied:*

(i) (M, g) *has a closed trapped surface (definition 1.4.1);*

(ii) (M, g) *has a point* p *such that any null geodesic starting at* p *reconverges*[5] *at some point in the past or future of* p*;*

(iii) (M, g) *has a compact spacelike hypersurface.* ■

1.6.1 A Singularity Theorem in the Presence of MOTS

Let $\Sigma^{n-1} \subseteq \mathcal{S}^n$ be connected manifolds. We say that Σ *separates* $\mathcal{S}$ if $\mathcal{S} \setminus \Sigma$ is not connected. In this case, we write $\mathcal{S} \setminus \Sigma = \mathcal{S}_+ \sqcup \mathcal{S}_-$, where $\mathcal{S}_\pm$ are open submanifolds of $\mathcal{S}$. We now give a singularity theorem for a spacetime in the presence of a MOTS that will be of crucial importance in the last part of this thesis.

Theorem 1.6.3. *Let* (M^{n+1}, g) *be a globally hyperbolic spacetime satisfying the lightlike convergence condition, let* $\mathcal{S}^n \subseteq M$ *be a spacelike Cauchy hypersurface, and* $\Sigma^{n-1} \subseteq \mathcal{S}$ *a connected, compact MOTS without boundary that separates* $\mathcal{S}$*, with* $\mathcal{S} \setminus \Sigma = S_+ \sqcup S_-$*, where* $S_\pm$ *are open disjoint sets in* S *and* $\overline{S_+}$ *is not compact. Assume that the principal eigenvalue for the MOTS stability operator of* Σ *is non zero. Under these hypotheses,* (M, g) *is null geodesically incomplete.*

Remark 1.6.4. Regarding the assumptions of this theorem, the separation of $\mathcal{S}$ by Σ is not essential,because one can show (the proof is analogous to that of prop. 14.48 of O'Neill (1983)) that there exists a smooth covering manifold of M for which there exists a spacelike Cauchy hypersurface $\tilde{\mathcal{S}}$ covering $\mathcal{S}$ and containing a isometric copy of Σ separating $\tilde{\mathcal{S}}$. Now, insofar as we are interested in geodesic incompleteness, one may as well work in covering manifolds.

Proof (sketch). Our hypotheses are a restriction of a more general study for singularities in the presence of MOTS done by Silva (2012), so we just give here an outline of the argument, referring to that reference for further details.

We first argue that there is a future-directed null Σ-ray $\eta : [0, a) \to M$ with $\eta'(0)$ being parallel to ℓ_+ at $\eta(0)$ (called an *outward-pointing* ray, with a *inward-pointing* ray being defined analogously for $\eta'(0)$ parallel to ℓ_-). Since the spacetime (M, g) is globally hyperbolic, $E^+(\Sigma) = \partial I^+(\Sigma)$. Given $p \in E^+(\Sigma) \setminus \Sigma$, consider $\eta : [0, 1] \to M$ a segment of a future-directed null generator of $\partial I^+(\Sigma)$ that starts at a point at Σ and ends in p, that is in particular normal do Σ, so $\eta'(0)$ is either parallel to ℓ_+ or ℓ_-. Denote by $\mathcal{H}_+$ the set of points in $E^+(\Sigma) \setminus \Sigma$ for which the case $\eta(0)$ parallel to ℓ_+ occurs, and $\mathcal{H}_-$ for the ℓ_- case. We then have $E^+(\Sigma) \setminus \Sigma = \mathcal{H}_+ \cup \mathcal{H}_-$,

One sees that $\mathcal{H}_+$ and $\mathcal{H}_-$ are disjoint sets: assuming they are not, taking a point p in their intersection we have η_+, η_- two future-directed outward- and inward-pointing, respectively, null Σ-rays starting at Σ and ending in p. We concatenate this two curves to a curve γ starting

[5]The more technically precise meaning of "reconvergence" in this context is the appearance of a conjugate point to p along any null geodesic.

at Σ and going to p following η_-, then going back to Σ following η_+ backwards. Then using a standard continuous function $\rho_X : \mathcal{S} \to M$ that fixes $\mathcal{S}$, where X is a timelike vector field over M such that all maximally extended integral curves intersect $\mathcal{S}$ exactly once (since $\mathcal{S}$ is a Cauchy hypersurface such X exists), and ρ_X is defined by "sliding points of M along the integral curves" to their intersection with $\mathcal{S}$ (see e.g. O'Neill (1983), prop. 14.31, or Espinoza (2020), thm. 3.8.12). One sees that the curve $\rho_X \circ \gamma$ intersects Σ in a point $q = \rho_X(\gamma(t_0))$ distinct from its endpoints, so $q \in \Sigma \cap I^-(p)$, implying $I^+(\Sigma) \cap \partial I^+(\Sigma) \neq \emptyset$, a contradiction. One also sees that $\rho_X(\mathcal{H}_+ \cup \Sigma) = \overline{\mathcal{S}_+}$, so by our hypothesis $\mathcal{H}_+ \cup \Sigma$ is not compact (see Silva (2012) prop. 2.4 for the details of these arguments). The construction of the desired Σ-ray follows an argument with the limit curve lemma laid out in Silva (2012), prop. 2.1, replacing $E^+(\Sigma)$ there with $\mathcal{H}_+ \cup \Sigma$.

It is worth pointing out that Σ need not be a MOTS in order to establish the existence of such a null Σ-ray, as this is relevant for the next step. Since we assume the stability operator of Σ has a non-zero principal eigenvalue $\lambda_0 \neq 0$ by proposition 1.5.6 its eigenfunction ϕ_0 can be chosen to be strictly positive. For the normal variation with vector field $V = \phi_0 \nu$, where $\boldsymbol{\nu}$ is the outward pointing unit normal vector field of Σ in $\mathcal{S}$, one can show that

$$\left.\frac{\partial \theta_+}{\partial t}\right|_{t=0} = L(\phi_0) = \lambda_0 \phi_0,$$

therefore under this variation we can deform Σ to be outer-trapped (i.e. $\theta_+ < 0$) by moving Σ along the variation either outwards or inwards according to the sign of λ_0. Still denoting this deformation as Σ, we then have an outward-pointing null Σ-ray $\eta : [0, a) \to M$ normal to this outer-trapped Σ as argued above, and this geodesic if future-incomplete because if it were complete, then by prop. 10.43 in O'Neill (1983), it would have a focal point of Σ, which is a contradiction since η is a Σ-ray. ∎

Example 1.6.5. In the four dimensional Minkowski space $\mathbb{R}^4_1$ with usual coordinates (t, x, y, z), consider $\mathcal{S} = \{t = 0\}$ a spacelike Cauchy hypersurface and the two dimensional plane $\Sigma = \{t = x = 0\}$. By identifying $y \sim y + 1$ and $z \sim z + 1$ in $\mathbb{R}^4_1$, the resulting spacetime (M, g) is geodesically complete, vacuum globally hyperbolic spacetime. The plane Σ under this identification wraps itself around to a totally geodesic 2-torus $\widetilde{\Sigma}$, therefore is a MOTS that separates $\widetilde{\mathcal{S}}$, but this does not contradict theorem 1.6.3. The MOTS operator of $\widetilde{\Sigma}$ is just the Laplacian operator over the flat 2-torus, then it has zero as a eigenvalue (Colbois (2010)). ◀

2 Genericity and Prevalent Properties for Spacetimes - Basic Notions

This chapter is dedicated to introduce terminology and standard notation for the main results in chapter 3. In particular, some key notions presented here seek to clarify what it means for a subset to be "small", or "negligible" in a topological space, especially in the absence of a natural measure-theoretic analogue of smallness ("measure zero") available on (finite-dimensional) manifolds. We will deal with a standard notion of topological genericity usually introduced in the context of Baire spaces, and also a less standard notion of *prevalence* to be introduced here, that is simple but will be frequent in chapter 3. We also briefly review the topological setting in the space of Lorentzian metrics and some key stability results introduced by Lerner (1973) that will be instrumental in the first part of this thesis.

2.1 The Notion of Topological Genericity

The goal of this work is to establish, under suitable conditions, that the appearance of singularities in a spacetime is generic, in the following precise way:

Definition 2.1.1. *Let X be topological space. A subset $A \subset X$ is said to be* residual *if it contains a countable union of open dense sets. A property of elements of X is said to be* generic *if the set of all those elements that possess it is residual.*

The complementary notion of a residual set is called a *meager set*, that is, a set contained in a countable union of nowhere dense sets (sets such that the interior of their closure is empty). If every residual set is dense, we say that the topological space is a *Baire space*. The *Baire category theorem* states that completely metrizable spaces or locally compact Hausdorff spaces are Baire spaces (John M. Lee (2011), Thm. 4.68).

While genericity implies density on Baire spaces, the converse is certainly false: the set of rational numbers is dense in the real line, but being countable it is also a meager set (being a countable union of singletons).

The idea of searching for generic properties is that its complementary meager set is *topologically negligible*: a nowhere dense set is so "topologically small" that the interior of its closure is empty, and a meager set is contained in a union of such "small" sets. While such notion of smallness doesn't seem to be as good as a measure-theoretical idea of a null set, it is widely adopted because there are no infinite-dimensional analogues of the Lebesgue measure (see Hunt, Sauer, and Yorke (1992) for a discussion on this topic and for an alternative notion of smallness in infinite dimensional linear spaces that is closer to measure theory, but that is delicate to adapt to infinite dimensional manifolds and won't be pursued here, and also Oxtoby (1980) for a survey of the similarities and shortcomings of Baire category in relation to measure theory). In any case, the notion residual sets and genericity and its dual notion of topological smallness have proven to be useful in many geometrical situations, giving rise to a number of important results as we shall see later.

We also introduce the following alternative notion of topological smallness that will be relevant for our first "topological smallness" theorem.

Definition 2.1.2. *Let X be a topological space and $C \subseteq X$ a closed set. We shall say that a set $A \subseteq C$ is* prevalent[1] *in C if $C \setminus A$ is a meager set in X.*

It is clear from this definition, for example, that the set of irrational numbers is prevalent in the real line, whereas the set of rationals is not.

The following straightforward topological lemma is relevant in this context, as it in particular gives a condition for prevalence.

Lemma 2.1.3. *Let C be a closed subset of a topological space X, and $U \subseteq C$ an open subset of X such that $int(C) \subseteq U$. Then $C \setminus U$ is nowhere dense in X (and U is prevalent in C).*

Proof. Since $C \setminus U$ is closed, $int\left(\overline{C \setminus U}\right) = int(C \setminus U)$. Now, $int(C \setminus U) \subseteq int(C) \setminus U$, and from our hypothesis, $int(C) \setminus U = \emptyset$, implying $int\left(\overline{C \setminus U}\right) = \emptyset$, therefore $C \setminus U$ is nowhere dense in X. ■

We see that every residual set is prevalent.

Proposition 2.1.4. *Let $C \subseteq X$ be a closed subset that is a Baire space in its subspace topology (this happens e.g. X is completely metrizable). Then every residual set of C (residual in the subspace topology) is prevalent in C (in the sense of definition 2.1.2).*

Proof. Let $A \subseteq C$ be residual in C. Then A contains $\bigcap_n O_n$ a countable collection of open dense subsets of C. We have $C \setminus A \subseteq \bigcup_n C \setminus O_n$, with each $C \setminus O_n$ being closed in C, therefore closed in X. Now $int_X(C \setminus O_n)$ is an open subset of C contained in $C \setminus O_n$ that doesn't intersect O_n, however O_n is dense in C, therefore $int_X(C \setminus O_n)$ must an empty set, so $C \setminus A$ is a meager set of X. ■

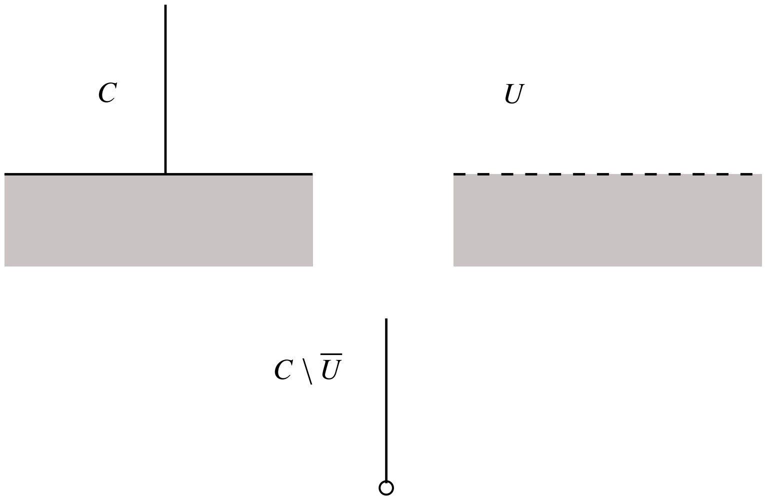
As the next example shows, the converse to proposition 2.1.4 is false.

Example 2.1.5. Let $X = \mathbb{R}^2$ with the usual topology, $C = \{y \leq 0\} \cup \{x = 0, y \geq 0\}$. For $U = \{y < 0\}$. Since (fig. 2.1) $C \setminus \overline{U} = \{x = 0, y > 0\}$, this closed line has empty interior, so $\overline{U}$ is prevalent in C.

Consider now the interior operation with respect to the subspace topology of C, that will be denoted by int_C. The set $C \setminus \overline{U} = \{x = 0, y > 0\}$ now has interior with respect to the subspace topology (this half line is open in C) so it cannot be meager in C, since C is a Baire space and any meager set therein must have empty interior. Therefore $\overline{U}$ cannot be residual in C (thus neither can U) with the subspace topology. ◀

[1]This nomenclature is inspired by the work of Hunt, Sauer, and Yorke (1992), but it is unrelated to the measure theoretic notion of prevalence introduced therein.

Figure 2.1: Sets C and U in example 2.1.5



2.2 Topological and Geometrical Settings

For the remaining of this chapter and we shall fix M a connected and non-compact smooth (i.e. C^∞) real manifold without boundary of dimension $n = m + k \geq 3$. Let $Sym^2(M)$ be the vector bundle of $(0, 2)$-type symmetric tensors on M. We denote by $L \subset Sym^2(M)$ the smooth subbundle whose sections are *Lorentzian metric tensors* (or *Lorentzian metrics* for short) on M, which is open in $Sym^2(M)$ when the latter is endowed with its standard manifold topology. In other words, a section of L is a map g which associates with each $p \in M$ a symmetric nondegenerate bilinear form $g_p : T_pM \times T_pM \to \mathbb{R}$ of index 1. Let $\Gamma^r(L)$ be the set of r-differentiable sections with $0 \leq r \leq \infty$, that is, the set of Lorentzian metrics g over M whose components g_{ij} in local coordinates have continuous partial derivatives up to order r for $r \geq 1$, or that are simply continuous when $r = 0$[2]. By a *metric* we always mean here a Lorentzian metric, unless stated otherwise, but we specify its degree of differentiability as needed.

Let us briefly describe the relevant topologies we shall adopt on $\Gamma^r(L)$ here (in appendix A we give brief description of the more technical aspects of such topologies). In general, given any other smooth manifold N, we denote by $C^r(M, N)$ the set of all r-differentiable functions $f : M \to N$. Over this set we shall always adopt the *strong* C^s *Whitney topologies*, $0 \leq s \leq r$. A basis for the (strong) C^s Whitney topology over $C^r(M, N)$ (s finite) can be most readily defined via jet bundles as follows. Given the bundle $J^s(M, N)$ of s-jets of r-differentiable maps, for each open set $\mathcal{O} \subseteq J^s(M, N)$ we set

$$B_s(\mathcal{O}) = \{f \in C^r(M, N) : j^s f(M) \subseteq \mathcal{O}\},$$

so the collection of all sets of this form is the desired basis. A basis for the so-called Whitney C^∞ topology on $C^\infty(M, N)$ is defined by taking as a basis the collection of all C^s-open sets, for

[2] We do not refer to these as "(of class) C^r" to avoid notational confusion with the Whitney topologies discussed ahead.

all $0 \leq s < \infty$.

In particular, consider a smooth fiber bundle E over M. Then the set $\Gamma^r(E)$ of its r-differentiable sections will also said to be given the C^s topology since we take $\Gamma^r(E) \subseteq C^r(M, E)$ and endow $\Gamma^r(E)$ with the induced C^s topology.

$\Gamma^r(L)$ will always be assumed to be endowed with the (induced) C^s topology as described. In most of our main results we shall take $s = 2 \leq r \leq \infty$. In case we fix some $s \in \mathbb{Z}_+$ in connection with the Whitney topology C^s in a statement, that statement is meant to hold separately on $\Gamma^r(L)$ for each $s \leq r \leq \infty$.

An alternative, perhaps more concrete way of defining Whitney topologies is by using local charts. Following Mukherjee (2015), for a given function $f \in C^s(M, N)$ with s finite, consider the following families:

(i) $\Phi = \{(U_i, \varphi_i)\}_{i \in I}$ a locally finite family of coordinate charts of M;

(ii) $\Psi = \{(V_i, \psi_i)\}_{i \in I}$ a family of coordinate charts of N;

(iii) $\mathcal{K} = \{K_i\}_{i \in I}$ a family of compact sets in M such that $K_i \subseteq U_i$ and $f(K_i) \subseteq V_i$ for all $i \in I$.

(iv) $\mathcal{E} = \{\varepsilon_i\}_{i \in I}$ a family of positive real numbers.

Denote by $\mathcal{N}^s(f, \Phi, \Psi, \mathcal{K}, \mathcal{E})$ the set of functions $g \in C^s(M, N)$ such that $g(K_i) \subseteq V_i$ for all $i \in I$ and for all multi-index α with $|\alpha| \leq s$,

$$\left\| \frac{\partial^\alpha(\psi_i \circ f \circ \varphi_i^{-1})}{\partial x^\alpha}(a) - \frac{\partial^\alpha(\psi_i \circ g \circ \varphi_i^{-1})}{\partial x^\alpha}(a) \right\| < \varepsilon_i,$$

for all $a \in \varphi_i(K_i)$. Then the collection of all sets $\mathcal{N}^s(f, \Phi, \Psi, \mathcal{K}, \mathcal{E}) \subseteq C^s(M, N)$ form a basis for the C^s topology (Mukherjee (2015), prop. 8.2.9).

In the seminal work of Lerner (1973), a number of key results using the strong Whitney topologies on $\Gamma^r(L)$ were obtained - some of which are briefly reviewed in the next section - and these have ever since been widely accepted as the natural topologies to adopt in the particular geometric setting of interest here. In any case, we work exclusively with the strong Whitney topologies on $\Gamma^r(L)$ for the main results in and from now on by C^s *topology* we always mean the strong C^s Whitney topology.

Strong Whitney topologies are not metrizable, not even first countable if the domain manifold is not compact (Mukherjee (2015), pgs. 239-240) so we will rely on net convergence arguments when needed.

2.2.1 How Generic is the Generic Condition for Spacetimes

In section 1.3 we looked at an algebraic condition for vectors and inextendible geodesics on spacetimes called the "generic condition" that is relevant for singularity theorems, this condition been seemingly unrelated to the notion of topological genericity.

The relation between generic condition and topological genericity has been expected to be positive since, heuristically speaking, physically reasonable space-times usually satisfy the generic condition (Hawking and Ellis (1973), p. 101). The work of Beem and Harris (1993) showed that the generic condition on each tangent space where the Riemann curvature is not zero is in fact generic in the sense of definition 2.1.1 (more specifically, the generic vectors at $p \in M$ form an open dense set in T_pM). More recently the masters thesis work of Larsson (2014) proved using differential topology methods, specifically the Thom transversality theorem, that the generic condition is indeed generic (Larsson (2014), Thm. 2.6.3): on the function space of Lorentzian metrics for a given manifold M of dimension $n \geq 4$ with strong Whitney topology C^r, $4 \leq r \leq \infty$, the set of metrics satisfying the generic condition forms a generic set (and is in particular dense, since spaces of suitable tensor fields endowed with Whitney topologies are Baire spaces).

2.3 Further Generic and Stability Properties

To keep the presentation reasonably self-contained as well as to establish further notation, we reproduce here, for later reference, some results established in Lerner (1973) that will be relevant for us later on.

Fix some $0 \leq r$ which is either an integer or else $r = \infty$. By a C^s *stable property* in $C^r(M, N)$, [resp. $\Gamma^r(E)$] we mean a property that is valid for all functions in a C^s open subset of $C^r(M, N)$ [resp. $\Gamma^r(E)$]. For each $t > s$ the C^t topology is finer than the C^s topology, so a stable property in C^s is also stable in C^t.

Denote by $\mathscr{S}\mathscr{T}^r \subset \Gamma^r(L)$ the set of r-differentiable metrics g such that (M, g) is time-orientable. The first relevant result is the C^0-stability of time-orientability.

Proposition 2.3.1 (Lerner (1973), prop. 4.7). *Let $X \in \mathfrak{X}(M)$ be an everywhere nonzero vector field. Then the set*

$$\mathscr{S}\mathscr{T}^r(X) := \{g \in \Gamma^r(L) \ : \ X \text{ is } g\text{-timelike}\}$$

is C^0-open in $\Gamma^r(L)$. In particular, $\mathscr{S}\mathscr{T}^r$ is C^0- open (and therefore C^s-open for each $0 \leq s \leq r$) in $\Gamma^r(L)$. ■

A convenient aspect of working with $\mathscr{S}\mathscr{T}^r(X)$ is that we can *simultaneously* choose the time-orientation in all of its elements so that X is future-directed, and we shall implicitly assume this choice from now on. Fix a codimension k submanifold $\Sigma \subseteq M$, and denote by $\mathscr{S}^r_\Sigma$ the set of metrics in $\Gamma^r(L)$ for which Σ is spacelike.

Proposition 2.3.2 (Lerner (1973), prop. 4.2). *$\mathscr{S}^r_\Sigma$ is C^0-open in $\Gamma^r(L)$.* ■

Consider also the set $\mathcal{SC}^r \subset \mathscr{S}\mathscr{T}^r$ of (time-orientable) metrics that are *stably causal*, meaning that $g \in \mathcal{SC}^r$ if there is a C^0-neighborhood $\mathcal{U} \ni g$ in $\Gamma^r(L)$ such that every $g' \in \mathcal{U}$

is a time-orientable causal metric. The set $\mathcal{SC}^r$ is nonempty because M is noncompact Lerner (1973, p. 27, item (a)). It is C^0-open by definition, and if we denote as $\mathcal{CH}^r \subset \mathscr{S}\mathscr{T}^r$ the set of chronological, time-orientable metrics, then this set is C^0-closed in $\mathscr{S}\mathscr{T}^r$, with $\overline{\mathcal{SC}} = \mathcal{CH}$ (closure in the C^0 topology, Lerner (1973, p. 27, item (b)).

Thus, *stably causal metrics is* C^0-generic *in the set of chronological metrics, in the sense that they form an open dense subset of the latter, or equivalently, that chronological but not stably causal metrics form a nowhere dense subset of the set of all chronological metrics.*

We emphasize, however, that unlike stability, this does *not* imply C^s-genericity for $s > 0$. This is so, of course, because while $\mathcal{SC}^r$ would still be C^s-open in $\mathcal{CH}^r$, it might no longer be dense in this finer topology. *A consequence for us here is that in order to obtain our higher-order genericity results we shall need to work with stably causal metrics even if the needed singularity theorems only require chronology.*

Assume now $r \geq 2$ and denote by $\mathcal{SE}^r$ the set of r-differentiable metrics $g \in \Gamma^r(L)$ for which the respective Ricci tensor, denoted by $Ric(g)$, satisfies

$$Ric(g)(v, v) > 0, \quad v \in TM \text{ } g\text{-causal.}$$

This set is C^2-open in $\Gamma^r(L)$ (Lerner (1973, Prop. 4.3). Importantly for us here, for each $g \in \mathcal{SE}^r$ all g-causal vectors are *generic* in (M, g) in the sense of definition 1.3.1.

Similarly, consider $\mathcal{E}^r$ the set of r-differentiable metrics $g \in \Gamma^r(L)$ satisfying

$$Ric(g)(v, v) \geq 0, \quad v \in TM \text{ } g\text{-causal.}$$

This set is C^2-closed, with $\overline{\mathcal{SE}^r} \subseteq \mathcal{E}^r$, where now the overbar indicates C^2-closure (Lerner (1973, p. 28, item 4.4)). The main result (Lerner (1973, Prop. 4.5)) we will need in our later arguments is the following relation between $\mathcal{E}^r$ and $\mathcal{SE}^r$:

Theorem 2.3.3. *In the C^2 strong Whitney topology, $int(\mathcal{E}^r) = \mathcal{SE}^r$, for all $2 \leq r \leq \infty$. In particular, $\mathcal{E}^r \setminus \overline{\mathcal{SE}^r}$ is C^2-nowhere dense in $\Gamma^r(L)$.* ■

Theorem 2.3.3 is the statement as proved by Lerner, but in the sense of definition 2.1.2, we see that the set $\mathcal{SE}^r$ is actually prevalent in $\mathcal{E}^r$.

Remark 2.3.4. The set $\mathcal{E}^r$ consists precisely of those metrics g satisfying the so-called *timelike convergence condition*: $Ric(g)(v, v) \geq 0, \forall v \in TM$ timelike (the null vectors case being obtained via limits). This is often referred to as the *strong energy condition* in the physics literature, because it arises via the Einstein field equation in the context of general relativity, by coupling the spacetime metric with physically relevant classical matter fields, most of which satisfy it. Hence, it is a very common assumption in singularity theorems. Theorem 2.3.3 has thus a simple but very suggestive meaning: only a "negligibly small" subset of the metrics satisfying the timelike convergence condition do not admit of an arbitrarily close approximation by a metric in $\mathcal{SE}^r$, a condition which in turn will often lead to causal geodesic incompleteness.

Part II

Prevalence of Singularities - Whitney Topologies

3 Genericity with Weakly Trapped Surfaces

Having understood what is meant to mean “topologically small” in the context of this work, and with the established stability and genericity results from Lerner (1973) all laid out, in this chapter we prove a prevalence result for future-trapped submanifolds inside the set of weakly trapped submanifolds, and use this condition, together with theorem 2.3.3 to establish a genericity theorem for metrics that have singularities,under the existence of future trapped surfaces (co-dimension 2) . With some extra control over the Riemann curvature as well, we can obtain a similar theorem for situations where higher codimensional submanifolds appear.

3.1 Main Results I: Codimension Two

We fix in this section a smooth embedded codimension $k \geq 2$ submanifold $\Sigma^m \subseteq M^{m+k}$. In some of our main results we shall need that Σ be compact and without boundary, and then we will simply say that Σ is *closed*, not to be confused standard topological closure. In order to simplify the notation, unless stated otherwise we work in $\Gamma^2(L)$ endowed with the C^2 Whitney topology, with the understanding that everything remains valid for r-differentiable metrics with $r \geq 2$, and thus omit any r superscripts in what follows.

Let $g \in \Gamma^2(L)$, and let $\nabla = \nabla^g$ denote its Levi-Civita connection. Recall if Σ is spacelike in (M, g), then we can define the second fundamental form tensor (or *shape tensor* for short) of Σ by

$$II^{\Sigma}(g)(V, W) := (\nabla_V W)^{\perp}, \quad \forall V, W \in \mathfrak{X}(\Sigma).$$

where $\perp$ denotes the normal part with respect to g. Given any local g-orthonormal frame $\{E_1, \ldots, E_m\} \subset \mathfrak{X}(\Sigma)$ on Σ, the associated *mean curvature vector of* Σ is

$$H^{\Sigma}(g) := tr_{\Sigma}\, II^{\Sigma}(g) = \sum_{i=1}^{m} II^{\Sigma}(g)(E_i, E_i).$$

When the metric is unambiguously understood, we shall denote the associated mean curvature vector simply by H^{Σ}. Straightforward coordinate computations and net convergence arguments similar to the ones given in the proof of Lerner (1973, Prop. 4.7(b)) show that the mapping

$$g \in \Gamma^r(L) \mapsto H^{\Sigma}(g) \in \Gamma^0(TM|_{\Sigma}) \tag{3.1}$$

is continuous in the C^r topology on $\Gamma^r(L)$ for all $1 \leq r \leq \infty$, where $TM|_{\Sigma}$ is the restriction of the tangent bundle TM to Σ.

Fix a vector field $X \in \mathfrak{X}(M)$. Our ambient topological space for the next definitions and results is the set $\mathscr{S}\mathscr{T}(X) \cap \mathscr{S}_\Sigma \cap \mathcal{SC}$ of C^2 stably-causal, time-oriented metrics in which X is future-directed timelike, and for which Σ is spacelike (which is C^2-open in $\Gamma(L)$ - conf. Props. 2.3.1 and 2.3.2). In that set, consider the subset $\mathcal{A}$ of metrics for which Σ is a future-trapped submanifold. Recall that Σ is *future-trapped* if and only if H^Σ_p is past-directed timelike for each $p \in \Sigma$ (O'Neill calls such a set *future-converging* O'Neill (1983, p. 435)).

More precisely, we define $\mathcal{A}$ as

$$\mathcal{A} = \{g \in \mathscr{S}\mathscr{T}(X) \cap \mathscr{S}_\Sigma \cap \mathcal{SC} : g(H^\Sigma, H^\Sigma) < 0, g(H^\Sigma, X) > 0\}.$$

Since H^Σ only depends on the metric coefficients and their first derivatives in suitable coordinates (conf. e.g. O'Neill (1983, Ex. 1, p. 123)) the definition of $\mathcal{A}$ readily implies (again by arguments entirely analogous to those in the proof of Lerner (1973, Prop. 4.7(b))) that this set is C^1-open in $\Gamma^2(L)$, and hence C^2-open therein as well. Consider also the set

$$\mathcal{FA} = \{g \in \mathscr{S}\mathscr{T}(X) \cap \mathscr{S}_\Sigma \cap \mathcal{SC} : g(H^\Sigma, H^\Sigma) \leq 0, g(H^\Sigma, X) \geq 0\}.$$

We say that Σ is *weakly future-trapped* for the metric g if $g \in \mathcal{FA}$. In the context of a fixed Σ as we have here, we informally refer to metrics in $\mathcal{A}$ as "future-trapped metrics," and analogously, we say the metrics in $\mathcal{FA}$ are "weakly future-trapped".

Observe that for $g \in \mathcal{FA}$ and a point $p \in \Sigma$, either $H^\Sigma_p(g)$ is past-directed and causal, or else it is zero. In particular, $\mathcal{FA}$ includes metrics g for which Σ is an *extremal submanifold*, i.e., $H^\Sigma(g) \equiv 0$ identically, which occurs for example if Σ is totally geodesic with respect to g.

Remark 3.1.1. For another key class of examples of elements of $\mathcal{FA}$, assume for the moment that Σ is closed and with codimension $k = 2$, and that the normal bundle of Σ is trivial. (This can be ensured independently of the choice of the ambient metric provided suitable orientability assumptions on Σ are made.) Then, given $g \in \mathscr{S}\mathscr{T}(X) \cap \mathscr{S}_\Sigma \cap \mathcal{SC}$ we may choose two normal future-directed null vector fields $\ell_\pm \in \mathfrak{X}^\perp(\Sigma)$ globally defined on Σ and spanning its normal bundle. The *null expansion scalars* $\theta_\pm \in C^1(\Sigma)$ associated with this choice are defined by

$$\theta_\pm := -g(H^\Sigma(g), K_\pm).$$

Observe that in this case $g \in \mathcal{A}$, i.e., it is future-trapped, if and only if $\theta_\pm < 0$. If by convention we say that ℓ_+ is *outward-pointing*, then Σ is a *marginally outer trapped surface* (MOTS) if $\theta_+ \equiv 0$ on Σ. Then, for each $p \in \Sigma$, since since $H^\Sigma_p, K_+(p) \in (T_p\Sigma)^\perp$ and the latter vector space is a two-dimensional Lorentz space, they can be orthogonal if and only if either H^Σ_p is zero or null (parallel to K_+), and hence $g \in \mathcal{FA}$. In other words, C^2 metrics for which Σ is a MOTS are all in $\mathcal{FA}$, i.e., they are weakly future-trapped.

Returning now to our main discussion, we evidently have $\mathcal{A} \subseteq \mathcal{FA}$. We will be able to prove much more regarding $\mathcal{A}$ and $\mathcal{FA}$, the main point being that $\mathcal{A}$ is prevalent in $\mathcal{FA}$. Informally, *weakly future-trapped spacetime metrics can "almost always" be arbitrarily C^2-approximated by future-trapped ones.*

A key tool for these proofs will be conformal perturbations of the metric, so let's briefly how the mean curvature vector of Σ transforms under a conformal change of metric. Let $g \in \Gamma^2(L)$ and consider the usual conformal transformation $\widehat{g} = e^{2f} g$, where $f : M \to \mathbb{R}$ is any smooth function. The Levi-Civita connection transforms as (conf. John M. Lee (2018), p. 217)

$$\widehat{\nabla}_X Y = \nabla_X Y + (Xf)Y + (Yf)X - g(X,Y) grad_g f,$$

for X, Y smooth vector fields over M. Then, the shape tensor of Σ associated with the metric $\widehat{g}$ is

$$\widehat{II}(V,W) = (\widehat{\nabla}_V W)^\perp = II(V,W) - g(V,W)(grad_g f)^\perp,$$

for V, W smooth vector fields tangent to Σ. It now readily follows that

$$\widehat{H}^\Sigma = e^{-2f} H^\Sigma - m e^{-2f} (grad_g f)^\perp, \tag{3.2}$$

and the scalar product of H^Σ transforms as

$$\begin{aligned} \widehat{g}(\widehat{H}^\Sigma, \widehat{H}^\Sigma) = e^{-2f} g(H^\Sigma, H^\Sigma) - 2e^{-2f} m g(H^\Sigma, grad_g f) \\ + e^{-2f} m^2 g((grad_g f)^\perp, (grad_g f)^\perp). \end{aligned} \tag{3.3}$$

We are now ready to prove the announced result.

Theorem 3.1.2. *$\mathcal{FA}$ is C^2-closed in $\mathcal{ST}(X) \cap \mathcal{S}_\Sigma \cap SC$, with $\mathcal{FA} = \overline{\mathcal{A}}$, and also we have $int(\mathcal{FA}) = \mathcal{A}$. In particular, $\mathcal{A}$ is prevalent in $\mathcal{FA}$.*

Proof. Firt lets show that $\mathcal{FA}$ is C^2-closed, implying $\overline{\mathcal{A}} \subseteq \mathcal{FA}$. Consider $\{g_\lambda\}_{\lambda\in\Lambda}$ a net of metrics in $\mathcal{FA}$ converging to a metric $g \in \mathcal{ST}(X) \cap \mathcal{S}_\Sigma \cap SC$ in the C^2 topology. Denoting by H^Σ_λ the mean curvature vector of Σ associated with the metric g_λ, by the continuity in (3.1) we have $H^\Sigma_\lambda \to H^\Sigma$ in the C^0 topology on $\Gamma(TM|_\Sigma)$, and since C^2 convergence implies C^0 convergence, we have a pointwise convergence $g_\lambda(H^\Sigma_\lambda, H^\Sigma_\lambda) \to g(H^\Sigma, H^\Sigma)$, which implies $g(H^\Sigma, H^\Sigma) \le 0$. Similarly, we have a pointwise convergence $g_\lambda(X, H^\Sigma_\lambda) \to g(X, H^\Sigma)$, thus showing that $g(X, H^\Sigma) \ge 0$, i.e., $g \in \mathcal{FA}$.

For $\mathcal{FA} \subseteq \overline{\mathcal{A}}$, let $g \in \mathcal{FA}$, and if we can find a sequence $g_n \in \mathcal{A}$ of metrics converging to g in the C^2 topology to g such that

$$g_n(H_n, H_n) < 0 \quad \text{and} \quad g_n(H_n, X) > 0, \tag{3.4}$$

then we will have $g \in \overline{\mathcal{A}}$.

We find such a sequence as follows: since g is stably causal, there exists a C^2 g-time cosmic function $t : M \to \mathbb{R}$ (Hawking and Ellis (1973), prop. 6.4.9), with $grad_g\, t$ being future-directed with respect to the fixed vector field X. With Σ being compact, we can find pre-compact open neighborhoods V and U of Σ such that $\overline{V} \subseteq U$. Consider a smooth bump function ϕ that is constant to 1 in $\overline{V}$ and with support in U. By defining $f = \phi t$, then f is zero outside of a compact set, and has the same g-gradient as t for points over Σ, so we define $f_n = f/n$ and $g_n = e^{2f_n} g$. Since $f_n \to 0$ in the C^2 topology by proposition A.3.8, we see that e^{2f_n} converges to 1 in the C^2 topology, and then also $g_n \to g$ in the C^2 topology.

To see g_n satisfies (3.4), we make some elementary computations. By eq. 3.3,

$$g_n(H_n, H_n) = e^{-2f_n} g(H^\Sigma(g), H^\Sigma(g)) - \frac{2me^{-2f_n}}{n} g(H^\Sigma(g), grad_g\, t) + \frac{m^2 e^{-2f_n}}{n^2} g((grad_g\, t)^\perp, (grad_g\, t)^\perp).$$

With $grad_g\, t$ future-directed and $H^\Sigma(g)$ being either zero or past directed (with respect to g and X), $g(H^\Sigma(g), grad_g\, t) \geq 0$, and also $g(H^\Sigma(g), H^\Sigma(g)) \leq 0$, but $g((grad_g\, t)^\perp, (grad_g\, t)^\perp) < 0$ strictly, and $g_n(H_n, H_n) < 0$. Similarly, by eq. (3.2),

$$g_n(H_n, X) = e^{-2f_n} g(H^\Sigma(g), X) - \frac{me^{-2f_n}}{n} g((grad_g\, t)^\perp, X),$$

and since $g(H^\Sigma(g), X) \geq 0$ and $g((grad_g\, t)^\perp, X) < 0$ because $grad_g\, t$ is also future directed, we obtained the desired sequence.

$\mathcal{A} \subseteq \mathcal{FA}$ is C^2-open in $\Gamma^2(L)$, so certainly $\mathcal{A} \subseteq int(\mathcal{FA})$. Consider now $g \in \mathcal{FA} \setminus \mathcal{A}$. We will exhibit a sequence $g_n \notin \mathcal{FA}$ with $g_n \to g$ in the C^2 topology, so that every C^2-open neighborhood of g will have a metric not in $\mathcal{FA}$, meaning $g \notin int(\mathcal{FA})$.

Since g is not a metric in $\mathcal{A}$, there exists a point $p \in M$ for which either $g(H_p^\Sigma, H_p^\Sigma) \geq 0$, or else $g(H_p^\Sigma, X_p) \leq 0$. But $g \in \mathcal{FA}$, so either $H_p^\Sigma(g)$ is lightlike, or otherwise $H_p^\Sigma(g) = 0$.

For the lightlike case, choose a past-directed lightlike vector $v \in T_p\Sigma^\perp$ not collinear to H_p^Σ (which exists because $k = codim\, \Sigma \geq 2$). We easily find on some relatively compact neighborhood U of p a smooth function $\phi \in C^\infty(U)$ such that $grad_g \phi|_p = v$. We then extend ϕ globally to smooth real-valued function M with support in U by a standard bump function argument, and define the sequence $\phi_n = \phi/n$. This then converges to the zero function in the C^2 topology, by proposition A.3.8, implying that $e^{2\phi_n}$ converges to the constant function 1 in the C^2 topology, and thus that $g_n = e^{2\phi_n} g$ converges to g in the C^2 topology on $\Gamma^2(L)$. Denoting by H^n the mean curvature vector of Σ associated with the metric g_n, we can now employ (3.3). Since $H_p^\Sigma(g)$ is (past-directed) g-lightlike, and $grad_g \phi|_p = v$ is also past-directed lightlike and g-normal to Σ, we obtain

$$g_n(H_p^n, H_p^n) = -\frac{2m}{n} e^{2\phi_n(p)} g(H_p^\Sigma(g), v) > 0.$$

The case $H_p^\Sigma = 0$ is very similar: we just choose $v \neq 0$ g-spacelike in $T_p\Sigma^\perp$, and the rest of the argument proceeds analogously, but now applying (3.3) we get

$$g_n(H_p^n, H_p^n) = \frac{m^2}{n} e^{-2\phi_n(p)} g(v, v) > 0.$$

In any case, we obtain $g_n \notin \mathcal{FA}$, as desired. ■

Remark 3.1.3. Some important remarks are appropriate here.

(1) By time-duality, we obviously have analogous results for *past-trapped* submanifolds, with an analogous notion of *weakly past-trapped* submanifolds.

(2) Consider now metrics in $\Gamma^\infty(L)$. Then $\mathcal{A}$ is still C^∞-open, $\mathcal{FA}$ is C^∞-closed, and $\overline{\mathcal{A}} \subseteq \mathcal{FA}$ (C^∞ closure). In the proof of theorem 3.1.2, one sees (because ϕ had compact support) that actually the sequence ϕ_n of smooth functions converges in any C^r topology (r finite) to 0, and thus g_n converges to g in any C^r topology. Therefore $g_n \to g$ in the C^∞ topology, and theorem 3.1.2 can be restated for the C^∞ topology. The exact same kind of proof inspection in Lerner (1973) gives an analogous "C^∞ statement" for theorem 2.3.3 provided one considers only smooth metrics.

(3) Since a pointwise converging net $\{g_\lambda\}$ of metrics in $\mathcal{FA}$ has its pointwise limit in $\mathcal{FA}$ this set is weakly closed therefore a Baire space (theorem A.3.11).

In Lerner (1973) it was questioned that perhaps $\overline{\mathcal{SE}} = \mathcal{E}$, but left the issue open in that reference, and this question doesn't seem to have been addressed in the literature yet. However, using later results on C^1-stability of causal geodesic completeness in globally hyperbolic spacetimes (Beem, Ehrlich, and Easley (1999, cor. 7.37)), we can now give a very simple counterexample that shows one may indeed have $\mathcal{E} \setminus \overline{\mathcal{SE}} \neq \emptyset$.

Example 3.1.4. Concretely, in the standard Minkowski spacetime $\mathbb{R}_1^{m+1}$ consider the quotient manifold $M = \mathbb{R}^{m+1}/\mathbb{Z}^m$ with induced metric g, where the $\mathbb{Z}^m$ isometric action defined by $(t, x^1, \ldots, x^m) \sim (t, x^1 + n_1, \ldots, x^m + n_m)$. Thus, (M, g) is a flat ($g \in \mathcal{E}$), globally hyperbolic geodesically complete spacetime. The spacelike Cauchy hypersurface $\Pi = \{t = 0\}$ in $\mathbb{R}^{m+1}$ quotients to an m-torus and thus is compact. Since global hyperbolicity, causal geodesic completeness and the spacelike character of Π are all C^1-stable via the cited result, there is a open C^2 neighborhood $\mathcal{U}$ of g such that all metrics in $\mathcal{U}$ satisfy these properties. Now, if we had $g \in \overline{\mathcal{SE}}$, then there would exist some $h \in \mathcal{SE} \cap \mathcal{U}$. However, in that case the spacetime (M, h) would satisfy all the conditions in the well-known Hawking-Penrose singularity theorem (conf. Beem, Ehrlich, and Easley (1999, Thm. 12.47)), while being causally geodesically complete, a contradiction. Therefore $g \in \mathcal{E} \setminus \overline{\mathcal{SE}}$. ◄

While the statement of theorem 3.1.2 is an analogous version of theorem 2.3.3 in the context of trapped and weakly trapped submanifolds, and the idea of the proof is also an adaptation of conformal perturbation techniques used by Lerner (1973), in example 3.1.4 we have constructed a situation where $\mathcal{E}^r \neq \overline{\mathcal{SE}^r}$. But in the setting of theorem 3.1.2, since we are dealing with stably causal metrics, we were actually able to prove $\mathcal{FA} = \overline{\mathcal{A}}$ with a similar sequence of conformal perturbations argument.

3.1.1 Singularities in Codimension Two

We can now combine these results with the "energy" conditions for the Ricci tensor (theorem 2.3.3). As pointed out in remark 3.1.3(ii), the results in theorem 2.3.3 and theorem 3.1.2 can be stated for the C^∞ topology. Because the latter assumption is the most common one in geometry, the following discussion will be carried out for smooth metrics, while easily restated for C^r metrics with the C^r topology with $r \geq 2$.

Recall that the set $\mathcal{A}$ of C^∞ metrics for which Σ is future trapped is C^∞-open in $\Gamma^\infty(L)$, and so is $\mathcal{SE}$. Therefore, $\mathcal{M} := \mathcal{A} \cap \mathcal{SE}$ is also C^∞-open in $\Gamma^\infty(L)$, and contained in $\mathscr{S}\mathscr{T}(X) \cap \mathscr{S}_\Sigma \cap \mathcal{SC}$.

Let $\mathcal{FM} := \mathcal{FA} \cap \mathcal{E}$ be the set of smooth spacetime metrics which are both weakly trapped and satisfy the timelike convergence conditions. This set is C^∞-closed in $\mathscr{S}\mathscr{T}(X) \cap \mathscr{S}_\Sigma \cap \mathcal{SC}$; thus $\overline{\mathcal{M}} \subseteq \mathcal{FM}$, and also

$$int(\mathcal{FM}) = int(\mathcal{FA}) \cap int(\mathcal{E}) = \mathcal{M},$$

therefore $\mathcal{M}$ is prevalent in $\mathcal{FM}$.

Remark 3.1.5. Just as in the discussion showing that $\mathcal{E} \setminus \overline{\mathcal{SE}}$ is not necessarily empty given in example example 3.1.4, the same spacetime given as counterexample therein works to show that $\mathcal{FM} \setminus \overline{\mathcal{M}}$ may also not be empty: just consider in addition the spacelike surface originating from the quotient of $S = \{t = x^1 = 0\}$. Therefore $g \in \mathcal{FM}$, but the possibility of $g \in \overline{\mathcal{M}}$ leads to the same contradiction.

This counterexample can now be used to display a similar phenomenon as that exhibited in example 2.1.5.

Example 3.1.6. Denote by N the underlying manifold in remark 3.1.5. We always have $\mathcal{M}$ prevalent in $\mathcal{FM}$, in particular $\mathcal{FM} \setminus \overline{\mathcal{M}}$ is nowhere dense in $\mathscr{S}\mathscr{T}(X) \cap \mathscr{S}_\Sigma \cap \mathcal{SC}$. However, we can show that $\mathcal{FM} \setminus \overline{\mathcal{M}}$ is not nowhere dense in the subspace topology of $\mathcal{FM}$ for this example.

In fact, since there is some $g \in \mathcal{FM} \setminus \overline{\mathcal{M}}$, consider an open neighborhood U of g such that $U \cap \overline{\mathcal{M}} = \varnothing$. The set $V = U \cap \mathcal{FM}$ is now an open neighborhood of g in $\mathcal{FM}$, and clearly $V \subseteq \mathcal{FM} \setminus \overline{\mathcal{M}}$; thus, it cannot be nowhere dense in $\mathcal{FM}$, as claimed. ◀

Now, consider the following key observation. Assume that (i) $g \in \mathcal{M}$, (ii) Σ is a closed submanifold of codimension $k = 2$ (following the physics usage we simply say that Σ is a *closed surface* in this case), and (iii) (M, g) is a chronological spacetime. Then, since in particular we will have $Ric(g)(v,v) > 0$ for any g-causal vector, the generic condition (definition 1.3.1), will be trivially satisfied (conf. Beem, Ehrlich, and Easley (1999, Prop. 2.12, p. 39)). But then all the hypotheses in the Hawking-Penrose singularity theorem (corollary 1.6.2) hold and therefore (M,g) must possess at least one incomplete causal geodesic[1].

Now, since our metrics are restricted to the set $\mathscr{S}\mathscr{T}(X) \cap \mathscr{S}_\Sigma \cap \mathcal{SC}$ (so in particular chronological), for every $g \in \mathcal{M}$, the spacetime (M,g) is causally geodesically incomplete. Since $\mathcal{M}$ is prevalent in $\mathcal{FM}$, we conclude that "nearly all" metrics in $\mathcal{FM}$ are arbitrarily C^∞-close to causally incomplete metrics. More precisely, we summarize the discussion of this subsection in the following theorem.

Theorem 3.1.7. *Let (M,g) be a spacetime of dimension ≥ 3 and smooth metric containing a spacelike closed surface Σ. Assume that the following conditions hold.*

1. *Σ is weakly future-trapped;*
2. *$Ric(g)(v,v) \geq 0$ for all g-timelike $v \in TM$;*
3. *(M,g) is stably causal.*

There exists a prevalent set χ in $\mathcal{FM}$ such that if $g \in \chi$, then there exists a net $(g_\lambda)_{\lambda\in\Lambda}$ of smooth metrics on M such that $g_\lambda \to g$ in the C^∞ topology and such that for each $\lambda \in \Lambda$, the spacetime (M,g_λ) satisfies the following conditions

1. *Σ is future-trapped;*
2. *$Ric(g_\lambda)(v,v) > 0$ for all g_λ-causal $v \in TM$;*
3. *(M,g_λ) is stably causal.*

In particular, (M,g_λ) has at least one incomplete causal geodesic. ■

3.2 Main results II: Higher Codimensions

Our goal in this final section for the chapter is to establish a version of Theorem 3.1.7 that is valid for all higher codimensions $k \geq 2$ of $\Sigma^m \subset M$. Since singularity theorems in the presence of higher co-dimensional submanifolds are more delicate, we must introduce some further notation and results as to have a better control of the Riemannian curvature as well.

For a Lorentz metric $g \in \Gamma^2(L)$ on M denote by $Riem(g)$ the covariant Riemann curvature $(0,4)$-tensor associated with g. Applying a jet bundle argument completely analogous to the one used to prove continuity of the Ricci tensor Ric as a function of g (conf. Lerner (1973, p. 23–24)), one readily sees that the function

$$Riem : g \in \Gamma^r(L) \mapsto Riem(g) \in \Gamma^0(T^{(0,4)}M)$$

[1]For this particular result, if (M,g) is chronological it is enough that $g \in \mathcal{FA} \cap \mathcal{SE}$ (Silva (2012)).

is continuous with respect to the C^r-topology on $\Gamma^r(L)$ for each $2 \leq r \leq \infty$, where now $T^{(0,4)}M$ denotes the smooth vector bundle of $(0,4)$-tensors over M. Consider in $\Gamma^2(L)$ the subset

$$\mathcal{P} = \{g \in \Gamma^2(L) : Riem(g)(w,v,v,w) > 0 \; \forall v \in TM \; g\text{-causal and all } w \text{ non-collinear to } v\}.$$

Proposition 3.2.1. *The set $\mathcal{P}$ is C^2-open in $\Gamma^2(L)$.*

Proof. Again, our arguments adapt some ideas in the proof of Lerner (1973, Prop. 4.3), but with a number of modifications of detail. Denote by $Curv(M)$ the vector subbundle of $T^{(0,4)}M$ of all *curvature-like tensors* over M, i.e., if $F \in Curv(M)$, then for all $p \in M$ and all vectors $w, x, y, z \in T_pM$ we have

CL1) $F(w,z,x,y) = -F(w,z,y,x)$;

CL2) $F(w,z,x,y) = -F(z,w,x,y)$;

CL3) $F(w,z,x,y) + F(w,x,y,z) + F(w,y,z,x) = 0$;

CL4) $F(x,y,w,z) = F(w,z,x,y)$.

Fix $g \in \mathcal{P}$. Denote by $\widehat{TM}$ the tangent bundle minus all zero vectors, and consider, for $C \in \Gamma^0(Curv(M))$, the set

$$V_C = \{v \in \widehat{TM} : C(w,v,v,w) > 0 \text{ for all } w \text{ non-collinear to } v\}.$$

We argue that V_C is open in $\widehat{TM}$. Indeed, suppose not. Then there is a sequence $v_n \in \widehat{TM} \setminus V_C$ converging to some vector $v \in V_C$. For each v_n, not being an element of V_C implies that there exists w_n not collinear to v_n and based at the same points respectively such that $C(w_n, v_n, v_n, w_n) \leq 0$. Now, fix some background Riemannian metric h for M. By taking into account the symmetries for C we can assume without loss of generality that all vectors are h-unitary, and each w_n is h-normal to v_n. By the compactness of the h-sphere bundle over compact subsets of M (since v_n is convergent), passing to a subsequence if necessary we can assume that w_n converges to a nonzero w with same base point in M as v, and such that $h(v,w) = 0$, implying non-collinearity. Thus, on the one hand, $v \in V_C$, and hence $C(w,v,v,w) > 0$, and on the other hand the convergence implies $C(w,v,v,w) \leq 0$, a contradiction.

Denote by $C_g \subseteq \widehat{TM}$ the set of all g-causal vectors. Now, since $g \in \mathcal{P}$, we have $C_g \subseteq V_{Riem(g)}$. However, C_g is evidently closed in $\widehat{TM}$, therefore, since the latter is a manifold and hence a normal topological space, there exists an open set $U \subseteq \widehat{TM}$ satisfying $C_g \subseteq U \subseteq \overline{U} \subseteq V_{Riem(g)}$. In addition, we can assume without loss of generality that U can be chosen to satisfy $v \in U \implies \alpha v \in U$, for nonzero α since both sets C_g and $V_{Riem(g)}$ have this property.

Denoting as $\pi : Curv(M) \to M$ the bundle standard projection, we now prove the following statement for this chosen U: for each point $p \in M$, there exists an open set $W \subseteq Curv(M)$ containing $Riem(g)(p)$ with the property that, if for $C \in \Gamma^0(Curv)$ and all $q \in \pi(W)$

we have $C(q) \in W$, then $\overline{U} \cap T_qM \subseteq V_C \cap T_qM$. Suppose by way of contradiction that this is false. Thus we have a nested sequence of sets $W_n \supseteq W_{n+1}$, all neighborhoods of $Riem(g)(p)$ in $Curv(M)$ with $\bigcap W_n = \{Riem(g)(p)\}$, a sequence $C_n \in \Gamma^0(Curv)$ and points $q_n \in \pi(W_n)$ such that there exists some $v_n \in (\overline{U} \setminus V_{C_n}) \cap T_{q_n}M$, which means that there is some $w_n \in T_{q_n}M$ not collinear to v_n satisfying $C_n(q_n)(w_n, v_n, v_n, w_n) \leq 0$. Since $C_n(q_n) \to Riem(g)(p)$ in $Curv(M)$ and $q_n \to p$ in M, an argument using a background Riemannian metric similar as the one in the first part of this proof shows that $v_n \to v$ and $w_n \to w$ up to a subsequence, for some nonzero $v \in \overline{U} \cap T_pM$, and $w \in T_pM$ not collinear to v. Thus, on the one hand, taking limits we get $Riem(g)(p)(w, v, v, w) \leq 0$, and on the other hand $v \in \overline{U} \subset V_{Riem(g)}$, so $Riem(g)(p)(w, v, v, w) > 0$, which is the desired contradiction.

The rest of the proof is quite similar to the final part that of Lerner (1973, Prop. 4.3, p. 28). Namely, using the local triviality of the bundle $Curv(M)$ and the neighborhoods W above, we can obtain a C^0 neighborhood $\mathcal{W} = \mathcal{W}(Riem(g)) \subset \Gamma^0(Curv(M))$ of $Riem(g)$ with the property that for any $C \in \mathcal{W}$ we have $\overline{U} \subset V_C$. Now, by the continuity of the map $Riem$ we have that $\mathcal{Z} := Riem^{-1}(\mathcal{W})$ is a C^2-open set containing g and by construction $\mathcal{Z} \subset \mathcal{P}$ as desired. ■

Now, let us define

$$\mathcal{FP} = \{g \in \Gamma^2(L) : Riem(g)(w, v, v, w) \geq 0\ \forall v \in TM\ g\text{-causal and all } w \text{ non-collinear to } v\}.$$

If we have a net $(g_\lambda)_{\lambda\in\Lambda}$ in $\mathcal{FP}$ converging in C^2 to g, we have that $Riem(g_\lambda) \to Riem(g)$ in the C^0 topology on $\Gamma^0(T^{(0,4)}M)$, whence we conclude that $g \in \mathcal{FP}$; that is, $\mathcal{FP}$ is C^2-closed. We now have:

Theorem 3.2.2. *In the C^2 topology on $\Gamma^2(L)$ we have $int(\mathcal{FP}) = \mathcal{P}$, so $\mathcal{P}$ is prevalent in $\mathcal{FP}$.*

Proof. Clearly, $\mathcal{P} \subset int(\mathcal{F})$. To prove the other inclusion, we adapt the arguments in the proof of theorem 3.1.2 above. Specifically, we fix $g \in \mathcal{FP} \setminus \mathcal{P}$, and build via conformal rescalings a sequence $g_n \notin \mathcal{FP}$ of metrics converging in C^2 to g. We will use the standard formulas for how global conformal rescalings of the metric change the Riemann curvature tensor. (Conf. e.g. John M. Lee (2018, Eq. 7.44), which although presented for Riemannian metric, does remain valid in this Lorentzian context.)

Since $g \notin \mathcal{P}$, we can pick a g-causal vector $v \in T_pM$ for some $p \in M$, and some $w \neq 0$ at the same base point not collinear to v with $Riem(g)(w, v, v, w) = 0$. Assume first that v is g-timelike, so w can be assumed to be spacelike and normal to v, and both can also be assumed to be g-unit vectors. Choose a g-normal neighborhood system $(x^1, \ldots, x^{dim\,M})$ centered at p, which can be chosen such that v has components $(1, 0, \ldots, 0)$, and also such that the first component of w is zero. Using such coordinates, consider the function $\xi(x^1, \ldots, x^{dim\,M}) := e^{x^1}$. We globally extend ξ with usual bump function arguments, (denoting the extension the same way) so that ξ is, in coordinates, e^{x^1} around p, and zero outside some compact subset of M containing p. Define, for each $n \in \mathbb{N}$, $\xi_n = \xi/n$, $g_n = e^{2\xi_n}g$, so that again $g_n \to g$ in the C^2 topology. With our choice of coordinates, a straightforward computation gives

$$Riem(g_n)_p(w, v, v, w) = -\frac{e^{2/n}}{n} < 0,$$

so $g_n \notin \mathcal{FP}$ as desired. The case when v is g-lightlike is slightly more involved. In this case we pick normal coordinates $(x^1, \ldots, x^{dim\,M})$ centered at p such that $e_i := \partial/\partial x^i(p)$ ($i = 1, \ldots, dim\,M$) form a g-orthonormal basis with e_1 timelike and $v = e_1 + e_2$. Let $\ell := e_1 - e_2$. ℓ is also null, and not collinear with v. Now, write

$$w = a \cdot v + b \cdot \ell + \sum_{i=3}^{dim\,M} w^i \cdot e_i.$$

Since w is not collinear with v, at least one of the b, w^i are nonzero, and since the part parallel to v gives no contribution to $Riem(g)(w, v, v, w)$ due to the curvature symmetries we can assume without loss of generality that $a = 0$. Observe that the vector $\sum_{i=3}^{dim\,M} w^i \cdot e_i$ is normal to both v and ℓ, and we have the possibility that either w is lightlike (when each $w^i \equiv 0$) or spacelike. Consider first the spacelike case. Define $\xi(x^1, x^2, \ldots, x^n) = (x^1 + x^2)^2$. Proceeding thenceforth just as in the timelike case, we now obtain

$$Riem(g_n)(w, v, v, w) = -\frac{4}{n} g(w, w) < 0.$$

Suppose now w is lightlike. Rescaling, we can assume $w = \ell$. Thus, define $\xi(x^1, x^2, \ldots, x^n) = (x^1)^2$. Ckmputing as before we obtain

$$Riem(g_n)(w, v, v, w) = -\frac{8}{n} < 0.$$

Therefore in each case we conclude that $g_n \notin \mathcal{FP}$ as desired, thus completing the proof. ■

Remark 3.2.3. Again, by arguments analogous to those in remark 3.1.3(ii), the theorem 3.2.2 remains valid for the C^∞ topology, provided we work with smooth metrics.

The following set is actually more relevant: let $O \subset \Gamma^2(L)$ be the the of 2-differentiable metrics for which the associated *tidal force operators* (see Beem, Ehrlich, and Easley (1999, pp. 35 and 38) for definitions and notation) along causal directions are positive semidefinite, that is, for $g \in \Gamma^2(L)$ and $v \in TM$ g-causal, the linear operators

$$\begin{cases} R^g_v : v^\perp \to v^\perp, & v \text{ for } g\text{-timelike}, \\ \overline{R^g}_v : \overline{v^\perp} \to \overline{v^\perp}, & v \text{ for } g\text{-lightlike}, \end{cases}$$

are both positive semidefinite. It is straightforward to see that

$$\mathcal{P} \subseteq \mathcal{SE}; \tag{3.5}$$

$$\mathcal{P} \subseteq \mathcal{O} \subseteq \mathcal{FP} \subset \mathcal{E} \tag{3.6}$$

Therefore, $\overline{\mathcal{P}} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{FP}$, and since $int(\mathcal{FP}) = \mathcal{P}$, then $int(\mathcal{FP}) \subseteq \mathcal{O}$, and the following corollary of theorem 3.2.2 is immediate.

Corollary 3.2.4. *In the C^2 topology, $\mathcal{O}$ is prevalent in $\mathcal{FP}$.* ■

Remark 3.2.5. It is straightforward to check that $g \in \mathcal{FP}$ if and only if

$$Riem(g)(w, v, v, w) \geq 0, \forall v \in TM \ g\text{-timelike and all } w \in v^{\perp_g}.$$

3.2.1 Singularities in Higher Codimensions

We consider a similar analysis as the one carried out in section 3.1.1 when the codimension k of the submanifold $\Sigma \subset M$ is higher than 2. However, we cannot use the Hawking-Penrose singularity theorem in this context, but rather use some analogous singularity theorem in the presence of future-trapped submanifolds of codimension higher than 2. Just such a result has been obtained by Galloway and Senovilla (2010). Also, similar to what we did in section 3.1.1 (cf. remark 3.2.3), the following discussion uses the C^∞ topology on the set of smooth metrics, and is easily adapted to C^2 metrics.

We need only to check that the positive-semidefiniteness of the tidal force operators along causal directions codified in the set $\mathcal{O}$ does implies the required conditions. Specifically, let $g \in \mathcal{A} \cap \mathcal{O}$ and assume that Σ is closed. Following Galloway and Senovilla (2010), let $\gamma : [0, b) \to M$ be some future-directed causal geodesic with $\gamma(0) \in \Sigma$ and $\gamma'(0)$ normal to Σ. Consider $e_1, \ldots, e_m$ some coordinate basis of tangent vectors for $T_{\gamma(0)}\Sigma$, and let $E_1, \ldots, E_m$ denote their parallel-transport vector fields along γ. Write $g_{ab} = g(E_a, E_b)$ (which is constant along γ). Along this geodesic, since we have assumed that $R_{\gamma'}$ (or $\overline{R_{\gamma'}}$ if γ is null) is positive semidefinite in the subspaces spanned by the vectors E_a, the trace

$$g^{ab} Riem(\gamma', E_a, E_b, \gamma') \geq 0.$$

But this is precisely the condition 3.1 in Galloway and Senovilla (2010). Observe also that the inclusions (3.6) imply that the positive semidefiniteness of the tidal forces also entails the timelike convergence condition. Therefore, if in addition we assume that (i) (M, g) is also chronological and (ii) satisfies the generic condition for causal vectors, then (M, g) is causally geodesically incomplete by Galloway and Senovilla (2010, Thm. 3). Therefore, the arguments in section 3.1.1 by taking proposition 3.2.1, theorem 3.2.2, remark 3.2.5 and the inclusions (3.5), (3.6) into account can be now adapted as follows.

Theorem 3.2.6. *Let (M, g) be a spacetime of dimension ≥ 3 and smooth metric containing a spacelike closed submanifold Σ of codimension $k \geq 2$. Assume that the following conditions hold.*

1. *Σ is weakly future-trapped;*
2. *$Riem(g)(w, v, v, w) \geq 0$ for all g-timelike $v \in TM$ and all w at the same base point g-orthogonal to v;*
3. *(M, g) is stably causal.*

There exists a prevalent subset χ in $\mathcal{FP}$ such that if$g \in \chi$, then there exists a net $(g_\lambda)_{\lambda\in\Lambda}$ of smooth metrics on M such that $g_\lambda \to g$ in the C^∞ topology and such that for each $\lambda \in \Lambda$, the spacetime (M, g_λ) satisfies the following conditions

1. *Σ is future-trapped;*
2. *$Ric(g_\lambda)(v, v) > 0$ for all g_λ-causal $v \in TM$, and thus the causal genericity condition holds in (M, g_λ);*
3. *the tidal force operators along g_λ-causal directions are all positive semidefin0ite;*
4. *(M, g_λ) is stably causal.*

In particular, each (M, g_λ) has at least one incomplete causal geodesic. ■

Part III

Genericity of Singularities - Initial Data

4 Genericity of Singularities in Initial Data sets with MOTS

The objective of this chapter is to obtain a genericity result of singular spacetimes under suitable conditions in similar fashion to the ones in chapter 3, but using the approach of *initial data* containing a MOTS. Under concrete situations to be discussed, we will have a separable Hilbert manifold structure for the set of triples $(h, \mathcal{K}, \psi)$, where $(h, \mathcal{K})$. is an initial data on a fixed connected n-manifold $\mathcal{S}$ and $\psi : \Sigma \to \mathcal{S}$ a MOTS embedding, where again Σ is a fixed compact and connected manifold (without boundary). The genericity statement we seek will follow from an abstract functional-analytic approach we will develop, together with the well-known Sard-Smale theorem.

4.1 Abstract Banach Manifold Genericity Result

In order to obtain our main genericity results, we shal give a suitable Banach manifold structure both on the set of initial data on a fixed connected n-manifold $\mathcal{S}$, and on the set of embeddings of a fixed connected compact $(n-1)$-manifold Σ into $\mathcal{S}$. In order to do that however, we shall first establish in this section a key but purely abstract result on Banach manifolds, adapting the main analysis in Biliotti, Javaloyes, and Piccione (2009). Our first lemma is an adaptation of lemma 2.2 in the latter reference, where we substitute the kernel condition therein for a suitable alternative on the orthogonal complement of images. For completeness we give proofs of other results from Biliotti, Javaloyes, and Piccione (2009), clearly indicating the source when we do so, as needed.

Here, for W a closed subspace in a Hilbert space H, we denote by $P_W : H \to W$ the orthogonal projection onto W.

We also recall that a bounded linear operator $T : E \to F$ between Banach spaces is said to be *Fredholm* if $\ker T$ and $\operatorname{coker} T$ have finite dimensions[1] , where $\operatorname{coker} T = E/\operatorname{Im} T$ (the dimension of the cokernel is also called the *codimension* of $\operatorname{Im} T$). The *index* of a Fredholm operator T is defined as $\operatorname{ind}(T) = \dim\ker T - \dim\operatorname{coker} T$. "For later reference, we also recall that if M, N are (perhaps infinite dimensional) Banach manifolds and a is an integer number, then a C^1 map $f : M \to N$ is said to a *Fredholm map of index a* if its derivative $df_p : T_pM \to T_{f(p)}N$ is a Fredholm bounded linear operator of fixed index a for every $p \in M$

Lemma 4.1.1. *Let E, F be Banach spaces, and let H be a Hilbert space with inner product $\langle\ ,\ \rangle$. Consider $T : E \to H$, $S : F \to H$ bounded linear operators, with* $\operatorname{Im} S$ *closed and such that*

[1] Some aauthors explicitly require the range of T to be closed; however, this condition follows easily from the finite dimensionality of the cokernel.

$P_{\operatorname{Im} S^\perp}(\operatorname{Im} T)$ is closed in $\operatorname{Im} S^\perp$ (this happens for example if S is a Fredholm operator). Then the sum operator $T \oplus S : E \times F \to H$ (given by $T \oplus S(x, y) = T(x) + S(y)$) is surjective if and only if $\operatorname{Im} T^\perp \cap \operatorname{Im} S^\perp = \{0\}$.

Proof. Assuming the sum operator to be surjective, given $z \in \operatorname{Im} T^\perp \cap \operatorname{Im} S^\perp$, then there is a pair (x, y) with $z = T(x) + S(y)$, but also by perpendicularity

$$\|z\|^2 = \langle z, z\rangle = \langle z, T(x) + S(y)\rangle = 0.$$

Assuming now $\operatorname{Im} T^\perp \cap \operatorname{Im} S^\perp = \{0\}$, take some $z \in H \setminus \{0\}$. With $\operatorname{Im} S$ closed, $H = \operatorname{Im} S \oplus \operatorname{Im} S^\perp$, so $z = S(x) + u$, $u \in \operatorname{Im} S^\perp$. We can also split $\operatorname{Im} S^\perp$ (since it is a Hilbert space under the restriction of the inner product and $P_{\operatorname{Im} S^\perp}(\operatorname{Im} T)$ is closed in $\operatorname{Im} S^\perp$) as follows:

$$\operatorname{Im} S^\perp = [P_{\operatorname{Im} S^\perp}(\operatorname{Im} T)] \oplus [(P_{\operatorname{Im} S^\perp}(\operatorname{Im} T))^\perp \cap \operatorname{Im} S^\perp].$$

Now, $u = P_{\operatorname{Im} S^\perp}(T(x)) + v$, where $v \in (P_{\operatorname{Im} S^\perp}(\operatorname{Im} T))^\perp \cap \operatorname{Im} S^\perp$. Also, $T(x) = S(\hat{y}) + P_{\operatorname{Im} S^\perp}(T(x))$, and we have $z = T(x) + S(y) - S(\hat{y}) + v$. If $v = 0$ we are done. Now, $v \in \operatorname{Im} S^\perp$, and for any $T(e)$, $e \in E$, again $T(e) = S(f) + P_{\operatorname{Im} S^\perp}(T(e))$, and we obtain,

$$\langle v, T(e)\rangle = \langle v, S(f)\rangle + \langle v, P_{\operatorname{Im} S^\perp}(T(e))\rangle = 0,$$

meaning $v \in \operatorname{Im} T^\perp$, implying $v = 0$. ∎

Remark 4.1.2. For the Banach adjoint $T^* : H^* \to E^*$, from standard functional analysis (Bachman and Narici (2000), p. 285) we have $\operatorname{Im} T^\perp = \ker T^*$, so the trivial intersection of images in lemma 4.1.1 can be substituted with trivial intersection of the adjoint kernels.

Recall also that in a Banach space E, a closed subspace $V \subseteq E$ is said to be *complemented* if there is another another closed subspace $W \subseteq E$, called the *complement of V*, satisfying $E = V \oplus W$.

Proposition 4.1.3 (Biliotti, Javaloyes, and Piccione (2009), lemma 2.4). *Let E, F, G be Banach spaces, and $T : E \to G$, $S : F \to G$ bounded linear operators, with $\ker S$ complemented in F and $\operatorname{Im} S$ finite codimensional in G (this happens for example if S is a Fredholm operator). Then the kernel of $T \oplus S$ is complemented in $E \times F$.*

Proof. One sees that $\operatorname{Im} T \cap \operatorname{Im} S \subseteq G$ has finite codimension in $\operatorname{Im} T$: for the quotient projection $\pi : G \to G/\operatorname{Im} S$, the restriction $\pi|_{\operatorname{Im} T} : \operatorname{Im} T \to G/\operatorname{Im} S$ has $\ker \pi|_{\operatorname{Im} T} = \operatorname{Im} T \cap \operatorname{Im} S$, thus inducing an injective linear map

$$[\pi|_{\operatorname{Im} T}] : \frac{\operatorname{Im} T}{\operatorname{Im} T \cap \operatorname{Im} S} \to \frac{G}{\operatorname{Im} S},$$

and $G/\operatorname{Im} S$ is finite dimensional, therefore $\operatorname{Im} T \cap \operatorname{Im} S$ has finite codimension as a subspace of $\operatorname{Im} T$.

Now, $\operatorname{Im} T \cap \operatorname{Im} S$ has a finite dimensional (therefore closed) algebraic complement Z in $\operatorname{Im} T$, that is, $(\operatorname{Im} T \cap \operatorname{Im} S) \oplus Z = \operatorname{Im} T$. Clearly $\ker T$ has finite codimension in $T^{-1}(Z)$, since the induced quotient of T mapping $[T] : T^{-1}(Z)/\ker T \to Z$ is surjective. we then set $T^{-1}(Z) = \ker T \oplus V$, $V \subseteq E$ finite dimensional. Lastly, $\ker S$ has a closed complement W in F from our hypothesis.

We claim $V \times W$ to be the desired complement of $\ker(T \oplus S)$. Take $(x, y) \in (V \times W) \cap \ker(T \oplus S)$, then $T(x) + S(y) = 0$. With $V \subseteq T^{-1}(Z)$, $T(x) \in Z$, but also $T(x) = -S(y) \in \operatorname{Im} S$, and since $(\operatorname{Im} T \cap \operatorname{Im} S) \cap Z = \{0\}$, it then follows $T(x) = S(y) =$, so $x \in \ker T \cap V = \{0\}$ and $y \in \ker S \cap W = \{0\}$, implying $(x, y) = 0$ and the intersection $(V \times W) \cap \ker(T \oplus S)$ is trivial.

To prove the sum $(V \times W) + \ker(T \oplus S)$ is all $E \times F$, for $(x, y) \in E \times F$ arbitrary, write $T(x) = T(e) + z$, where $e \in E$, $T(e) \in \operatorname{Im} S$ and $z \in Z$. Since $Z \subseteq \operatorname{Im} T$, by the decomposition of $T^{-1}(Z)$ we see that $z = T(v)$ for some $v \in V$, and then $x = e + v + a$, for some $a \in \ker T$. Also, $T(e) = S(w)$ for some $w \in W$. Setting $y = b + w'$, where $b \in \ker S$ and $w' \in W$. Then, the vectors $(e + a, b - w) \in \ker(T \oplus S)$ and $(v, w' + w) \in V \times W$ are such that

$$(e + a, b - w) + (v, w' + w) = (e + v + a, b + w') = (x, y),$$

thus proving our claim. ■

Proposition 4.1.4 (Biliotti, Javaloyes, and Piccione (2009), lemma 2.3). *Let U, V be vector spaces, $L : U \to V$ a linear map, and $S \subseteq V$ a finite codimensional subspace. Then $L^{-1}(S)$ has finite codimension in U, with*

$$\operatorname{codim}_U (L^{-1}(S)) = \operatorname{codim}_V (S) - \operatorname{codim}_V (S + \operatorname{Im} L). \tag{4.1}$$

Proof. Considering the quotient projection $\pi : V \to V/S$, the composition $\pi \circ L : U \to V/S$ has kernel $L^{-1}(S)$, so the induced quotient map $[\pi \circ L] : U/L^{-1}(S) \to V/S$ is injective, with proves $L^{-1}(S)$ has finite codimension in U.

For (4.1), using standard dimensional analysis from linear algebra applied to the linear map $[\pi \circ L] : U/L^{-1}(S) \to V/S$ we have

$$\begin{aligned} \operatorname{codim}_V S = \dim (V/S) &= \dim (\operatorname{Im}[\pi \circ L]) + \operatorname{codim}_{V/S} (\operatorname{Im}[\pi \circ L]) \\ &= \dim (U/L^{-1}(S)) + \operatorname{codim}_V (\operatorname{Im}(L) + S). \end{aligned}$$ ■

For the abstract Banach manifold result we'll need,we have announced, we fix a separable Banach manifold X, a Hilbert manifold Y, and a Hilbert space V. We apply the general functional-analytic results above to prove the following proposition, an adaptation of Biliotti, Javaloyes, and Piccione (2009), prop. 3.1 with more suitable hypotheses for our concrete case of interest.

Proposition 4.1.5. *Consider $A \subseteq X \times Y$ an open set, and let $f : A \to V$ be a C^k function, $k \geq 1$. For every $(x_0, y_0) \in A$ such that $f(x_0, y_0) = 0$, assume that the partial derivative $\frac{\partial f}{\partial y}(x_0, y_0) : T_{y_0}Y \to V$ is a Fredholm operator. Then 0 is a regular value of f if and only if*

$$\left(\operatorname{Im} \frac{\partial f}{\partial x}(x_0, y_0)\right)^{\perp} \cap \left(\operatorname{Im} \frac{\partial f}{\partial y}(x_0, y_0)\right)^{\perp} = \{0\}. \tag{4.2}$$

Proof. 0 is a regular value of f if and only if for all $(x_0, y_0) \in f^{-1}(0)$, $df_{(x_0,y_0)} : T_{x_0}X \times T_{y_0}Y \to V$ is surjective with complemented kernel. Now $df_{(x_0,y_0)}$ can be written as the sum operator

$$df_{(x_0,y_0)} = \frac{\partial f}{\partial x}(x_0, y_0) \oplus \frac{\partial f}{\partial y}(x_0, y_0).$$

Applying lemma 4.1.1 since $\frac{\partial f}{\partial y}(x_0, y_0)$ is Fredholm, it follows that if $df_{(x_0,y_0)}$ is surjective then (4.2) holds. Conversely if (4.2) holds, by lemma 4.1.1 we obtain surjectivity, and also by proposition 4.1.3 the kernel of $df_{(x_0,y_0)}$ is complemented. ■

Remark 4.1.6. For the sake of brevity, we will call the trivial intersection condition (4.2) as the *T condition*. As stated in remark 4.1.2, we can substitute this condition for

$$\ker\left(\frac{\partial f^*}{\partial x}(x_0, y_0)\right) \cap \ker\left(\frac{\partial f^*}{\partial y}(x_0, y_0)\right) = \{0\}$$

at points $(x_0, y_0) \in f^{-1}(0)$. This might be helpful in concrete situations as it directly states the condition in the form of system of equations for the adjoints.

Under the hypothesis of proposition 4.1.5, assuming (4.2) to be valid, then the level set $M = f^{-1}\{0\} \subseteq A$ is an embedded submanifold or $X \times Y$, with tangent space at some $(x_0, y_0) \in M$ given by

$$T_{(x_0,y_0)}M = \left\{ (v, w) \in T_{x_0}X \times T_{y_0}Y : \frac{\partial f}{\partial x}(x_0, y_0)(v) + \frac{\partial f}{\partial y}(x_0, y_0)(w) = 0 \right\}.$$

The main abstract genericity result we'll need is the following (adapted from Biliotti, Javaloyes, and Piccione (2009, corollary 3.4)).

Theorem 4.1.7. *Under the hypothesis of proposition 4.1.5, assume also that $\frac{\partial f}{\partial y}$ is a Fredholm operator of index zero for points in M. Denoting by $\Pi : X \times Y \to X$ the canonical projection, then,*

(i) $\Pi|_M : M \to X$ *is Fredholm map of index zero.*

(ii) The critical points of $\Pi|_M$ *are the points* $(x_0, y_0) \in M$ *such that* y_0 *is a critical point of the functional*

$$y \in A_{x_0} \mapsto f(x_0, y) \in V,$$

where $A_x = \{y \in Y : (x, y) \in A\}$.

(iii) Assuming X, Y *and* V *to be separable, the set of points* $x \in X$ *such that the functional* $y \in A_x \mapsto f(x, y) \in V$ *has no critical points is generic in* $\Pi(A)$.

Proof. For (i), given $(x_0, y_0) \in M$, we have $\ker d\Pi|_{T_{(x_0,y_0)}M} = T_{(x_0,y_0)}M \cap (\{0\} \times T_{y_0}Y)$, and this vector space is isomorphic to $\ker\left(\frac{\partial f}{\partial y}(x_0, y_0)\right)$ so it is finite dimensional. Also, one readily sees that

$$\operatorname{Im} d\Pi|_{T_{(x_0,y_0)}M} = \left(\frac{\partial f}{\partial x}(x_0, y_0)\right)^{-1}\left(\operatorname{Im}\frac{\partial f}{\partial y}(x_0, y_0)\right),$$

so from proposition 4.1.4, $\operatorname{Im} d\Pi|_{T_{(x_0,y_0)}M}$ is finite codimensional, and since $df_{(x_0,y_0)}$ is surjective, by eq. (4.1)

$$\operatorname{codim}(\operatorname{Im} d\Pi|_{T_{(x_0,y_0)}M}) = \operatorname{codim}\left(\frac{\partial f}{\partial y}(x_0, y_0)\right).$$

Since we are assuming $\frac{\partial f}{\partial y}(x_0, y_0)$ to be Fredholm of index zero, it follows that $d\Pi|_{T_{(x_0,y_0)}M}$ is also Fredholm of index zero.

For (ii), it is easy to see that $(x_0, y_0) \in M$ is a regular point of $\Pi|_M$ if and only if

$$\operatorname{Im}\left(\frac{\partial f}{\partial x}(x_0, y_0)\right) \subseteq \operatorname{Im}\left(\frac{\partial f}{\partial y}(x_0, y_0)\right).$$

Taking orthogonal complements, this is equivalent to

$$\operatorname{Im}\left(\frac{\partial f}{\partial y}(x_0, y_0)\right)^{\perp} \subseteq \operatorname{Im}\left(\frac{\partial f}{\partial x}(x_0, y_0)\right)^{\perp},$$

and by condition (4.2), equivalent to $\operatorname{Im}\left(\frac{\partial f}{\partial y}(x_0, y_0)\right)$ be a zero codimensional subspace, in turn equivalent to the triviality of $\ker\left(\frac{\partial f}{\partial y}(x_0, y_0)\right)$.

To summarize, $(x_0, y_0) \in M$ is a regular point of $\Pi|_M$ if and only if $\ker\left(\frac{\partial f}{\partial y}(x_0, y_0)\right) = \{0\}$, that is, if and only if y_0 is a regular point of the functional $y \in A_{x_0} \mapsto f(x_0, y) \in V$. Item (iii) now follows, because the set of points $x \in X$ such that $y \in A_x \mapsto f(x, y) \in V$ has no critical points coincides with the set of regular values for $\Pi|_M$, and genericity follows by the Sard-Smale theorem (Smale (1965)). ■

Remark 4.1.8. We make here an important comment on the plausibility of the hypotheses in theorem 4.1.7.

The condition T together with the Fredholm condition is equivalent to 0 being a regular value of a C^k function $f : A \subseteq X \times Y \to V$. Now, suppose X, Y, V were all finite-dimensional.

Then, the simplest version of elementary transversality theorem (corollary A.3.15) applied to the submanifold $W = \{0\} \subset V$ would mean the the set of C^k functions $f : A \to V$ for which 0 is a regular value is residual with respect to a Whitney topology. This suggests that the condition T might not be restrictive.

While we expect to have an analogous situation for our infinite dimensional manifolds, tranversality theory is far harder for the infinite dimensional case, and we were unable to find an analogous Thom transversality theorem in the literature applied to our case. Therefore the T condition will remain a technical condition for us.

4.2 Manifold Structures

The function we will concretely analyze in order to apply theorem 4.1.7 is the expansion scalar θ_+ defined on a suitable open set (when manifold structures are defined) around a triple $(h_0, \mathcal{K}_0, \psi_0)$, with $(h_0, \mathcal{K}_0)$ an initial data set for which $\psi_0 : \Sigma \to \mathcal{S}$ is a MOTS embedding. With a domain well defined, the expansion scalar is a function of the form

$$(h, \mathcal{K}, \psi) \mapsto \theta_+(h, \mathcal{K}, \psi) = \operatorname{tr}_h \mathcal{K} \circ \psi + H^{\psi}_{h,\boldsymbol{\nu}}.$$

First,we describe possible infinite dimensional manifold structures on the set of vacuum initial data, then discuss a Hilbert manifold structure for the set of embeddings $\Sigma \to \mathcal{S}$.

These infinite dimensional structures are very technical in nature, and describing them precisely is out of the scope of this work. Fortunately, we shall need only very general feature of these here, so we shall content ourselves to briefly summarize the ideas behind them. With this structures well established, we then define the θ_+ on this "suitable open set".

4.2.1 Manifold of Initial Data Sets

Denote by $ID(\mathcal{S})$ the set of vacuum initial data $(h, \mathcal{K})$. We set to establish a reasonable structure for the initial data set $ID(\mathcal{S})$.

A technical condition that arises frequently in the study of initial data sets is the existence of the so called *Killing initial data* (abbreviated to *KID*). For a initial data set $(\mathcal{S}, h, \mathcal{K})$, given an open set $\Omega \subseteq \mathcal{S}$, consider the following operator[2] for Y a vector field and N a function over Ω,

$$P^*(Y, N) = \begin{pmatrix} 2\left(\nabla_{(i}Y_{j)} - \nabla^l Y_l h_{ij} - \mathcal{K}_{ij}N + \operatorname{tr}\mathcal{K} N h_{ij}\right) \\ \nabla^l Y_l \mathcal{K}_{ij} - 2\mathcal{K}^l{}_{(i}\nabla_{j)}Y_l + \mathcal{K}^q{}_l \nabla_q Y^l h_{ij} - \Delta N h_{ij} + \nabla_i \nabla_j N \\ + \left(\nabla^p \mathcal{K}_{lp} h_{ij} - \nabla_l \mathcal{K}_{ij}\right) Y^l - N \operatorname{Ric}(h)_{ij} + 2N\mathcal{K}^l_i K_{jl} - 2N(\operatorname{tr}\mathcal{K})\mathcal{K}_{ij} \end{pmatrix}.$$

The equations $P^*(Y, N) = 0$ are the called the *vacuum KID equations* on Ω. A KID on Ω is

[2]This equation derived as the formal adjoint of the linearization for the constraint equation (cf. Chruściel and Delay (2003), section 2).

then defined as the pair (Y, N) on Ω that is a nontrivial solution of $P^*(Y, N) = 0$. If there are no nontrivial solutions of $P^*(Y, N) = 0$, the initial data set $(\mathcal{S}, h, \mathcal{K})$ satisfies the *no initial Killing data* (*no KIDs condition*) on Ω. As shown by Moncrief (1975), solutions to the vacuum KID equations are in one-to-one correspondence with Killing vector fields in the maximal Cauchy development of the initial data.

Since we want conditions on $\mathcal{S}$ as to apply theorem 1.6.3, $\mathcal{S}$ cannot be compact. For a concrete manifold structure under such conditions we will settle for $\mathcal{S}$ to be a non-compact, oriented *asymptotically flat manifold* without boundary. Asymptotically flatness here means that $\mathcal{S}$ has a compact set K such that $\mathcal{S} \setminus K$ is the union of a finite number of regions $\mathcal{S}_1, \ldots, \mathcal{S}_\ell$ called the *ends* of $\mathcal{S}$, each diffeomorphic to $\mathbb{R}^n \setminus B$, where B denotes the closed ball of radius 1 centered at zero[3].

For the sake of simplicity we here assume $\mathcal{S}$ has only one end. Then, under these conditions we find a few options in literature, but for our practical purposes we choose the smooth Hilbert manifold structure laid out by Bartnik (2005). This structure[4] is locally modeled on Sobolev spaces $H^2 \times H^1$, therefore is separable.

4.2.2 Manifold of Embeddings

Infinite dimensional structures for functions spaces $\Sigma \to \mathcal{S}$ with Σ compact is a well-established idea in the literature. Here we follow Alias and Piccione (2011), section 3.

The idea of the process is to choose a “well suitable” class of regularity $\mathscr{R}$ less regular than C^∞ regularity, that densely contains C^∞ and can be continuously embedded into those of $C^{2,\alpha}$ regularity (Alias and Piccione (2011), section 2), then model $\mathscr{R}(\Sigma, \mathcal{S})$ as a Banach or Hilbert manifold of such regularity, and finally argue that the set embeddings $Emb(\Sigma, \mathcal{S})$ is open in $\mathscr{R}(\Sigma, \mathcal{S})$.

Here we choose Sobolev H^{k+2} regularity (a Hilbert space), with $k, r \in \mathbb{N}$ and $\alpha \in (0, 1)$ (we take $r \geq 2$ since at least two derivatives are computed in curvature tensors) satisfying $(k+2-r-\alpha)/n > 1/2$ so that the Sobolev space H^{k+2} is continuously embedded into the Hölder space $C^{r,\alpha}$ (Aubin (1998), thms. 2.10 and 2.20), so even if it seems we are losing too much regularity by locally modeling the embeddings over Sobolev spaces, these can be identified as to have enough regularity. More precisely, Alias and Piccione (2011) show that $Emb(\Sigma, \mathcal{S})$ in this regularity can be viewed as a Hilbert manifold locally modeled[5] on the Hilbert space $H^{k+2}(\Sigma)$.

[3] The precise definition of asymptotic flatness also involves detailed falloff conditions for the metric and its derivatives up to second order, that will need not concern us here.

[4] Originally we wanted to model using a *Banach* manifold structure following Chruściel and Delay (2003), since their work is not limited to the asymptotically flat case. However their Banach structure is of the form $H^l \cap C^{r,\alpha}$ (both Sobolev and Hölder) using a Sobolev-Hölder sum norm, and it is unclear to us if the Banach manifold locally modeled on this space is separable, as this condition is crucial to apply the Sard-Smale theorem.

[5] Integrals on Σ are computed with respect to some background Riemannian metric h, but the resulting Sobolev space does not depend on this choice. See Aubin (1998) for details.

4.2.3 Null Expansion Scalar Function θ_+

To make technical sense of the expansion scalar as a smooth function on the set of initial data sets and two-sided, framed embeedings, it requires a chosen outward normal direction *a priori*, therefore we can only work on a neighborhood of a fiducial $(h_0, \mathcal{K}_0, \psi_0)$.

To be more precise, consider $(h_0, \mathcal{K}_0) \in ID(\mathcal{S})$ an asymptotically flat initial data set satisfying the no KIDs condition, and $\psi_0 \in Emb(\Sigma, \mathcal{S})$ a smooth embedding that is a MOTS with respect to initial data set $(h_0, \mathcal{K}_0)$, with $\mathbf{v}_0 \in \mathfrak{X}^{\perp}(\psi_0)$ the chosen outward pointing unit normal vector field of the embedding ψ_0 under h_0. Then

$$\theta_+(h_0, \mathcal{K}_0, \psi_0) = \operatorname{tr}_{h_0} \mathcal{K}_0 \circ \psi_0 + H^{\psi_0}_{h_0, \mathbf{v}_0} = 0.$$

The process is now to "propagate" the normal vector $\mathbf{v}_0$ over a small neighborhood around $\psi_0(\Sigma)$. Since $\psi_0(\Sigma)$ is an smooth compact submanifold of $\mathcal{S}$, there is a smooth vector field $\mathcal{V} \in \mathfrak{X}(\mathcal{S})$ such that $\mathcal{V} \circ \psi_0 = \mathbf{v}_0$. Restricting to a small enough neighborhood around $\psi_0(\Sigma)$, we can assume $\mathcal{V}_p \notin (d\psi_0)_p(T_p\Sigma)$ at all points in this neighborhood. We can then choose a open neighborhood around our embedding ψ_0 in $Emb(\Sigma, \mathcal{S})$ such that all embeddings ψ in this neighborhood have $\mathcal{V} \circ \psi$ nowhere tangent. (These embeddings are called *framed embeddings*, and we see via this argument that this is a stable property). With initial data $(h, \mathcal{K})$ close enough to $(h_0, \mathcal{K}_0)$, we consider the h-normal part of $\mathcal{V} \circ \psi$:

$$\mathbf{v}(h, \psi) = \frac{(\mathcal{V} \circ \psi)^{\perp, h}}{\|(\mathcal{V} \circ \psi)^{\perp, h}\|_h}.$$

Denote by $\mathcal{A}_0$ the open set in $ID(\mathcal{S}) \times Emb(\Sigma, \mathcal{S})$ around $(h_0, \mathcal{K}_0, \psi_0)$ as describe above. Then we can define

$$\theta_+ : (h, \mathcal{K}, \psi) \in \mathcal{A}_0 \mapsto \operatorname{tr}_h \mathcal{K} \circ \psi + H^{\psi}_{h, \mathbf{v}(h,\psi)}, \tag{4.3}$$

4.2.4 Linearization of θ_+

With θ_+ suitably defined, the formal partial linearization of θ_+ with respect to the embedding is the MOTS stability operator (definition 1.5.5): if $\psi : \Sigma \to \mathcal{S}$ is a MOTS embedding for the initial data $(h, \mathcal{K})$,

$$\left.\frac{\partial \theta_+}{\partial \psi}\right|_{(h,\mathcal{K},\psi)} (f) = L(f) = -\Delta_h f + 2\langle X, \operatorname{grad}_h f\rangle_h + (Q + \operatorname{div}_h X - \|X\|_h^2) f,$$

where

$$Q = \frac{1}{2}\mathrm{Scal}_\Sigma - [J(\mathbf{v}) + \rho] - \frac{1}{2}|\mathcal{K}_\mathbf{v} + \mathcal{K}|^2,$$

and X the vector field h-dual to the one form $\mathcal{K}(\mathbf{v}, \cdot)$

An issue that needs addressing in other to apply the analytical machinery developed is the Fredholm property for the MOTS stability operator. However, since L is a second order linear eliptic operator with a specific form, we have the following general result for the Fredholm propriety of such operators.

Proposition 4.2.1 (Dan A. Lee (2019), coro. A.9). *Let (Σ, h) be a compact Riemannian manifold without boundary, and consider L a second order linear elliptic operator on (Σ, h) of the form*

$$L(f) = -\Delta_h f + \langle V, grad_h f\rangle_h + qf,$$

with $V \in \mathfrak{X}(M), q \in C^\infty(\Sigma)$. Then

$$\begin{aligned} L &: W^{k+2,p}(\Sigma) \longrightarrow W^{k,p}(\Sigma), \\ L &: C^{k+2,\alpha}(\Sigma) \longrightarrow C^{k,\alpha}(\Sigma) \end{aligned}$$

are Fredholm operators of index zero for each $k \in \mathbb{N}$. ∎

For completeness, we can also compute the formal partial linearization of θ_+ with respect to the variable of initial data: given (b, Q) two $(0,2)$-symmetric tensors over $\mathcal{S}$, for simplicity we calculate this assuming Σ is embedded in $\mathcal{S}$ ($i : \Sigma \hookrightarrow \mathcal{S}$ is the embedding; for the general case we will have to carry objects over the map ψ as in section 1.1.2).

For a perturbation $h + \delta b$ of the metric h, the first order expansion of the inverse is

$$(h + \delta b)^{ij} = h^{ij} - \delta h^{ik} b_{kl} h^{lj} + o(\delta^2).$$

We first analyze the trace term of θ_+ in (4.3). A perturbation $(h + \delta b, \mathcal{K} + \delta Q)$ has first order expansion

$$\begin{aligned} \mathrm{tr}_{h+\delta b}(\mathcal{K} + \delta Q) &= \mathrm{tr}_{h+\delta b}(\mathcal{K}) + \delta\, \mathrm{tr}_{h+\delta b}(Q) \\ &= \mathrm{tr}_h \mathcal{K} - \delta h^{ik} b_{kl} h^{lj} \mathcal{K}_{ij} + \delta\, \mathrm{tr}_h Q + o(\delta^2), \end{aligned}$$

subtracting $\mathrm{tr}_h \mathcal{K}$, dividing by δ, and taking the limit $\delta \to 0$,

$$\lim_{\delta \to 0} \frac{\mathrm{tr}_{h+\delta b}(\mathcal{K} + \delta Q) - \mathrm{tr}_h \mathcal{K}}{\delta} = \mathrm{tr}_h Q - h^{ik} b_{kl} h^{lj} \mathcal{K}_{ij}, \tag{4.4}$$

and $h^{ik} b_{kl} h^{lj} \mathcal{K}_{ij}$ is the h-inner product of the two tensors b and $\mathcal{K}$, which we denote by $\langle b, \mathcal{K}\rangle_h$.

The second term in θ_+ is a bit trickier. Since Σ has a normal unitary vector field $\boldsymbol{\nu}$, that is dependent on the metric h. Denote by $\boldsymbol{\nu}(\delta) = \boldsymbol{\nu}(h + \delta b)$, with a first order expansion $\boldsymbol{\nu}(\delta) = \boldsymbol{\nu} + \delta V + o(\delta^2)$. The scalar mean curvature on the direction $\boldsymbol{\nu}(\delta)$ has the form (here we denote $\nabla^\delta_{ij} = \nabla^{h+\delta b}_{\partial_i} \partial_j$,)

$$\begin{aligned} H_{h+\delta b, \boldsymbol{\nu}(\delta)} &= (h + \delta b)^{ij} (h + \delta b)(\nabla^\delta_{ij}, \boldsymbol{\nu}(\delta)) \\ &= h^{ij} h(\nabla^\delta_{ij}, \boldsymbol{\nu}(\delta)) + \delta h^{ij} b(\nabla^\delta_{ij}, \boldsymbol{\nu}(\delta)) - \delta h^{ik} b_{kl} h^{lj} h(\nabla^\delta_{ij}, \boldsymbol{\nu}(\delta)) + o(\delta^2) \end{aligned}$$

First we expand the Christoffel symbols ($\Gamma^k_{ij}(\delta)$ denotes the perturbed symbol, Γ^k_{ij} corresponds to h):

$$\Gamma^k_{ij}(\delta) = \Gamma^k_{ij} + \frac{\delta}{2} h^{km}\gamma^{(b)}_{mij} + o(\delta^2),$$

where $\gamma^{(b)}_{mij} = b_{mi;j} + b_{mj;i} - b_{ij;m}$ (covariant derivative with respect to h), and we denote by $T^{(b)}$ the (1,2)-tensor over $\mathcal{S}$ with components $h^{km}\gamma^{(b)}_{mij}$. The expansion of ∇^δ_{ij} is then

$$\nabla^\delta_{ij} = \nabla_{ij} + \frac{\delta}{2} T^{(b)}_{ij} + o(\delta^2),$$

and the perturbed scalar mean curvature is

$$\begin{aligned} H_{h+\delta b, \boldsymbol{\nu}(\delta)} = {} & h^{ij} h(\nabla_{ij}, \boldsymbol{\nu}(\delta)) + \frac{\delta}{2} h^{ij} h(T^{(b)}_{ij}, \boldsymbol{\nu}(\delta)) \\ & + \delta h^{ij} b(\nabla_{ij}, \boldsymbol{\nu}(\delta)) - \delta h^{ik} b_{kl} h^{lj} h(\nabla_{ij}, \boldsymbol{\nu}(\delta)) + o(\delta^2), \end{aligned}$$

Expanding $\boldsymbol{\nu}(\delta)$ as well

$$\begin{aligned} H_{h+\delta b, \boldsymbol{\nu}(\delta)} = {} & H_{h,\boldsymbol{\nu}} + \delta h^{ij} h(\nabla_{ij}, V) + \frac{\delta}{2} h(\operatorname{tr}_h T^{(b)}, \boldsymbol{\nu}) \\ & + \delta h^{ij} b(\nabla_{ij}, \boldsymbol{\nu}) - \delta h^{ik} b_{kl} h^{lj} h(\nabla_{ij}, \boldsymbol{\nu}) + o(\delta^2). \end{aligned}$$

The term $h^{ik} b_{kl} h^{lj} h(\nabla_{ij}, \boldsymbol{\nu})$ is $\langle \mathcal{K}^h_\nu, b\rangle_h$, where $\mathcal{K}^h_\nu$ is the scalar second fundamental form of Σ in the direction $\boldsymbol{\nu}$. Observe also that, since $(h+\delta b)(\boldsymbol{\nu}(\delta), \boldsymbol{\nu}(\delta)) = 1$, we have $h(\boldsymbol{\nu}, V) = -1/2 b(\boldsymbol{\nu}, \boldsymbol{\nu}) + o(\delta)$. Applying this, and also observing that $h^{ij}\nabla_{ij} = H_{h,\boldsymbol{\nu}}\boldsymbol{\nu} + h^{ij}\nabla^\top_{ij}$ (the last term denoting the tangent part to Σ), and also decomposing $V = V^\top + V^{\perp,h}$, we finally obtain

$$\begin{aligned} H_{h+\delta b, \boldsymbol{\nu}(\delta)} = {} & H_{h,\boldsymbol{\nu}} - \delta \langle \mathcal{K}^h_\nu, b\rangle_h + \frac{\delta}{2} h(\operatorname{tr}_h T^{(b)}, \boldsymbol{\nu}) + \frac{\delta}{2} H_{h,\boldsymbol{\nu}} b(\boldsymbol{\nu}, \boldsymbol{\nu}) \\ & + \delta h^{ij} h(\nabla^\top_{ij}, V^\top) + \delta h^{ij} b(\nabla^\top_{ij}, \nu) + o(\delta^2) \end{aligned}$$

so the linearization is

$$\begin{aligned} \lim_{\delta \to 0} \frac{H_{h+\delta b, \boldsymbol{\nu}(\delta)} - H_{h,\boldsymbol{\nu}}}{\delta} = {} & - \langle \mathcal{K}^h_\nu, b\rangle_h + \frac{1}{2} h(\operatorname{tr}_h T^{(b)}, \boldsymbol{\nu}) + \frac{1}{2} H_{h,\boldsymbol{\nu}} b(\boldsymbol{\nu}, \boldsymbol{\nu}) \\ & + h^{ij} h(\nabla^\top_{ij}, V^\top) + h^{ij} b(\nabla^\top_{ij}, \nu). \end{aligned} \tag{4.5}$$

Adding (4.4) and (4.5),

$$\begin{aligned} \left.\frac{\partial \theta_+}{\partial(h, \mathcal{K})}\right|_{(h,\mathcal{K},i)} (b, \mathcal{Q}) = {} & \operatorname{tr}_h \mathcal{Q} - \langle b, \mathcal{K}\rangle_h - \langle \mathcal{K}^h_\nu, b\rangle_h + \frac{1}{2} h(\operatorname{tr}_h T^{(b)}, \boldsymbol{\nu}) + \frac{1}{2} H_{h,\boldsymbol{\nu}} b(\boldsymbol{\nu}, \boldsymbol{\nu}) \\ & + h^{ij} h(\nabla^\top_{ij}, V^\top) + h^{ij} b(\nabla^\top_{ij}, \nu). \end{aligned}$$

4.3 Geometric Interpretation

With the abstract analytical machinery in place, a suitable infinite dimensional manifold structure and θ_+ all well established, we now give our main results.

To simplify our language, let us introduce some terminology: say that a triple $(h, \mathcal{K}, \psi)$ is a *MOTS triple* if $\theta_+(h, \mathcal{K}, \psi) = 0$ (i.e. the embedding ψ is a MOTS in the initial data $(h, \mathcal{K})$), and call a MOTS triple $(h, \mathcal{K}, \psi)$ *non-degenerate* if the associated MOTS stability operator has a non-zero principal eigenvalue (in such case the stability operator is said to be *non-degenerate*).

Theorem 4.3.1. *Let $(h_0, \mathcal{K}_0)$ be initial data set satisfying the no KIDs condition, $\psi_0 : \Sigma \to \mathcal{S}$ an embedding that is a MOTS for the initial data $(h_0, \mathcal{K}_0)$ on $\mathcal{S}$, and consider the neighborhood $\mathcal{A}_0$ where θ_+ is defined. Assume θ_+ satisfies the T condition (remark 4.1.6). Then there exists an open neighborhood $\mathcal{O} \subseteq ID(\mathcal{S})$ around $(h_0, \mathcal{K}_0)$ where the set*

$$\mathcal{G} = \{(h, \mathcal{K}) \in \mathcal{O} : \text{every MOTS in } (h, \mathcal{K}) \text{ are non-degenerate}\} \tag{4.6}$$

is a residual set in $\mathcal{O}$.

Proof. Since the MOTS stability operator is a Fredholm operator of index 0, we are in conditions to apply theorem 4.1.7: here, $M = \theta_+^{-1}(0)$ is the submanifold in $\mathcal{A}_0$ of MOTS triples. By item (iii), it is a generic property in $\Pi(\mathcal{A}_0) = \mathcal{O}$ the following: initial data $(h, \mathcal{K})$ such that the functional

$$\psi \in (A_0)_{(h,\mathcal{K})} \mapsto \theta_+(h, \mathcal{K}, \psi)$$

has no critical points. Having no critical points, then any MOTS embedding ψ for such $(h, \mathcal{K})$ is such that the associated stabability operator is non-degenerate. ■

Remark 4.3.2. What theorem 4.3.1 says about the residual set $\mathcal{G} \subseteq \mathcal{O}$ is that if *there is a MOTS in the initial data $(h, \mathcal{K}) \in \mathcal{G}$, then this MOTS is non-degenerate*. However our method cannot guarantee that such a MOTS exists for initial data in $\mathcal{G}$. In order to obtain this last condition, we have to make extra assumptions about the fiducial MOTS triple $(h_0, \mathcal{K}_0, \psi_0)$. We cite two distinct situations where this is possible:

(1) Assume that $(h_0, \mathcal{K}_0, \psi_0)$ is itself non-degenerate. Then by condition T the partial derivative with respect to the embeddings variable at $(h_0, \mathcal{K}_0, \psi_0)$ is an isomorphism (this can be more clearly seen from the abstract case in theorem 4.1.7). Then by a straightforward application of the implicit function theorem in Banach spaces (cf. Abraham, Marsden, and Ratiu (2012), thm. 2.5.7) We can reduce $\mathcal{A}_0$ to a open set of the form $\mathcal{U}_0 \times \mathcal{V}_0$, $(h_0, \mathcal{K}_0) \in \mathcal{U}_0$, $\psi_0 \in \mathcal{V}_0$, where now all initial data in $\mathcal{O} = U_0$ have a non-degenerate MOTS, so when the fiducial MOTS triple is non-degenerate we have obtained that existence of non-degenerate MOTS for initial data in $\mathcal{U}_0$ is *stable*. (Actually, this argument bypasses the generic step (iii) in theorem 4.1.7 thanks to condition T and the implicit function theorem.)

(2) For this second case, we restrict $(h_0, \mathcal{K}_0)$ and the dimension and topology of $\mathcal{S}$ to the following theorem due to Andersson, Eichmair, and Metzger (2011):

Theorem 4.3.3 (Andersson, Eichmair, and Metzger (2011), thm. 3.3). *Let $(h, \mathcal{K})$ be a initial data set for $\mathcal{S}$, where $3 \leq \dim \mathcal{S} \leq 7$. Assume that there is a connected bounded open set $\Omega \subseteq \mathcal{S}$ with smooth embedded boundary $\partial\Omega$. Assume this boundary consists of two non-empty closed hypersurfaces $\partial_+\Omega$ and $\partial_-\Omega$, possibly consisting of several components, so that*

$$H_{\partial_+\Omega} - \mathrm{tr}_{\partial_+\Omega}\mathcal{K} > 0 \quad \textit{and} \quad H_{\partial_-\Omega} + \mathrm{tr}_{\partial_-\Omega}\mathcal{K} > 0, \tag{4.7}$$

where the mean curvature scalar is computed as the tangential divergence of the unit normal vector field that is pointing out of Ω. Then there exists a smooth closed embedded MOTS[6] *$\Sigma \subseteq \Omega$ for the initial data $(h, \mathcal{K})$ that is homologous to $\partial_+\Omega$.* ■

By restricting the topology and dimension of $\mathcal{S}$ as in theorem 4.3.3, consider an initial data $(h_0, \mathcal{K}_0)$ for $\mathcal{S}$, and then a MOTS ψ_0 for this initial data is obtained by the latter theorem. This MOTS triple $(h_0, \mathcal{K}_0, \psi_0)$ is now our fiducial MOTS triple, that may or may not be non-degenerated. The point is that conditions (4.7) are open, so there is an open set $\mathcal{U}_0$ around $(h_0, \mathcal{K}_0)$ where *all initial data in $\mathcal{U}_0$ have a MOTS* by theorem 4.3.3. Applying theorem 4.3.1 under these conditions, the residual set $\mathcal{G}$ will have the property that *all initial data $(h, \mathcal{K})$ in $\mathcal{G}$ have a non-degenerate MOTS.*

Now let us organize theorem 4.3.1 together with conditions laid out in remark 4.3.2 as to obtain a generic condition for incompleteness.

Consider a MOTS triple $(h_0, \mathcal{K}_0, \psi_0)$ and its open neighborhood $\mathcal{A}_0$ where θ_+ is defined, assume now that $\psi_0(\Sigma)$ separates $\mathcal{S}$ as defined section 1.6.1. Reducing $\mathcal{A}_0$ to a smaller open set we can assume that all embeddings separate $\mathcal{S}$. By theorem 4.3.1 there is an open set $\mathcal{O}$ around $(h_0, \mathcal{K}_0)$ where the property (4.6) is generic in $\mathcal{O}$. If $(h, \mathcal{K}) \in \mathcal{G}$ *has a MOTS*, this MOTS is non-degenerate, therefore we are in conditions to apply theorem 1.6.3, and MOTS for $(h, \mathcal{K})$ can be deformed to be outer-trapped, then the maximal Cauchy development of $(h, \mathcal{K})$ (cf. theorem 1.5.3) is null incomplete.

Following remark 4.3.2, by suitably reducing $\mathcal{A}_0$, under condition (1) null imcompleteness is now *stable* around $(h_0, \mathcal{K}_0)$, and under condition (2) it is *generic* in an open subset around $(h_0, \mathcal{K}_0)$. We summarize this in the following theorem:

Theorem 4.3.4. *Let $(h_0, \mathcal{K}_0)$ be initial data set satisfying the no KIDs condition, $\psi_0 : \Sigma \to \mathcal{S}$ an embedding that is a MOTS for the initial data $(h_0, \mathcal{K}_0)$, where $\psi_0(\Sigma)$ separates $\mathcal{S}$. Consider the open set $\mathcal{A}_0$ around $(h_0, \mathcal{K}_0, \psi_0)$ where θ_+ is defined and all embeddings separates $\mathcal{S}$, and also assume that θ_+ satisfies the T condition. Under these assumptions,*

[6] Other properties of Σ are obtained in Andersson, Eichmair, and Metzger (2011), but we need not concerned about them here.

(1) *if* $(h_0, \mathcal{K}_0, \psi_0)$ *is itself non-degenerate, then there is an open set* $\mathcal{O}$ *around* $(h_0, \mathcal{K}_0)$ *where all initial data* $(h, \mathcal{K}) \in \mathcal{O}$ *are such that their maximal Cauchy development of* $(h, \mathcal{K})$ *is null incomplete (null incompleteness is* stable*);*

(2) *if* $(h_0, \mathcal{K}_0, \psi_0)$ *is a MOTS triple originating from theorem 4.3.3 (where the topology and dimensions of* $\mathcal{S}$ *have to be further restricted) then there is an open set* $\mathcal{O}$ *around* $(h_0, \mathcal{K}_0)$ *and a residual set* $\mathcal{G} \subseteq \mathcal{O}$ *where all initial data* $(h, \mathcal{K}) \in \mathcal{G}$ *are such that their maximal Cauchy development of* $(h, \mathcal{K})$ *is null incomplete (null incompleteness is* generic*).* ■

A Whitney Topologies

In this appendix we review the elements of differential topology pertinent to this thesis on jet bundles and Whitney topologies and the associated Baire proprieties of such topologies (for more on this subject see, e.g., Hirsch (1976) or Mukherjee (2015) for a comprehensive introduction to Whitney topologies, and Mather (1969) for more specific technical results).

A.1 Notation

We establish basic notation that recurs while discussing jets of functions. If $f : U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is a function of class C^r, we denote by $D^k f_p$, $k \leq r$ its kth order derivative at a point $p \in U$. This can be identified with a symmetric k-multilinear mapping. We denote the set of symmetric k-multilinear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$ by $S^k(m, n)$. In particular, $S^1(m, n)$ is the set of linear mappings $\mathbb{R}^m \to \mathbb{R}^n$.

With k-multilinear mappings we can define polynomials. Given $A \in S^l(m, n)$, $x \in \mathbb{R}^m$, we denote by $A(x)^l = A(x, \dots, x)$ (the same vector x repeated l times as an argument of A). For $A_j \in S^j(m, n)$, $j \leq k$ and $y_0 \in \mathbb{R}^n$, a polynomial of several variables of degree k is a mapping $p : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$p(x) = y_0 + A_1(x) + A_2(x)^2 + \cdots + A_k(x)^k.$$

To define jets it is convenient to work with polynomials without the constant term ($y_0 = 0$). We denote the set of such k-degree polynomials by $P^k(m, n)$. There is a natural identification

$$P^k(m, n) \cong S^1(m, n) \times S^2(m, n) \times \cdots \times S^k(m, n).$$

To simplify the notation of polynomials in several variables and multiple partial derivatives, we introduce the idea of *m-multi-indices* which is an m-tuple of positive integers $\alpha = (\alpha_1, \dots, \alpha_m)$. We write $|\alpha| = \alpha_1 + \cdots + \alpha_m$, and for any $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ we use the notation $x^\alpha = x_1^{\alpha_1} \dots x_m^{\alpha_m}$. A partial derivative of order $|\alpha| \leq r$ for the function f is denoted by

$$\frac{\partial^\alpha f}{\partial x^\alpha} = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}}.$$

A.2 Space of Jets

Here we present a brief summary of the construction of space of r-jets as a smooth fiber bundle; the main reference here is Mukherjee (2015) (see chapter 8 for the details and omitted computations).

Let M and N be smooth manifolds of dimensions m and n, respectively, both with empty boundary[1]. Given $p \in M$ and $r \in \mathbb{N}$, consider the following relation between functions $f, g \in C^\infty(M, N)$: $f \overset{p}{\underset{r}{\sim}} g$ if

- $f(p) = g(p)$;
- Given local charts (U, φ) and (V, ψ) around p and $f(p)$, respectively, such that $f(U), g(U) \subseteq V$, and for any $k \in \mathbb{N}$ with $k \leq r$, one has

$$D^k(\psi \circ f \circ \varphi^{-1})_{\varphi(p)} = D^k(\psi \circ g \circ \varphi^{-1})_{\varphi(p)}.$$

Using standard multivariable calculus arguments, it can be shown that this relation does not depend on the choice of local charts, and therefore it is well-defined. It is also immediately observed that this relation is an equivalence relation.

Remark A.2.1. The last item in the definition of $\overset{p}{\underset{r}{\sim}}$ is equivalent to the following statement: for all m multi-indice α with $|\alpha| \leq r$,

$$\frac{\partial^\alpha(\psi \circ f \circ \varphi^{-1})}{\partial x^\alpha}(\varphi(p)) = \frac{\partial^\alpha(\psi \circ g \circ \varphi^{-1})}{\partial x^\alpha}(\varphi(p)).$$

The equivalence class of a function $f \in C^\infty(M, N)$ under the relation $\overset{p}{\underset{r}{\sim}}$ is denoted by $j^r_p f$ and called the *r-jet of f at the point p*. The set of all r-jets at the point p is denoted by $J^r_p(M, N)$. Given $q \in N$, the set of all r-jets $j^r_p f$ with $f(p) = q$ is denoted by $J^r_p(M, N)_q$. The *space of r-jets*,denoted by $J^r(M, N)$, is the disjoint union

$$J^r(M, N) = \bigsqcup_{(p,q) \in M \times N} J^r_p(M, N)_q.$$

We establish the following terminology in jet theory:

Given $j^r_p f \in J^r(M, N)$, the point $p \in M$ is called the *source* of the jet, and the point $q = f(p)$ is called the *target* of the jet. The functions

$$\begin{aligned} \sigma : J^r(M, N) &\to M & \tau : J^r(M, N) &\to N \\ j^r_p f &\mapsto p, & j^r_p f &\mapsto f(p), \end{aligned}$$

are called, respectively, the *source* and *target* maps, and are clearly surjective. The *r-jet prolongation* for $f \in C^\infty(M, N)$ is the map

$$\begin{aligned} j^r f : M &\to J^r(M, N) \\ p &\mapsto j^r_p f. \end{aligned}$$

[1]The space of jets can be developed in a similar way for manifolds with boundary, but then the resulting manifold will have corners, as developed in Michor (1980), chapter 2, or Margalef-Roig and Outerelo Domínguez (1992), chapter 1. Since our primary interest are manifolds without boundary, we will avoid such complications.

Jets as Fiber Bundles

The set $J^r(M,N)$ is endowed with the structure of a manifold and, moreover, of a fiber bundle, in the following way: given non-empty open sets $U \subseteq M$ and $V \subseteq N$, denote by $J^r(U,V) \subseteq J^r(M,N)$ the subset of r-jets $j^r_p f$ of smooth functions $f : U \to V$. Consider charts (U,φ) and (V,ψ) of M and N, respectively. It can be verified that the function $h_{U,V} : J^r(U,V) \to \varphi(U) \times \psi(V) \times P^r(m,n)$ defined by

$$h_{U,V}(j^r_p f) = (\varphi(p), \psi(f(p)), D(\psi \circ f \circ \varphi^{-1})_{\varphi(p)}), \ldots, D^r(\psi \circ f \circ \varphi^{-1})_{\varphi(p)}))$$

is a bijection (Mukherjee (2015), lemma 8.1.3). For a collection of charts $\{(U_i,\varphi_i)\}$ and $\{(V_\alpha,\psi_\alpha)\}$ of M and N that cover these manifolds, the collection of sets $J^r(U_i,V_\alpha)$ covers $J^r(M,N)$. A more extensive analysis (Mukherjee (2015), lemma 8.6.1 and theorem 8.6.2) shows that the transition maps $h_{U_i,V_\alpha} \circ h^{-1}_{U_j,V_\beta}$ are smooth diffeomorphisms. Thus, by means of usual results on the construction of smooth manifolds (for example, John M. Lee (2012), lemma 1.35), we obtain a smooth manifold structure for $J^r(M,N)$, the collection $\{(J^r(U_i,V_\alpha), h_{U_i,V_\alpha})\}$ thus being a collection of smooth charts that covers $J^r(M,N)$. With this structure, a simple computation in coordinates (Mukherjee (2015), theorem 8.6.2) shows that the source σ and target τ are smooth submersions, and also that the prolongation $j^r f$ is a smooth embedding.

Such charts, together with the source and target maps, induce a fiber bundle structure on the manifold $J^r(M,N)$ in the following way: consider the projection $\pi : J^r(M,N) \to M \times N$ defined by $\pi(j^r_p f) = (\sigma(j^r_p f), \tau(j^r_p f)) = (p, f(p))$, which is certainly surjective. We observe that $\pi^{-1}(U \times V) = J^r(U,V)$. Considering (U,φ), (V,ψ) charts of M and N, respectively, with $h_{U,V}$ the chart in $J^r(M,N)$ associated, consider the application $\Phi_{U,V} : \pi^{-1}(U \times V) \to U \times V \times P^r(m,n)$ setting[2] $\Phi_{U,V} = (\varphi^{-1} \times \psi^{-1} \times Id_{P^r(m,n)}) \circ h_{U,V}$. This mapping is a smooth diffeomorphism, being a composition of diffeomorphisms. It is readily verified that, for $pr : U \times V \times P^r(m,n) \to U \times V$ the coordinate projection, the diagram

$$\begin{array}{ccc} J^r(U,V) & \xrightarrow{\Phi_{U,V}} & U \times V \times P^r(m,n) \\ & {\scriptstyle\pi}\searrow \qquad \swarrow{\scriptstyle pr} & \\ & U \times V & \end{array}$$

commutes, showing $\Phi_{U,V}$ has the local trivialization property for the projection π, which allows us to interpret $J^r(M,N)$ as a smooth fiber bundle with base space $M \times N$ and fiber given by the polynomials $P^r(m,n)$.

[2] The Cartesian product of functions is defined by $(f \times g)(p,q) = (f(p), g(q))$.

A.3 Whitney Topologies

The main interest in jet theory is due to its use in defining a class of topologies on the set of smooth functions between two manifolds, the so-called *Whitney topologies*. The first contact with such topologies is often made by using derivatives of maps with respect to concretely chosen atlases as in the classical reference of differential topology from Hirsch (1976); indeed, it is often more useful when computations are needed; but for the purposes of this work the approach via jets is more advantageous because it gives us an invariant construction of these topologies, and such a construction allows us to easily import topological properties of the known topologies in the space of continuous functions, which we will describe shortly.

The equivalence of the coordinate version with the one via jets is not evident only from the definitions; in fact, the proof is somewhat laborious and can be found in Mukherjee (2015), propositions 8.2.8 and 8.2.9.

A.3.1 Weak and Strong Topologies in the Space of Continuous Functions

To construct Whitney topologies by means of jets, an intermediate step requires us to understand the topologies on the set of continuous functions. We briefly describe the necessary theory.

If X and Y are topological spaces, we denote by $C^0(X,Y)$ the set of continuous functions with domain in X and codomain in Y. Following Munkres (2000), chapter 7, the first relevant topologies on the set of continuous functions are given by the following two definitions.

Definition A.3.1. *If X and Y are topological spaces, the* compact-open topology *in $C^0(X,Y)$ is generated by subbases of the form*

$$S(K,V) = \{f \in C^0(X,Y) : f(K) \subseteq V, K \subseteq X \text{ compact and } V \subseteq Y \text{ open}\}. \tag{A.1}$$

Definition A.3.2. *If X and Y are topological spaces with Y metrizable with a metric d, the* compact convergence topology *on $C^0(X,Y)$ is defined by the basis sets*

$$B(f,K,\varepsilon) = \{g \in C^0(X,Y) : \sup_{x\in K} d(f(x),g(x)) < \varepsilon, K \subseteq X \textit{ compact}\}. \tag{A.2}$$

When Y is a metric space these two topologies for $C^0(X,Y)$ coincide (Munkres (2000), theorem 46.8). As an immediate corollary, if Y is metrizable, then the compact convergence topology does not depend on the choice of metric on Y.

When X,Y are manifolds, we refer to the compact-open topology (or equivalently, the compact convergence topology) as the *weak topology*, denoted by $C^0_W(X,Y)$. Since manifolds are completely metrizable topological spaces[3], the following theorem (Hirsch (1976), theorem 4.1) will be relevant.

Theorem A.3.3. *Let X be a second countable topological space with each connected component being locally compact, and Y a completely metrizable topological space. Then $C^0_W(X,Y)$ is completely metrizable.* ■

We also define another relevant topology on $C^0(X,Y)$. Here, $\mathrm{Gr}(f)$ denotes the graph set of the function f, and we consider the usual product topology on $X \times Y$.

Definition A.3.4. *The* graph topology *or* strong topology *on $C^0(X,Y)$ is the topology generated by basis*

$$B(W) = \{f \in C^0(X,Y) : \mathrm{Gr}(f) \subseteq W, W \subseteq X \times Y \text{ open}\}.$$

We will denote the strong topology by $C^0_S(X,Y)$.

With X Hausdorff and paracompact, Y metrizable, we have an alternative basis[4] associated with the metric d on Y (Mukherjee (2015), proposition 8.2.5) given as follows: for $\epsilon : X \to (0, +\infty)$ a continuous function, the basis is formed by sets of the form

$$S(f,\epsilon) = \{g \in C^0(X,Y) : d(f(x), g(x)) < \epsilon(x) \text{ for all } x \in X\}. \tag{A.3}$$

More general topological questions about the strong topology can be found in Naimpally (1966). In particular, we note that *the strong topology is finer than the weak topology*, and that if X is compact and Hausdorff, the strong and the weak topologies coincide (Mukherjee (2015), lemmas 8.2.2-8.2.4 or Naimpally (1966), theorem 4.2).

When the domain X is not compact, we find an important difference: *the strong topology is not in general metrizable.* Proposition 2 in Krikorian (1969) shows that in general $C^0(X,Y)$ is not first countable (and the topological conditions of this result include manifold topologies).

A.3.2 C^r Whitney Topologies

With the main results of the weak and strong topologies in the space of continuous functions given, we define the Whitney topologies by inducing the topology of $C^0(M, J^r(M,N))$ on $C^\infty(M,N)$ as follows:

Definition A.3.5. *Let M and N be smooth manifolds, and $J^r(M,N)$ the corresponding space of r-jets. Consider the function*

$$\begin{aligned} j^r : C^\infty(M,N) &\to C^0(M, J^r(M,N)) \\ f &\mapsto j^r f, \end{aligned}$$

[3] A topological space is *completely metrizable* if it is metrizable with a complete metric d.

[4] In this metrizable case, the topology generated by such bases is also found in the literature as the *fine topology*, for example Krikorian (1969).

which is known to be injective. The weak (Whitney) C^r topology *on* $C^\infty(M,N)$ *is the topology induced on* $C^\infty(M,N)$ *by the function* j^r *considering the weak topology on* $C^0(M,J^r(M,N))$. *We will denote the weak* C^r *topology by* $C^\infty_{W,r}(M,N)$.

Similarly, the strong (fine) C^r topology *on* $C^\infty(M,N)$ *is induced by the function* j^r *considering the strong topology on* $C^0(M,J^r(M,N))$. *We will denote the strong* C^r *topology by* $C^\infty_{S,r}(M,N)$.

Since j^r is injective, the map $j^r : C^\infty_{W,r}(M,N) \to C^0_W(M,J^r(M,N))$ becomes a topological embedding (the analogue for the strong C^r topology). The weak C^r topology is induced by j^r from the weak topology $C^0_W(M,J^r(M,N))$, and it is described by the subbases in (A.1); therefore a subbase for $C^\infty_{W,r}(M,N)$ is

$$\begin{aligned} S_r(K,V) &= (j^r)^{-1}(S(K,V)) \\ &= \{f \in C^\infty(M,N) : j^r f(K) \subseteq V, K \subseteq M \text{ compact and } V \subseteq J^r(M,N) \text{ open}\}. \end{aligned}$$

For the strong topology, with some more work (cf. Michor (1980), 4.4.1), one can verify that a basis for $C^\infty_{S,r}(M,N)$ can be given via sets of the form

$$B_r(V) = \{f \in C^\infty(M,N) : j^r f(M) \subseteq V, V \subseteq J^r(M,N) \text{ open}\}.$$

For a fixed metric d_r on $J^r(M,N)$, the same reasoning as in the weak C^r topology, inducing with j^r the basis in (A.2) for the C^0 weak topology, we see that it has a basis set given by

$$B_r(f,K,\varepsilon) = \{g \in C^\infty(M,N) : \sup_{p\in K}\{d_r(j^r f(p), j^r g(p))\} < \varepsilon, K \subseteq M \text{ compact}\}, \quad \text{(A.4)}$$

and similarly to (A.3), the strong C^r topology has a base

$$S_r(f,\epsilon) = \{g \in C^\infty(M,N) : d_r(j^r f(p), j^r g(p)) < \epsilon(p), \epsilon \in C^0(M,(0,+\infty))\}.$$

Remark A.3.6. Since we are inducing the C^r Whitney topologies by the function j^r via the strong or the weak topology on $C^0(M,J^r(M,N))$, if M is compact then these two topologies coincide for $C^0(M,J^r(M,N))$, therefore C^r *weak and strong topologies are equal if* M *is compact.*

Definition A.3.7 (Whitney C^∞ Topology). *Denote* $\mathcal{T}_r^W$ *and* $\mathcal{T}_r^S$ *(*$r = 0,1,2\ldots,$*) as the weak and strong* C^r *topologies, respectively. The* strong Whitney C^∞ topology *is defined by the base* $\mathcal{B}^S = \bigcup_{r=0}^\infty \mathcal{T}_r^S$, *and the* weak Whitney C^∞ topology *is defined analogously.*

Since $\mathcal{T}_s^S \subseteq \mathcal{T}_r^S$ if $s \le r$, (and the analogous statement for the weak topologies), we see that these topologies are well-defined. The strong and weak C^∞ topologies will be denoted, respectively, by $C^\infty_S(M,N)$ and $C^\infty_W(M,N)$. Also, by remark A.3.6 we see that the C^∞ topologies coincide if M is compact.

A.3.3 Convergence of Sequences

Convergence of sequences in the weak C^r topology is straightforward: using the basis set in (A.4), one easily sees that a sequence $f_n \in C^\infty(M,N)$ converges to f in the weak C^r topology if and only if, for some metric d_r defined in $J^r(M,N)$, $j^r f_n$ converges d_r-uniformly to f on each compact set of M.

For the strong C^r topology, convergence of sequences is trickier. We need to control all derivatives of the sequence (codified in the r-jet prolongation) "at infinity" (over all M), not just on compact sets, like with the weak topology. When M is compact, as already said, this is not a problem (as a matter of fact strong and weak topologies coincide in this case), the issue then lies with noncompact M. The following result gives a necessary and sufficient condition for convergence in the strong C^r topology (see Golubitsky and Guillemin (1973), pgs. 43-44, for a proof).

Proposition A.3.8. *Let M be noncompact. A sequence $f_n \in C^\infty(M,N)$ converges to $f \in C^\infty(M,N)$ in the strong C^r topology if and only if there is a compact set $K \subseteq M$ such that $j^r f_n = j^r f$ for all outside of K for all but a finite number of elements in the sequence, and $j^r f_n$ converges to $j^r f$ uniformly on K with respect to some metric d_r in $J^r(M,N)$.* ■

A.3.4 Baire Property

Defining the Whitney topologies as we have done so far allows one to verify the Baire property of the weak and strong topologies via the respective properties on $C^0(X,Y)$.

Definition A.3.9. *A topological space X is said to be a* Baire space *if for any countable collection of dense open sets $\{A_n\}$ of X the intersection $\bigcap_n A_n$ is dense in X.*

The Baire category theorem then states that completely metrizable spaces are Baire spaces, so from theorem A.3.3:

Corollary A.3.10. *If X is second countable and locally compact, and Y is completely metrizable, then $C^0_W(X,Y)$ is a Baire space.* ■

As observed earlier, the strong topology is not metrizable, but we still have the Baire property. If Y is metrizable, then a subset $F \subseteq C^0(X,Y)$ is said to be *uniformly closed* if every sequence in F that converges uniformly is convergent to a function in F. In particular, weakly closed sets of $C^0(X,Y)$ (i.e. closed in the weak topology) are uniformly closed. We have the following general result (Hirsch (1976), theorem 4.2):

Theorem A.3.11. *Let X be paracompact and Y be metrizable. If $F \subseteq C^0(X,Y)$ is uniformly closed, then F is a Baire space in the strong topology (i.e. in the subspace topology of $C^0_S(X,Y)$). In particular $C^0_S(X,Y)$ is a Baire space.* ■

Connecting these results with the Whitney topologies goes through the following lemma (Michor (1980), lemma 4.2 or Hirsch (1976), theorem 4.3):

Lemma A.3.12. *The image of the function* $j^r : C^\infty(M,N) \to C^0(M, J^r(M,N))$ *is closed in the weak topology.* ■

With $\operatorname{Im} j^r$ closed in $C^0_W(M, J^r(M,N))$ a completely metrizable space, $\operatorname{Im} j^r$ is completely metrizable as a subspace in the weak topology, and therefore a Baire space by corollary A.3.10. Also, $\operatorname{Im} j^r$ in the weak topology is homeomorphic to $C^\infty_{W,r}(M,N)$, therefore *the weak* C^r *topology is completely metrizable and a Baire space*.

For the strong C^r topology, we have $\operatorname{Im} j^r$ weakly closed in $C^0_S(M, J^r(M,N))$, therefore a Baire space as a subspace in the strong topology by theorem A.3.11, and it being again homeomorphic to $C^\infty_{S,r}(M,N)$, the latter is a Baire space.

We can also verify the Baire property for C^∞ topologies. For the weak topology, a straightforward way is to work with *projective limits* for topological spaces (cf. Ribes and Zalesskii (2010), chapter 1). Consider the collection of maps

$$I_{r,s} : C^\infty_{W,r}(M,N) \to C^\infty_{W,s}(M,N), \quad s \le r,$$

where $I_{r,s}$ is the identity map. It can be easily verified that the projective limit of the collection $\{C^\infty_{W,r}(M,N), I_{r,s}, \mathbb{N}\}$ is $C^\infty_W(M,N)$ (with the same being true for the strong topology).

In the case of the weak topology, since all the $C^\infty_{W,r}(M,N)$ are completely metrizable, the Cartesian product of these spaces is completely metrizable. Now the projective limit is homeomorphic to a closed subset of the Cartesian product (Ribes and Zalesskii (2010, lemma 1.1.2)), and it follows that $C^\infty_W(M,N)$ is completely metrizable, and in particular a Baire space.

Despite the strong C^∞ topology not having the good properties of the weak topology, it is also true that $C^\infty_S(M,N)$ is a Baire space, the proof (see Gravesen (1983)) follows an argument with similar strucure to the proof of the usual Baire category theorem for complete metric spaces.

A.3.5 Thom's Tranversality

We finish with a brief note on transversality. We leave the general notions and basic results of transversality to standard manifold theory literature (e.g. John M. Lee (2012), chapter 6.)

Definition A.3.13. *Let* $F : M \to N$ *be a smooth function and let* $S \subseteq N$ *be an embedded submanifold. We say that* F is transverse to S, *denoted by* $F \pitchfork S$, *if for every* $p \in M$, *either* $F(p) \notin S$, *or else*

$$dF_p(T_pM) + T_{F(p)}S = T_{F(p)}N.$$

Theorem A.3.14 (Thom Transversality Theorem). *Let $Z \subseteq J^r(M, N)$ be an embedded submanifold. Then the set*

$$\mathcal{T}(Z) = \{f \in C^\infty(M, N) : j^r f \pitchfork Z\}$$

is a residual set in $C^\infty_S(M, N)$, and is in particular dense. If Z is closed, then $\mathcal{T}(Z)$ is an open dense set ■

The proof of the Thom transversality theorem is fairly elaborated and can be seen in Mukherjee (2015), section 8.7, or Golubitsky and Guillemin (1973), chapter 2-§5.

Corollary A.3.15 (Elementary Transversality Theorem). *Let Z be a submanifold of N. Then*

$$\mathcal{W}(Z) = \{f \in C^\infty(M, N) : f \pitchfork Z\}$$

is a dense set in $C^\infty_S(M, N)$. If Z is closed, then $\mathcal{W}(Z)$ is an open dense set. ■

B MOTS Stability Operator

Here we give a brief survey of MOTS and the MOTS stability operator, and present some of its main properties.

B.1 MOTS Stability Operator

We now return to the notion of the MOTS stability operator and its properties. This section is an expansion on the discussion done in section 1.5.2, where the necessary theory for the main text was given. A fairly detailed account with explicit calculations of the results cited here can be found in Hafemann (2023) (especially appendix A, where an in depth computation of the time derivative of θ_+ is given).

We recall that a MOTS on a spacetime (M, g) is a surface Σ, or more generally a codimension two immersion $\psi : \Sigma \to M$, with trivial normal bundle such that its null expansion scalar θ_+ is zero on Σ (cf. section 1.4). To transition for the initial data version of MOTS, the following result gives us a way to reinterpret θ_+ solely from the information of the immersion.

Proposition B.1.1. *Let (M^{n+1}, g) be a spacetime, $\phi : \mathcal{S}^n \to M^{n+1}$ be a spacelike immersion with $\mathbf{u} \in \mathfrak{X}^\perp(\phi)$ the unique unit future-directed timelike normal vector field and $\psi : \Sigma^{n-1} \to S^n$ a two-sided immersion with $\mathbf{v} \in \mathfrak{X}^\perp(\psi)$ a unit normal. Defining the the future-directed null vector fields as $\ell_\pm := (\mathbf{u} \circ \psi \pm d\phi \circ \mathbf{v})/\sqrt{2} \in \mathfrak{X}^\perp(\phi \circ \psi)$, it follows that $\phi \circ \psi$ is a spacelike immersion of codimension two with trivial normal bundle and the null expansion given by*

$$\theta_\pm = \operatorname{tr}_\Sigma \mathcal{K} \circ \psi \pm H^\psi_{\mathbf{v}}, \tag{B.1}$$

where $H^\psi_{\mathbf{v}}$ is the mean curvature scalar of ψ with respect to the normal $\mathbf{v}$ and $\mathcal{K}$ is the second fundamental form of ϕ with respect to the normal $\mathbf{u}$ and the partial trace is respect to the induced metric. ■

With this for of the null expansions scalars, the definition of a MOTS in initial data sets is well motivated.

Definition B.1.2 (Null expansion and MOTS - Initial data version). *Let $(\mathcal{S}, h, \mathcal{K})$ be an initial data set and $\psi : \Sigma \to \mathcal{S}$ be a two-sided immersion with $\mathbf{v} \in \mathfrak{X}(\psi)$ as the outward pointing unit normal vector field of ψ. The outward* null expansion *θ_+ [resp. inward* null expansion *θ_-] of Σ is defined as*

$$\theta_\pm = \operatorname{tr}_\Sigma \mathcal{K} \circ \psi \pm H^\psi_{\mathbf{v}}, \tag{B.2}$$

where $H^\psi_{\mathbf{v}}$ is the mean curvature scalar of ψ with respect to the normal $\mathbf{v}$ and $\mathcal{K}$ is the second fundamental form of ψ with respect to the normal $\mathbf{v}$ and the partial trace is in respect to the induced metric. For the sign of θ_+ we then define Σ to be

- *outer trapped if* $\theta_+ < 0$,
- *weakly outer trapped if* $\theta_+ \leq 0$,
- *marginally outer trapped if* $\theta_+ = 0$.

Some special classes of MOTS are important in some contexts.

Definition B.1.3 (Homologous Surfaces). *Let* $(\mathcal{S}^n, h, \mathcal{K})$ *be an initial data. A pair of codimension one surfaces* Σ *and* Σ' *in* $\mathcal{S}$ *are said to be* homologous *if there exists a smooth map* $\Phi : (a, b) \times \Sigma \to \mathcal{S}$ *satisfying*

(i) $[0, 1] \subseteq (a, b)$,

(ii) *for each* $t \in (a, b)$*, the map* $\phi_t : x \in \Sigma \mapsto \Phi(t, x) \in \mathcal{S}$ *is an embedding,*

(iii) $\phi_0 = id_\Sigma$ *and* $\phi_1(\Sigma) = \Sigma'$.

We say that Σ *and* Σ' *are* outward homologous *if the variation vector field* $V = \frac{\partial\Phi}{\partial t}\big|_{t=0}$ *is equal to* $V = f\boldsymbol{\nu}$*, where* $\boldsymbol{\nu}$ *the outward pointing unit normal vector field of* ϕ_0*, for some strictly positive function* $f \in C^\infty(\Sigma)$.

Definition B.1.4 (Outermost MOTS). *Let* Σ *be a MOTS in an initial data set* $(\mathcal{S}^n, h, \mathcal{K})$ *with* $\boldsymbol{\nu}$ *an outward pointing unit normal vector field of* Σ *in* $\mathcal{S}$.

(i) *We say that* Σ *is* outermost MOTS *in* $\mathcal{S}$ *if there are no outer trapped* $(\theta_+ < 0)$ *or marginally outer trapped* $(\theta_+ = 0)$ *surfaces outward homologous to* Σ.

(ii) *We say that* Σ *is a* weakly outermost MOTS *in* $\mathcal{S}$ *provided there are no outer trapped surfaces* $(\theta_+ < 0)$ *outward homologous to* Σ.

Given a spacetime (M^{n+1}, g) let $\Sigma^{n-1} \subseteq M$ be a smooth closed (compact without boundary) codimension two spacelike submanifold with trivial normal bundle. For $\ell_\pm$ the future-directed normal null vector fields on Σ normalized with $g(\ell_+, \ell_-) = -1$, we shall assume that Σ is a MOTS with respect to ℓ_+ $(\theta_+ = 0)$. For convenience, we also define on Σ the normal unit timelike vector field $\mathbf{u} = (\ell_+ + \ell_-)/\sqrt{2}$ and normal unit spacelike vector field $\boldsymbol{\nu} = (\ell_+ - \ell_-)/\sqrt{2}$. Finally, let $\Phi : (-t_0, t_0) \times \Sigma \to M$ be a smooth variation of Σ in M with a normal variation vector field V. As a result, the normal vector field V can be decomposed into

$$V = \frac{\partial\Phi}{\partial t}\bigg|_{t=0} = \phi\ell_+ + \psi\boldsymbol{\nu}, \quad \phi, \psi \in C^\infty(\Sigma).$$

We shall in addition assume that a smooth choice was made on each $\Sigma_t = \phi_t(\Sigma)$ of future-directed null normal vector fields $\ell_\pm(t)$ so that $\ell_\pm(0) = \ell_\pm$, and $\langle\langle \ell_+(t), \ell_-(t)\rangle\rangle = -1$. Thus, denote by $\theta_+(t)$ the null expansion with respect to $\ell_+(t)$ (that is, $\theta_+(t) = -g(\ell_+(t), H^{\phi_t})$). With this convention we obtain, after some hefty computations for the derivative of θ_+, the key form of $\theta'_+(0)$.

Proposition B.1.5. *Let Σ^{n-1} be a MOTS within a spacetime (M^{n+1}, g). Let $\Phi : (-t_0, t_0)\times\Sigma \to M$ be a variation with normal variation vector field $V = \phi\ell_+ + \psi\boldsymbol{\nu}$. Then, the variation of the null expansion scalar $\theta_+(t)$ on Σ in the direction of the variation vector field V is given by*

$$\theta'_+(0) = -(|\chi_+|^2 + Ric_g(\ell_+, \ell_+)) \cdot \phi + L(\psi), \tag{B.3}$$

where

$$L(\psi) = -\Delta\psi + 2\langle X, \operatorname{grad}\psi\rangle + (Q + \operatorname{div} X - \|X\|^2)\psi, \tag{B.4}$$

$$Q = \frac{1}{2}\mathrm{Scal}_\Sigma - [J(\boldsymbol{\nu}) + \rho] - \frac{1}{2}|\chi_+|^2. \tag{B.5}$$

The differential operators and scalar curvature defined for the induced metric in Σ, X is the vector field on Σ metrically dual - also with the induced metric - to the one-form $\mathcal{K}_{\mathbf{u}}(\boldsymbol{\nu}, \cdot)|_{T\Sigma}$ and where ρ and J are, respectively, the energy density and energy-momentum density associated with the timelike vector field **u**. ■

Proposition B.1.5 allows us to define a linear second-order elliptic differential operator L, called the *MOTS stability operator*. Furthermore, although we have considered a MOTS in a spacetime, we can give a purely initial-data description. This is motivated by the following observation.

Suppose that, in addition to the conventions adopted for proposition B.1.5, we have $\Sigma^{n-1} \subseteq \mathcal{S}^n$, where is a $\mathcal{S}$ spacelike hypersurface in the spacetime (M^{n+1}, g), with the unique unit normal future-directed timelike normal vector field U, induced metric h and second fundamental form $\mathcal{K}$ with respect to U. Assume also that an h-unit normal vector field $\boldsymbol{\nu}$ is chosen on Σ, so that $\ell_\pm = (U|_\Sigma \pm \boldsymbol{\nu})/\sqrt{2}$. Setting $\mathbf{u} = U|_\Sigma$ and noting that $\mathcal{X}_+ = \mathcal{K}_{\boldsymbol{\nu}} + \mathcal{K}_{\mathbf{u}}$, where $\mathcal{K}_{\boldsymbol{\nu}}$ is the second-fundamental form of Σ associated with the normal vector field $\boldsymbol{\nu}$. This motivates the following definition.

Definition B.1.6 (MOTS Stability Operator - Initial Data Version). *Let Σ be a closed MOTS (compact without boundary) within an initial data $(\mathcal{S}^n, h, \mathcal{K})$. We define the* MOTS stability operator $L : C^\infty(\Sigma) \to C^\infty(\Sigma)$ *to be*

$$L(\psi) = -\Delta\psi + 2\langle X, \operatorname{grad}\psi\rangle + (Q + \operatorname{div} X - \|X\|^2)\psi, \tag{B.6}$$

$$Q = \frac{1}{2}\mathrm{Scal}_\Sigma - [J(\boldsymbol{\nu}) + \rho] - \frac{1}{2}|\mathcal{K}_{\boldsymbol{\nu}} + \mathcal{K}|^2, \tag{B.7}$$

where the geometric quantities are defined on Σ, $\boldsymbol{\nu}$ is the outward pointing unit normal vector field on Σ, $\mathcal{K}_{\boldsymbol{\nu}}$ is the scalar second fundamental form of Σ wrt. the induced metric from $(\mathcal{S}, h)$ on the direction $\boldsymbol{\nu}$, X is the vector field dual to the one-form $\mathcal{K}(\boldsymbol{\nu}, \cdot)$ along Σ and where ρ and J are defined as in Definition 1.5.2.

It is worth mentioning that the MOTS stability operator can be derived from normal variations in the initial data. In the case of time-symmetric initial data ($\mathcal{K} = 0$), the operator L reduces to the self-adjoint, classic stability (or Jacobi) operator of the minimal surface theory, which consists of the second variation of the volume. Although the operator L is not self-adjoint in general, the operator possesses crucial properties for its spectrum.

Proposition B.1.7 (Andersson, Mars, and Simon (2008), Galloway (2018)). *Let Σ be a closed MOTS (compact without boundary) within an initial data set $(\mathcal{S}^n, h, \mathcal{K})$. The following statements hold for the MOTS stability operator L.*

(1) *There is a real eigenvalue $\lambda_1 = \lambda_1(L)$, called the* principal eigenvalue *of L, such that for any other eigenvalue μ, $Re(\mu) \geq \lambda_1$. The associated eigenfunction $\phi \in C^\infty(\Sigma)$, $L\phi = \lambda_1\phi$, is unique up to a multiplicative constant, and can be chosen to be strictly positive.*

(2) *$\lambda_1 \geq 0$ (resp., $\lambda_1 > 0$) if only if there exist some $\psi \in C^\infty(\Sigma), \psi > 0$, such that $L(\psi) \geq 0$ (resp., $L(\psi) > 0$).* ■

References

ABRAHAM, Ralph; MARSDEN, Jerrold E; RATIU, Tudor. **Manifolds, tensor analysis, and applications**. [S.l.]: Springer Science & Business Media, 2012. v. 75.

ALIAS, Luis J; PICCIONE, Paolo. On the manifold structure of the set of unparameterized embeddings with low regularity. **Bulletin of the Brazilian Mathematical Society, New Series**, Springer, v. 42, p. 171–183, 2011.

ANDERSSON, Lars; EICHMAIR, Michael; METZGER, Jan. Jang's equation and its applications to marginally trapped surfaces. In: COMPLEX Analysis and Dynamical Systems IV: General relativity, geometry, and PDE (Contemporary Mathematics, 554). [S.l.: s.n.], 2011. P. 13–46.

ANDERSSON, Lars; MARS, Marc; SIMON, Walter. Stability of marginally outer trapped surfaces and existence of marginally outer trapped tubes. **Advances in Theoretical and Mathematical Physics**, International Press of Boston, v. 12, n. 4, p. 853–888, 2008.

AUBIN, Thierry. **Some nonlinear problems in Riemannian geometry**. Berlin ; New York: Springer, 1998. P. 394. (Springer monographs in mathematics).

BACHMAN, G.; NARICI, L. **Functional Analysis**. [S.l.]: Dover Publications, 2000. (Academic Press textbooks in mathematics).

BARTNIK, Robert A. Phase space for the Einstein equations. **Communications in Analysis and Geometry**, International Press of Boston, Inc., v. 13, n. 5, p. 845–885, 2005.

BEEM, John K.; EHRLICH, Paul E.; EASLEY, Kevin L. **Global Lorentzian Geometry**. 2. ed. New York: CRC Press, 1999. P. 656.

BEEM, John K.; HARRIS, Steven G. The generic condition is generic. **General Relativity and Gravitation**, v. 25, n. 9, p. 939–962, Sept. 1993. DOI: 10.1007/BF00759194.

BERNAL, Antonio N; SÁNCHEZ, Miguel. On Smooth Cauchy Hypersurfaces and Geroch's Splitting Theorem. **Communications in Mathematical Physics**, v. 243, n. 3, p. 461–470, 2003.

BILIOTTI, Leonardo; JAVALOYES, Miquel Angel; PICCIONE, Paulo. Genericity of nondegenerate critical points and Morse geodesic functionals. **Indiana Univ. Math. J.**, v. 58, p. 1797–1830, 4 2009.

CHOQUET-BRUHAT, Yvonne; GEROCH, Robert. Global aspects of the Cauchy problem in general relativity. **Communications in Mathematical Physics**, v. 14, n. 4, p. 329–335, Dec. 1969. DOI: 10.1007/BF01645389.

CHRUŚCIEL, Piotr T; GALLOWAY, Gregory J. Outer trapped surfaces are dense near MOTSs. **Classical and Quantum Gravity**, v. 31, n. 4, p. 11, 2014. DOI: 10.1088/0264-9381/31/4/045013.

CHRUŚCIEL, Piotr T.; DELAY, Erwann. Manifold structures for sets of solutions of the general relativistic constraint equations. **Journal of Geometry and Physics**, v. 51, n. 4, p. 442–472, 2004. DOI: https://doi.org/10.1016/j.geomphys.2003.12.002.

________. On mapping properties of the general relativistic constraints operator in weighted function spaces, with applications. en. Société mathématique de France, n. 94, 2003. DOI: 10.24033/msmf.407. Available from:
<http://www.numdam.org/item/MSMF_2003_2_94__1_0/>.

COLBOIS, Bruno. **Laplacian on Riemannian manifolds**. 2010. Available from:
<https://faculty.fiu.edu/~lhermi/dido/colbois-course.pdf>.

COOK, Gregory B. Initial Data for Numerical Relativity. **Living Reviews in Relativity**, Springer Science and Business Media, v. 3, n. 1, 2000. DOI: 10.12942/lrr-2000-5.

COSTA E SILVA, Ivan Pontual. **Lecture Notes on Semi-Riemannian Geometry**. [S.l.: s.n.]. No prelo 2020.

EINSTEIN, Albert. Die Feldgleichungen der Gravitation. **Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin**, p. 844–847, 1915.

ESPINOZA, Victor Luis. **Linhas e raios geodésicos causais em espaços-tempos com aplicações à relatividade**. 2020. Dissertação de Mestrado – Universidade Federal de Santa Catarina, Florianópolis. Available from:
<https://repositorio.ufsc.br/handle/123456789/216502>.

GALLOWAY, Gregory J; SENOVILLA, José M M. Singularity theorems based on trapped submanifolds of arbitrary co-dimension. **Classical and Quantum Gravity**, v. 27, n. 15, p. 152002, 2010. DOI: 10.1088/0264-9381/27/15/152002.

GALLOWAY, Gregory J. Rigidity of outermost MOTS: the initial data version. **General Relativity and Gravitation**, Springer Science and Business Media LLC, v. 50, n. 3, Feb. 2018.

GOLUBITSKY, Martin; GUILLEMIN, Victor. **Stable mappings and their singularities**. Berlin Heidelberg New York, N.Y: Springer, 1973. (Graduate texts in mathematics, 14).

GRAVESEN, Jens. Whitney C^∞-Topologies and the Baire Property. **Mathematica Scandinavica**, Mathematica Scandinavica, v. 52, n. 1, p. 58–60, 1983. Available from:
<http://www.jstor.org/stable/24491467>.

HAFEMANN, Eduardo. **Geometry and topology of black hole horizons**. 2023. Masters Thesis – Universidade Federal de Santa Catarina. Available from:
<https://repositorio.ufsc.br/handle/123456789/251595>.

HAWKING, S. W. The Occurrence of Singularities in Cosmology. **Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences**, The Royal Society, v. 294, n. 1439, p. 511–521, 1966.

HAWKING, S. W.; PENROSE, R. The Singularities of Gravitational Collapse and Cosmology. **Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences**, The Royal Society, v. 314, n. 1519, p. 529–548, 1970. DOI: 10.1098/rspa.1970.0021.

HAWKING, S.W; ELLIS, G.F.R. **The Large Scale Structure of Space-Time**. 1. ed. Cambridge: Cambridge University Press, 1973. P. 391. (Cambridge Monographs on Mathematical Physics).

HIRSCH, Morris W. **Differential Topology**. New York, NY: Springer New York, 1976. v. 33. (Graduate Texts in Mathematics).

HUNT, Brian R.; SAUER, Tim; YORKE, James A. Prevalence: a translation-invariant "almost every" on infinite-dimensional spaces. **Bulletin of the American Mathematical Society**, v. 27, p. 217–238, 1992. DOI: https://doi.org/10.1090/S0273-0979-1992-00328-2.

KRIKORIAN, Nishan. A note concerning the fine topology on function spaces. **Compositio Mathematica**, Wolters-Noordhoff Publishing, v. 21, n. 4, p. 343–348, 1969. Available from: <http://www.numdam.org/item/CM_1969__21_4_343_0/>.

LARSSON, Eric. **Lorentzian Cobordisms, Compact Horizons and the Generic Condition**. 2014. Master of Science Thesis – KTH Royal Institute of Technology, Stockholm. Available from: <http://kth.diva-portal.org/smash/record.jsf?pid=diva2%3A723418&dswid=6483>.

LEE, Dan A. **Geometric relativity**. Providence, Rhode Island: American Mathematical Society, 2019. P. 361. (Graduate studies in mathematics, volume 201). ISBN 9781470450816.

LEE, John M. **Introduction to Riemannian Manifolds**. 2. ed. [S.l.]: Springer, 2018. v. 176, p. 437. (Graduate Texts in Mathematics).

________. **Introduction to Smooth Manifolds**. 2. ed. New York: Springer, 2012. v. 218, p. 708. (Graduate Texts in Mathematics).

________. **Introduction to Topological Manifolds**. 2. ed. [S.l.]: Springer, 2011. v. 176, p. 433. (Graduate Texts in Mathematics).

LERNER, David E. The space of Lorentz metrics. **Communications in Mathematical Physics**, v. 32, n. 1, p. 19–38, 1973. DOI: 10.1007/BF01646426.

MARGALEF-ROIG, J.; OUTERELO DOMÍNGUEZ, E. **Differential Topology**. Amsterdam: North-Holland, 1992.

MATHER, John N. Stability of C^∞ Mappings: II. Infinitesimal Stability Implies Stability. **Annals of Mathematics**, Annals of Mathematics, v. 89, n. 2, p. 254–291, 1969. Available from: <http://www.jstor.org/stable/1970668>.

MICHOR, Peter W. **Manifolds of differentiable mappings**. Orpington: Shiva Pub, 1980. (Shiva mathematics series ; 3).

MONCRIEF, Vincent. Spacetime symmetries and linearization stability of the Einstein equations. I. **Journal of Mathematical Physics**, American Institute of Physics, v. 16, n. 3, p. 493–498, 1975.

MUKHERJEE, Amiya. **Differential topology**. 2. ed. Cham Heidelberg: Birkhäuser, 2015.

MUNKRES, James R. **Topology**. 2. ed. Upper Saddle River, NJ: Prentice Hall, Inc, 2000.

NAIMPALLY, Somashekhar Amrith. Graph topology for function spaces. **Transactions of the American Mathematical Society**, v. 123, n. 1, p. 267–272, 1966. DOI: 10.1090/S0002-9947-1966-0192466-4.

O'NEILL, Barrett. **Semi-Riemannian Geometry**: With Applications to Relativity. 1. ed. San Diego: Academic Press, 1983. v. 108. (Pure and Applied Mathematics).

OXTOBY, John C. **Measure and category: a survey of the analogies between topological and measure spaces**. 2. ed. New York ; Heidelberg ; Berlin: Springer-Verlag, 1980. (Graduate Texts in Mathematics, 2). OCLC: 1204344049.

PENROSE, Roger. Gravitational Collapse and Space-Time Singularities. **Phys. Rev. Lett.**, American Physical Society, v. 14, p. 57–59, 1965.

RIBES, Luis; ZALESSKII, Pavel. **Profinite groups**. 2nd ed. Berlin ; New York: Springer, 2010. (Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, v. 40).

SILVA, I P Costa e. On the geodesic incompleteness of spacetimes containing marginally (outer) trapped surfaces. **Classical and Quantum Gravity**, IOP Publishing, v. 29, n. 23, p. 15, 2012. DOI: 10.1088/0264-9381/29/23/235008.

SMALE, S. An Infinite Dimensional Version of Sard's Theorem. **American Journal of Mathematics**, JSTOR, v. 87, n. 4, p. 861–866, 1965.

WALD, Robert M. **General Relativity**. 1. ed. Chicago: The University of Chicago Press, 1984. P. 491.