PARTIAL ACTIONS OF INVERSE SEMIGROUPS AND THEIR ASSOCIATED ALGEBRAS

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RESUMO

Estudamos a interação entre álgebras de Steinberg e *skew* álgebras de grupos parciais e caracterizamos isomorfismos de *skew* álgebras de grupos que preservam diagonal, sobre álgebras comutativas, em termos de equivalência contínua de órbitas das ações parciais associadas. Mostramos que qualquer álgebra de Steinberg, associada a um grupoide amplo e Hausdorff, pode ser visto como uma skew álgebra de semigrupos inverso.

Provamos que dada uma ação parcial de um semigrupo inverso S em um anel comutativo A, o *skew* anel de semigrupo inverso $A \rtimes S$ é simples se, e somente se, A é um subanel comutativo maximal de $A \rtimes S$ e A é S-simples. Aplicamos este resultado no contexto de ações de semigrupos inversos topológicos para conectar a simplicidade do *skew* anel de semigrupo inverso associado com propriedades topológicas da ação, e apresentamos uma nova prova do critério de simplicidade para uma álgebra de Steinberg associada a um grupoide amplo e Hausdorff (ver [13, Corollary 4.6]).

De maneira semelhante à Exel em [28], construímos o grupoide de germes associado a uma ação parcial de semigrupo inverso. Descrevemos a álgebra de Steinberg de um grupoide de germes amplo e Hausdorff como uma *skew* álgebra de semigrupo inverso. Também provamos que, sob hipóteses naturais, a direção oposta é válida. Finalizamos esta tese com uma descrição e estudo de equivalência contínua de órbitas para ações parciais topologicalmente principais de semigrupos inversos, e aplicamos nossos resultados em álgebras de caminhos de Leavitt.

Esta tese for baseada nos artigos: [5], [3] and [2].

Palavras chaves: Semigrupos inversos. Ações parciais. *skew* algebras de semigrupos inversos. Álgebras de Steinberg. Grupoides de germes. Ações topologicalmente principais. Equivalência contínua de órbitas.

ABSTRACT

We study the interplay between Steinberg algebras and partial skew group algebras and we characterize diagonal-preserving isomorphisms of partial skew group algebras, over commutative algebras, in terms of continuous orbit equivalence of the associated partial actions. We show that any Steinberg algebra, associated to an ample Hausdorff groupoid, can be seen as a skew inverse semigroup algebra.

We prove that given a partial action of an inverse semigroup S on a commutative ring A, the skew inverse semigroup ring $A \rtimes S$ is simple if, and only if, A is a maximal commutative subring of $A \rtimes S$ and A is S-simple. We apply this result in the context of topological inverse semigroup actions to connect simplicity of the associated skew inverse semigroup ring with topological properties of the action, and we present a new proof of the simplicity criterion for a Steinberg algebra associated with a Hausdorff ample groupoid (see [13, Corollary 4.6]).

In a manner similar to Exel em [28] we construct the groupoid of germs associated to a partial action of inverse semigroups. We describe the Steinberg algebra of an ample Hausdorff groupoid of germs as a partial skew inverse semigroup algebra. We also prove that, under natural hypotheses, the converse holds. We finish this thesis with a description and study of orbit equivalence for partial actions of inverse semigroups, and we apply our results in Leavitt path algebras.

This thesis is built on the three articles: [5], [3] and [2].

Key-words: Inverse semigroups. Groupoids. Partial actions. Skew inverse semigroup algebras. Steinberg algebras. Groupoid of germs. Topologically principal actions. Continuous orbit equivalence.

RESUMO EXPANDIDO

Introdução

A noção de ação parcial de grupos em C*-álgebras e a construção de seu C*-produto cruzado associado foram inicialmente introduzidas por Exel em [26]. Estas C*-álgebras se mostraram ferramentas poderosas no estudo de diversas C*-álgebras, por exemplo, álgebras de Cuntz-Krieger [31], álgebras de Cuntz-Li [6], C*-álgebras de grafos [11], C*-álgebras de ultragrafos [41, 38] e álgebras associadas a diagramas de Bratteli [34, 39], para citar algumas.

Os resultados de [28] provam que ações parciais de grupos podem ser interpretadas como ações de semigrupos inversos, que foram introduzidas em [68]. Além disso, ações de semigrupos inversos podem ser usadas para descrever certas C*-álgebras como produtos cruzados [60, Teorema 3.3.1]. Embora as abordagens acima sejam semelhantes em alguns aspectos, cada uma delas tem suas vantagens e desvantagens - por exemplo, ações de semigrupos inversos respeitam a operação completamente, enquanto grupos têm, em geral, uma estrutura algébrica melhor do que semigrupos inversos.

Ações parciais de grupos e ações de semigrupos inversos podem ser generalizadas simultaneamente pela noção de ações parciais de semigrupos inversos. Definida em [10], uma ação parcial de um semigrupo inverso S em um conjunto X é um homomorfismo parcial θ de semigrupos inversos de S no semigrupos inverso de todas as bijeções parciais de X. Em contraste com as ações de semigrupos, não exigimos que a operação do semigrupo S seja completamente respeitada - apenas que $\theta(ts)$ seja uma extensão de $\theta(t)\theta(s)$, para quaisquer $t, s \in S$.

Em um contexto puramente algébrico, os *skew* anéis de grupos parciais introduzidos por Dokuchaev e Exel em [24], são uma generalização dos clássicos *skew* anéis de grupos, e também são um análogo algébrico de C*-produtos cruzados parciais. Assim como no nível de C*-álgebras, algumas classes importantes de álgebras, tais como álgebras de caminhos de Leavitt de grafos e ultragrafos, podem ser descritas como *skew* anéis de grupos parciais (ver [40, 43]). Dokuchaev apresenta uma visão abrangente dos desenvolvimentos na teoria de ações parciais de grupos em [23].

Baseado no trabalho de Nándor Sieben (ver [68]), a classe de *skew* álgebras de semigrupos inversos foi introduzida por Exel e Vieira em [33]. De fato, os resultados de [33] provam que as *skew* álgebras de grupos parciais são isomorfas a certas *skew* álgebras de semigrupos inversos (ver [33, Teorema 3.7]).

As álgebras de Steinberg, introduzidas em [73], são versões algébricas das C*-álgebras de grupoides amplos (possivelmente não Hausdorff), previamente introduzidas por Renault [64]. Independentemente, Clark et al. introduziram em [16] a mesma classe de álgebras, porém restritas à classe de grupoides amplos e Hausdorff. O desenvolvimento da teoria das álgebras de Steinberg tem atraído muita atenção ultimamente, em vista do fato que álgebras de Steinberg incluem, em particular, as álgebras de Kumjian-Pask de grafos de grau alto (*higher rank graphs*) introduzidas em [62] (que por sua vez incluem álgebras de caminhos de Leavitt). Ver [13], [17] e [74] para mais detalhes sobre o desenvolvimento da teoria.

Objetivos

Vincular a teoria dos *skew* anéis de grupos parciais com a teoria das álgebras de Steinberg, da mesma forma que a teoria dos C*-produtos cruzados parciais está ligada à teoria das C*-álgebras de grupoides. Em particular, fornecer uma versão algébrica do resultado de Abadie (ver [1]) que mostra que qualquer produto cruzado parcial, associado a uma ação parcial de um grupo em um espaço topológico, pode ser visto como uma C*-álgebra de grupoide. A versão algébrica desse teorema nos permitirá unir resultados de Li (ver [52]), sobre equivalência contínua de órbitas de ações parciais de grupos em espaços topológicos, e resultados de Carlsen e Rout (ver [12]), sobre isomorfismo que preservam diagonal entre álgebras de Steinberg, para apresentar resultados referentes a isomorfismos de *skew* álgebras (comutativas) de grupos parciais que preservam diagonais. Provar no contexto algébrico os teoremas [60, Teorema 3.3.1] e [63, Teorema 8.1].

Estudar e caracterizar a simplicidade de *skew* anéis parciais associados à ações parciais de semigrupos inversos em anéis comutativos, generalizando os resultados apresentados em [59] e [37]. Aplicar estes resultados no contexto de ações parciais topológicas de semigrupos inversos para conectar a simplicidade do *skew* anel de semigrupo inverso parcial associado com propriedades topológicas da ação parcial. Além disso, usar nosso resultado e os resultados descritos no primeiro parágrafo para apresentar uma nova prova do critério de simplicidade para uma álgebra de Steinberg associada a um grupoide amplo e Hausdorff (ver [13, Corollary 4.6]).

Generalizar os primeiros resultados obtidos sobre ações topológicas parciais de grupos para ações topológicas parciais de semigrupos inversos: Descrever a álgebra de Steinberg de um grupoide de germes amplo e Hausdorff como uma *skew* álgebra de semigrupo inverso parcial; Definir e estudar equivalência contínua de órbitas para ações parciais de semigrupos inversos, e se possível, dar uma caracterização equivalente em termos de isomorfismo de *skew* álgebras de semigrupos inversos que preservam diagonais. Analisar sob quais condições as *skew* álgebras de semigrupos inversos parciais podem ser realizados como álgebras de Steinberg. Por fim, conectar as noções de equivalência contínua de órbitas de ações de semigrupos inversos, equivalência contínua de grafos (ver [9, Definição 3.1]), e isomorfismo entre álgebras de caminho de Leavitt.

Metodologia

Pesquisa bibliográfica, principalmente artigos publicados em jornais conceituados, e discussões frequentes sobre os objetivos e resultados já obtidos, bem como os problemas a serem resolvidos e dificuldades encontradas, com o orientador e demais pesquisadores envolvidos neste trabalho.

Resultados e Discussão

Dada uma ação parcial de um grupo discreto em um espaço topológico localmente compacto, Hausdorff e zero-dimensional, provamos que a álgebra de Steinberg do grupoide de transformação associado a esta ação parcial é isomorfo à *skew* álgebra de semigrupo inverso. Em seguida, aplicamos esta interpretação e caracterizamos isomorfismos que preservam diagonais entre *skew* álgebras (comutativas) de grupos parciais em termos de equivalência contínua de órbitas quando considerando ações parciais topologicamente principais (ver Teorema 2.2.16). Mostramos que qualquer álgebra de Steinberg, associada a um grupoide amplo e Hausdorff, pode ser descrita como uma *skew* álgebra de semigrupo inverso parcial (ver Teorema 2.3.1).

Provamos que dada uma ação parcial de um semigrupo inverso S em um anel comutativo A, o *skew* anel de semigrupo inverso parcial $A \rtimes S$ é simples se, e somente se, A é uma subanel comutativo maximal de $A \rtimes S$ e A é S-simples (ver Teorema 3.1.5). Aplicamos este resultado no contexto de sistemas dinâmicos topológicos: dada uma ação parcial topológica de um semigrupo inverso em um espaço localmente compacto, Hausdorff e zero-dimensional, mostramos que o *skew* anel de semigrupo inverso parcial associado é simples se, e somente se, a ação parcial é minimal, topologicamente principal e satisfaz uma certa condição sobre a existência de funções com suporte não vazio nos ideais do *skew* anel de semigrupo inverso. (ver Teorema 3.2.18). Além do mais, com o resultado principal deste capítulo (ver Teorema 3.1.5), conseguimos apresentar uma nova prova de [13, Corollary 4.6], como desejado.

Extendemos a construção de grupoides de germes de [28] para ações parciais de semigrupos inversos, de modo a também extender grupoides de transformação de ações parciais de grupos. Como desejado, descrevemos a álgebra de Steinberg de um grupoide de germes amplo e Hausdorff como uma *skew* álgebra de semigrupos inverso parcial (ver Teorema 4.3.4) e, portanto, generalizamos os Teoremas 2.1.1 e 2.3.1 apresentados anteriormente. Descrevemos equivalência contínua de órbitas para ações parciais topologicamente principais de semigrupos inversos, e damos uma caracterização equivalente em termos da existência de um isomorfismo preservando as diagonais entre as *skew* álgebras de semigrupos inversos parciais associadas, de isomorfismos preservando diagonais entre as álgebras de Steinberg dos grupoides de germes associados, bem como de isomorfismos entre os pseudogrupos topológicos plenos (ver Teorema 4.5.11). Finalizamos este trabalho com uma aplicação de nossos resultados, interpretando álgebras de caminho de Leavitt como *skew* álgebras de semigrupos inversos, e com isto caracterizamos equivalência contínua de órbitas de gráfos em termos de equivalência contínua de órbitas de certas ações de semigrupos inversos associadas.

Considerações Finais

A definição de uma *skew* álgebra de semigrupo inverso parcial envolve um quociente por um certo ideal, que é motivado pela definição C*-algébrica de produtos cruzados por semigrupos inversos. Em geral, este quociente faz com que estas álgebras não sejam graduadas, diferentemente do caso das *skew* álgebra de grupos (e de fato só ocorre nesta situação). Este fato amplia as dificuldades para provar resultados análogos aos existentes no contexto de ações parciais de grupos.

De modo geral obtemos os resultados desejados, embora alguns pontos tenham sido inesperados. À medida em que generalizamos alguns conceitos, claramente perdemos alguns resultados, por exemplo, na caracterização de simplicidade do *skew* anel de um semigrupo inverso parcial proveniente de uma ação parcial topológica de semigrupo inverso não é suficiente que ação parcial envolvida seja minimal e topologicamente efetiva (como no caso de ações parciais de grupos). Foi necessária a adição de uma condição sobre o suporte de algumas funções (ver Teorema 3.2.18). Aliás, fica a pergunta: Quais são as condições necessárias e suficientes apenas sobre a ação parcial topológica para que o *skew* anel de semigrupo inverso parcial associado seja simples? A teoria sobre *skew* álgebras de semigrupos inversos (parciais) ainda é muito recente e ainda há muito a ser explorada.

Palavras chaves: Semigrupos inversos. Ações parciais. *skew* algebras de semigrupos inversos. Álgebras de Steinberg. Grupoides de germes. Ações topologicalmente principais. Equivalência contínua de órbita.

Contents

INTRODUCTION

The well-know Wagner-Preston representation theorem states that every inverse semigroup can be faithfully represented by an inverse subsemigroup of $\mathcal{I}(X)$, where $\mathcal{I}(X)$ is the inverse semigroup formed by all partial bijections for some set X. More precisely, an inverse semigroup S embeds into $\mathcal{I}(S)$ by means of the homomorphism $\gamma : S \to \mathcal{I}(S)$ given by $\gamma(s) = \gamma_s : X_{s^*} \to X_s$, where $X_s = \{t \in S : tt^* \leq ss^*\}$ and $\gamma_s(x) = sx$ (see Theorem 1.1.7). An action of an inverse semigroup S on a set X is simply an inverse semigroup homomorphism $\theta : S \to \mathcal{I}(X)$. In particular, the map γ is an example of an inverse semigroup action.

An inverse semigroup may act, for example, on a topological space or on an algebra, and in these settings we consider actions which preserve the topological or algebraic structure at hand. More specifically, we will be particularly interested in two classes of topological actions: the canonical action of the inverse semigroup of compact-open bisections $\mathcal{G}^a$ of an ample Hausdorff groupoid $\mathcal{G}$ on its unit space $\mathcal{G}^{(0)}$ (see Example 1.5.9); and, given a directed graph E, the canonical action of the "graph inverse semigroup" $\mathcal{S}_E$ on the boundary path space ∂E (see Example 1.5.13).

In an algebraic context, inverse semigroup actions on algebras (or rings) induce other algebras (rings), which we will call skew inverse semigroup algebras (rings), which retain information about the initial dynamical system. From the two actions mentioned in the previous paragraph (the first associated to a groupoid $\mathcal{G}$ and the other associated to a directed graph E) we obtain, in a similar manner, actions on the algebras consisting of all locally constant, compactly supported functions on $\mathcal{G}^{(0)}$ and on ∂E, respectively, taking values in a given (unital commutative) ring R. The skew inverse semigroup algebras obtained in this manner are isomorphic to the Steinberg algebra of $\mathcal{G}$ and to the

Leavitt path algebra of E, respectively (see Theorem 2.3.1 and Proposition 4.6.6).

Closely related with actions of inverse semigroups are the *partial actions* of groups. The notion of partial action of groups on C*-algebras, and the construction of their associated crossed product C*-algebra was initially introduced by Exel in [26]. These have proven to be a powerful tool in the study of many C*-algebras, e.g. Cuntz-Krieger algebras [31], Cuntz-Li algebras [6], graph C*-algebras [11], ultragraph C*-algebras [41, 38], and algebras associated with Bratteli diagrams [34, 39], to name a few. In fact, the results of [28] prove that partial group actions can be regarded as actions of inverse semigroups, which were already considered in [68]. Furthermore, actions of inverse semigroups can be used to describe groupoid C*-algebras as crossed products by inverse semigroups [60, Theorem 3.3.1]. Although the above approaches are similar in some respects, each of them has its advantages and drawbacks – for example, actions of inverse semigroups respect the operation completely, whereas groups have, overall, a better algebraic structure than general inverse semigroups.

Partial actions of groups and actions of inverse semigroups can be simultaneously generalized by the notion of partial actions of inverse semigroups. Defined in [10], a partial action of the inverse semigroup S on a set X is a partial homomorphism (or dual pre-homomorphism) of inverse semigroups $\theta : S \to \mathcal{I}(X)$. In contrast with semigroup actions, we require only that $\theta(ts)$ is an extension of $\theta(t)\theta(s)$, for all $t, s \in S$.

In a purely algebraic context, partial skew group rings were introduced by Dokuchaev and Exel [24], as a generalization of classical skew group rings and as an algebraic analogue of partial crossed product C*-algebras. The theory has attracted strong interest lately, as some important classes of algebra, such as graph and ultragraph Leavitt path algebras, have been shown to be partial skew group algebras (see [40, 43]). See [23] for a comprehensive overview of developments in the theory of partial actions of groups.

Based on work by Nándor Sieben (see [68]), the class of skew inverse semigroup algebras was introduced first by Exel and Vieira in [33], in the case of global actions, and then later by Shourijeh and Rahni [67] in the case of partial actions. In fact, the results of [33] prove that partial skew group algebras are isomorphic to certain skew inverse semigroups algebras (see [33, Theorem 3.7]).

Steinberg algebras are isomorphic to a certain skew inverse semigroup algebras (see Theorem 2.3.1). These algebras, introduced by Steinberg in [73], are associated to (possibly non-Hausdorff) ample groupoids and are the "algebraisation" of Renault's C*-algebras of groupoids. Independently, Clark et al. in [16] introduced the same class of algebras, however restricted to the class of Hausdorff ample groupoids. The development of the theory of Steinberg algebras has attracted a lot of attention lately. In particular, Steinberg algebras include the Kumjian–Pask algebras of higher-rank graphs introduced in [62] (which in turn include Leavitt path algebras). See [13], [17] and [74] for a few examples of the development of the theory.

In this thesis, we shall be concerned with partial dynamical systems of inverse semigroups as well as the properties of their associated partial skew inverse semigroup algebras (rings). It is important to emphasize that this thesis is built on three articles written during my doctorate: [5], [3] and [2]). Chapters 2, 3 and 4 are similar to each of these articles, respectively. The work is organized as follows:

In the first chapter we set up the notation and conventions that we use throughout out the thesis. We recall some important properties of inverse semigroups, topological groupoids and Steinberg algebras associated to ample groupoids. We discuss different ways of defining partial actions of inverse semigroups, and the construction of the partial skew inverse semigroup algebras (and rings). We present the universal property for both Steinberg algebras as well as for skew inverse semigroup algebras.

In the second chapter we study the interplay between Steinberg algebras and skew group algebras: For a partial action of a group in a Hausdorff, locally compact and zero-dimensional topological space, we realize the associated partial skew group algebra as a Steinberg algebra over the transformation groupoid attached to the partial action. We then apply this realization to characterize diagonal-preserving isomorphisms of partial skew group algebras, over commutative algebras, in terms of continuous orbit equivalence of the associated partial actions. We finish this chapter by showing that any Steinberg algebra, associated to an ample Hausdorff groupoid, can be seen as a partial skew inverse semigroup algebra.

We have already mentioned that given a partial action θ of an inverse semigroup S on a ring A one may construct its associated skew inverse semigroup ring $A \rtimes_\alpha S$. In the third chapter, our main result asserts that, when A is commutative, the ring $A \rtimes_\alpha S$ is simple if, and only if, A is a maximal commutative subring of $A \rtimes_\alpha S$ and A is S-simple (see Theorem 3.1.5). We apply this result in the context of topological inverse semigroup actions to connect simplicity of the associated skew inverse semigroup ring with topological properties of the action. Furthermore, we use our result to present a new proof of the simplicity criterion for a Steinberg algebra $A_R(\mathcal{G})$ associated with a Hausdorff ample groupoid $\mathcal{G}$.

In the last chapter, we construct the groupoid of germs associated to a partial action of inverse semigroups in a manner similar to Exel's groupoid of germs and which generalize transformation groupoids. We describe the Steinberg algebra of an ample Hausdorff groupoid of germs as a partial skew inverse semigroup algebra (see Theorem 4.3.4), and therefore we generalizae Theorems 2.1.1 and 2.3.1. We also prove that, under natural hypotheses, the converse holds, that is, partial skew inverse semigroup algebras (of appropriate algebras) may be realized as Steinberg algebras. We describe and study the orbit equivalence for partial actions of inverse semigroups, and we give an equivalent characterization in terms of diagonal preserving isomorphism of skew inverse semigroup algebras, as well of topological full pseudogroups. We finish this thesis with an application of our results, by realizing Leavitt path algebras as skew inverse semigroup algebras, and we characterize orbit equivalence of directed graphs in terms of continuous orbit equivalence of associated actions. This chapter generalizes previous work of the second chapter as well as from others, which dealt mostly with actions of semigroups or partial actions of groups.

1 PRELIMINARIES

In this chapter, we will introduce concepts and notations that will be used throughout this thesis and several results, which are well-known from the literature. The most important and used results will be proven, while the trivial be referred.

Throughout this chapter, we will introduce the concepts of inverse semigroups, groupoids, Steinberg algebras, partial actions of inverse semigroup and the construction of partial skew inverse semigroup algebras.

1.1 Inverse semigroups

We will start by presenting the basic theory of inverse semigroups. Before this, recall that a *poset* (partially ordered set) $(P, \leq)$ is:

- $\wedge$-*semilattice* (read "meet semilattice") if every pair of elements $s, t \in P$ admits infimum, and we denote it by $s \wedge t$.
- $\vee$-*semilattice* (read "join semilattice") if every pair of elements $s, t \in P$ admits supremum, and we denote it by $s \vee t$.
- *lattice* if it is both $\wedge$-semilattice and $\vee$-semilattice.

A *semigroup* is a set S endowed with an associative binary operation

$$(s,t) \mapsto st.$$

Many interesting semigroups have a zero element, that is, there is an element $0 \in S$ such that

$$0s = s0 = 0, \quad \text{for all } s \in S.$$

We say that semigroup S is a *monoid* if it has a *unit*, that is, if there is an element $1 \in S$ such that

$$1s = s1 = s, \quad \text{for all } s \in S.$$

Both the zero and unit of a semigroup are unique, when they exist.

An *inverse* of an element $s \in S$ is an element $t \in S$ such that

$$sts = s \qquad \text{and} \qquad tst = t.$$

A semigroup is *regular* if every element s admits an inverse, and we say that S is an *inverse semigroup* if every $s \in S$ admits a *unique* inverse, which we denote by s^* in this case. In an inverse semigroup, the inverse operation defines an involution, that is,

$$(s^*)^* = s \quad \text{and} \quad (st)^* = t^* s^*, \quad \text{for all } s, t \in S.$$

A *subsemigroup* of a semigroup S is a subset $P \subseteq S$ which is closed under the semigroup operation. Every subsemigroup P of an inverse semigroup S, which is itself regular is, in fact, closed under inverses of S, and so is an inverse semigroup on its own right. We call such P an *inverse subsemigroup* of S.

Given an inverse semigroup S, one may prove that the collection of *idempotent* elements in S, namely

$$E(S) = \{e \in S \mid e^2 = e\},$$

is a commutative inverse subsemigroup of S (see [48, Theorem 1.1.3]). It immediately follows that if e in $E(S)$, then $e^* = e$, and hence e can be thought of as a "projection". Notice that if $s \in S$, then

$$(ss^*)^2 = s(s^* s s^*) = ss^*,$$

so $ss^* \in E(S)$. On the other hand, if $e \in E(S)$, then $e = ee^*$. Hence,

$$E(S) = \{ss^* \mid s \in S\}.$$

Notice that meet semilattices are precisely the inverse semigroups in which every element is an idempotent (see [48, Proposition 1.4.9]).

Example 1.1.1. Groups are precisely the inverse semigroups which have only one idempotent. Indeed the only idempotent of a group is its unit. Conversely, if an inverse semigroup S has only one idempotent, denoted by e, then, for all $s \in S$, $s^*s = e = ss^*$, and $es = (ss^*)s = s = s(s^*s) = se$, which means that S is a group.

We define a partial order on $E(S)$ by

$$e \leq f \quad \Longleftrightarrow \quad e = ef, \quad \text{for all } e, f \in E(S),$$

which makes $E(S)$ a $\wedge$-*semilattice*, meaning that, for every e and f in $E(S)$, there exists a largest element which is smaller than e and f, namely ef. The order on $E(S)$ extends to S as the so-called *natural partial order* defined by

$$s \leq t \quad \Longleftrightarrow \quad s = ts^*s \quad \Longleftrightarrow \quad s = ss^*t.$$

It is compatible with the product and inverse operations in the sense that

$$s \leq t, \ \ u \leq v \quad \Longrightarrow \quad su \leq tv,$$

and,

$$s \leq t \quad \Longleftrightarrow \quad s^* \leq t^*$$

(see [48, Proposition 1.2.7.(3)]).

Example 1.1.2. Groups are the inverse semigroups for which the natural partial order is equality. Indeed, the natural partial order of a group is easily seen to be equality. Conversely, suppose that S is an inverse semigroup whose natural partial order is the equality relation. If e and f are two idempotents, then $ef \leq e$ and $ef \leq f$, but the natural partial order is equality, so $e = ef = f$. Thus S has exactly one idempotent and, by Example 1.1.1, S is a group.

Example 1.1.3. Let X be an arbitrary set. It is easy to see that the power set $\mathcal{P}(X)$ is a commutative inverse semigroup with respect to intersection of sets. Moreover, $E(\mathcal{P}(X)) = \mathcal{P}(X)$, the natural partial order is the inclusion of sets and $\mathcal{P}(X)$ has unit and zero, which are X and $\varnothing$, respectively.

Example 1.1.4. Every $\wedge$-semilattice is an inverse semigroup with the meet as the operation: $xy = x \wedge y$.

Example 1.1.5. Let $S = \mathbb{N} \cup \{\infty, z\}$ be the disjoint union of the lattice $\mathbb{N}$ and a two-elements set $\{\infty, z\}$ with the product given by, for $m, n \in \mathbb{N}$,

$$nm = \min\{n, m\}, \quad n\infty = \infty n = nz = zn = n,$$

$$z\infty = \infty z = z \quad \text{and} \quad zz = \infty\infty = \infty.$$

In other words, S is the inverse semigroup obtained by adjoining the lattice $\mathbb{N}$ to the group of order 2 $\{\infty, z\}$, in a way that every element of $\mathbb{N}$ is smaller than z and ∞. Notice that $E(S) = \mathbb{N} \cup \{\infty\}$, 0 is the zero element of S and $s^* = s$ for all $s \in S$.

Inverse semigroups are most easily understood in terms of partial bijections. Let X be a set. By a partially defined map on X, we mean a map $f : A \to B$, where A and B are subsets of X. We denote the set of all partial bijections of X, including the empty function $\varnothing \to \varnothing$, by $\mathcal{I}(X)$. Given $f \in \mathcal{I}(X)$, we denote the domain and image of f by $\mathrm{dom}(f)$ and $\mathrm{im}(f)$, respectively. Let f and g be two partial bijections of X. Then their composite is a partial function $f \circ g$, where the domain of $f \circ g$ is given by

$$g^{-1}(\mathrm{dom}(f) \cap \mathrm{im}(g))$$

and if $x \in \mathrm{dom}(f \circ g)$ then $(f \circ g)(x) = f(g(x))$. The image of $f \circ g$ is

$$f(\mathrm{dom}(f) \cap \mathrm{im}(g)).$$

The case where $\mathrm{dom}(f)$ and $\mathrm{im}(g)$ have an empty intersection causes no problems: $f \circ g$ is just the empty function. All identity maps of subsets of X are partial bijections. If f is a partial bijection of X then the inverse function f^{-1} is the inverse element of f.

The composition operation defined above endows $\mathcal{I}(X)$ with a structure of an inverse monoid with zero $\varnothing \to \varnothing$, called the *symmetric inverse semigroup of* X. The idempotents of $\mathcal{I}(X)$ are precisely the identity maps defined of subsets of X. The partial order among general elements of $\mathcal{I}(X)$ is the order given by "extension", meaning that, for any f and g in $\mathcal{I}(X)$, we have that $f \leq g$ if, and only if,

$$\mathrm{dom}(f) \subseteq \mathrm{dom}(g) \quad \text{and} \quad f(x) = g(x), \quad \text{for all } x \in \mathrm{dom}(f).$$

Just as an analogue to Cayley's theorem for groups, the *Wagner-Preston representation theorem* states that every inverse semigroup can be realized as an inverse subsemigroup of some $\mathcal{I}(X)$ (see Theorem 1.1.7 below).

Example 1.1.6. Let X be a non-empty set and let f be a bijection of X. We denote by $[[f]]$ the subset of $\mathcal{I}(X)$ consisting of all partial bijections g of X for which there is a finite partition $X_1, \cdots, X_k$ of $\mathrm{dom}(g)$ and a set of integers $\{n_1, \cdots, n_k\}$ such that $g|_{X_i} = f^{n_i}|_{X_i}$, for all $i \in \{1, \cdots, k\}$. Then $[[f]]$ is an inverse subsemigroup of $I(X)$.

1.1.1 Partial homomorphisms

Homomorphisms between inverse semigroups S and T are just semigroup homomorphisms, that is, maps $\varphi : S \to T$ such that

$$\varphi(sr) = \varphi(s)\varphi(r), \qquad \text{for all } s, r \in S.$$

It is easy to see that if $\varphi : S \to T$ is a homomorphism between inverse semigroups then:

- if $e \in E(S)$, then $\varphi(e) \in E(T)$;
- φ preserves the involution, that is, $\varphi(s^*) = \varphi(s)^*$, for all $s \in S$;
- φ preserves the order, that is, $\varphi(s) \leq \varphi(r)$ whenever $s \leq r$;
- $\mathrm{im}(\varphi)$ is an inverse subsemigroup of T;
- if R is an inverse subsemigroup of T, then $\varphi^{-1}(R)$ is an inverse subsemigroup of S.

Theorem 1.1.7. *(Wagner-Preston representation theorem) Let S be an inverse semigroup. Then there is a set X and an injective homomorphism $\varphi : S \to \mathcal{I}(X)$.*

Proof. For each element $s \in S$, we define $D_s = \{t \in S : tt^* \le ss^*\}$, and a map $\gamma_s : D_{s^*} \to D_s$ by

$$\gamma_s(r) = sr.$$

This is well-defined, since for all $s \in S$ and $r \in D_{s^*}$,

$$(sr)(sr)^* = srr^*s^* \le ss^*.$$

Notice that γ_{s^*} is a map from D_s to D_{s^*}, that $\gamma_{s^*} \circ \gamma_s$ is the identity on D_{s^*} and that $\gamma_s \circ \gamma_{s^*}$ is the identity on D_s. Hence γ_s is a bijection and $\gamma_s^{-1} = \gamma_{s^*}$. We may thus define $\gamma : S \to \mathcal{I}(S)$ by $\gamma(s) = \gamma_s$.

In order to prove that $\gamma_t \circ \gamma_s = \gamma_{ts}$, we need to check that the domains coincide, that is,

$$\gamma_s^{-1}(D_s \cap D_{t^*}) = D_{(ts)^*}.$$

If $r \in \gamma_s^{-1}(D_s \cap D_{t^*}) = D_{s^*} \cap \gamma_s^{-1}(D_{t^*})$, then $r \in D_{s^*}$ and $sr = \gamma_s(r) \in D_{t^*}$, which implies, respectively,

$$rr^* \le s^*s \quad \text{and} \quad sr(sr)^* \le t^*t.$$

Then

$$rr^* = rr^*(s^*s)(s^*s) = s^*(srr^*s^*)s \le s^*t^*ts = (ts)^*(ts),$$

and so $r \in D_{(ts)^*}$.

On the other hand, if $r \in D_{(ts)^*}$, then $rr^* \le (ts)^*(ts) = s^*t^*ts$ and $rr^* \le s^*s$. This implies that $r \in D_{s^*}$, and

$$\gamma_s(r)(\gamma_s(r))^* = srr^*s^* \le s(s^*t^*ts)s^* = t^*t,$$

that is, $\gamma_s(r) \in D_{t^*}$. We can conclude that $r \in D_{s^*} \cap \gamma_s^{-1}(D_{t^*}) = \gamma_s^{-1}(D_s \cap D_{t^*})$.

It is immediate from the definitions that $\gamma_t\gamma_s$ and γ_{ts} have the same effect on elements of $D_{(ts)^*}$ and so γ is a homomorphism.

Finally, suppose that $\gamma_s = \gamma_t$. Then

$$t = \gamma_t(t^*)t = \gamma_s(t^*)t = st^*t \leq s,$$

and symmetrically, $s \leq t$. Therefore, $\gamma : S \to \mathcal{I}(S)$ is an isomorphism of S on to inverse subsemigroup of $\mathcal{I}(S)$. □

Partial homomorphisms are a generalization of homomorphisms of inverse semigroup.

Definition 1.1.8. [10, Definition 2.11] Let S be an inverse semigroup and let T be a semigroup (not necessarily inverse). A *partial homomorphism* of S on T is a map $\varphi : S \to T$ that satisfies, for all $s, r \in S$,

(i) $\varphi(s)\varphi(r)\varphi(r^*) = \varphi(sr)\varphi(r^*)$,

(ii) $\varphi(s^*)\varphi(s)\varphi(r) = \varphi(s^*)\varphi(sr)$,

(iii) $\varphi(s)\varphi(s^*)\varphi(s) = \varphi(s)$.

Notice that, if T happens to be an inverse semigroup, then Definition 1.1.8 (iii), applied to both s and s^*, together with the uniqueness of inverses, immediately implies that

$$\varphi(s^*) = \varphi(s)^*, \quad \text{for all } s \in S. \tag{iii'}$$

Hence, if T is an inverse semigroup, the axioms (i) - (iii) in Definition 1.1.8 are equivalent to (i)-(ii) plus (iii').

Lemma 1.1.9. *Let S be an inverse semigroup, let T be a semigroup and let $\varphi : S \to T$ be a partial homomorphism. Then*

(a) $\varphi(e) \in E(T)$, for all $e \in E(S)$,

(b) $\varphi(e)\varphi(s) = \varphi(es)$ and $\varphi(s)\varphi(e) = \varphi(se)$, for all $s \in S$ and $e \in E(S)$.

Proof. (a) If $e \in E(S)$, then

$$\varphi(e) \overset{(iii)}{=} \varphi(e)\varphi(e^*)\varphi(e) \overset{(i)}{=} \varphi(ee^*)\varphi(e) = \varphi(e)\varphi(e) = \varphi(e)^2.$$

(b) If $s \in S$ and $e \in E(S)$, then

$$\begin{aligned}\varphi(e)\varphi(s) &\overset{(a)}{=} \varphi(e)\varphi(e)\varphi(s) \overset{(ii)}{=} \varphi(e)\varphi(es)\\ &\overset{(iii)}{=} \varphi(e)\varphi(es)\varphi((es)^*)\varphi(es) \overset{(i)}{=} \varphi(es)\varphi((es)^*)\varphi(es)\\ &\overset{(iii)}{=} \varphi(es).\end{aligned}$$

Similarly, $\varphi(s)\varphi(e) = \varphi(se)$. □

Proposition 1.1.10. *[10, Proposition 3.1] Let S, T be inverse semigroups and let $\varphi : S \to T$ be a map. Then φ is a partial homomorphism if, and only if, for all $s, r \in S$,*

(i') $\varphi(s)\varphi(r) \leq \varphi(sr)$,

(ii') $\varphi(s) \leq \varphi(r)$ whenever $s \leq r$,

(iii') $\varphi(s^*) = \varphi(s)^*$.

Proof. We assume that φ is a partial homomorphism. We have already seen that (iii') holds. Let $s, r \in S$. Then

$$\begin{aligned}\varphi(s)\varphi(r) &\overset{(iii)}{=} \varphi(s)\varphi(r)\varphi(r^*)\varphi(r) \overset{(i)}{=} \varphi(sr)\varphi(r^*)\varphi(r)\\ &\overset{(iii')}{=} \varphi(sr)\varphi(r)^*\varphi(r) \leq \varphi(sr),\end{aligned}$$

proving (ii'). For (i'), suppose that $s \leq r$. Then $s = rs^*s$, and by Lemma 1.1.9 (a), $\varphi(s^*s)$ is idempotent. Thus

$$\varphi(s) = \varphi(rs^*s) \overset{1.1.9\ (b)}{=} \varphi(r)\varphi(s^*s) \leq \varphi(r).$$

Conversely, suppose that (i') - (iii') hold. The axiom (iii) is immediate from (iii'). Given $s, r \in S$, we have that

$$\begin{aligned}\varphi(s)\varphi(r)\varphi(r^*) &\overset{(i)}{\leq} \varphi(sr)\varphi(r^*) \overset{(iii')}{=} \varphi(sr)\varphi(r^*)\varphi(r)\varphi(r^*)\\ &\overset{(i')}{\leq} \varphi(srr^*)\varphi(r)\varphi(r^*) \leq \varphi(s)\varphi(r)\varphi(r^*),\end{aligned}$$

where, in the last step, we used (ii') and the fact that $srr^* \leq s$. This proves (i), and (ii) follows similarly. □

Remark 1.1.11. Maps $\varphi : S \to T$ satisfying conditions (i') - ((iii')) of Proposition 1.1.10 are sometimes called *dual pre-homomorphisms* by some authors (for instance, in [51]).

Example 1.1.12. Let $\varphi : S \to T$ be a homomorphism between inverse semigroups and let e be an idempotent of T. Then the map $\varphi_e : S \to T$, defined by $\varphi_e(s) = e\varphi(s)e$, is a partial homomorphism, called the "restriction" or "compression" of φ to e.

1.2 Groupoids

A *groupoid* consists of sets $\mathcal{G}$ and $\mathcal{G}^{(2)} \subseteq \mathcal{G} \times \mathcal{G}$, endowed with a product map $(b, c) \mapsto bc$ from $\mathcal{G}^{(2)}$ to $\mathcal{G}$, and an inverse map $b \mapsto b^{-1}$ from $\mathcal{G}$ to itself, such that the following conditions hold:

(i) if (b, c) and (c, d) are in $\mathcal{G}^{(2)}$, then so are (bc, d) and (b, cd), and the equality $(bc)d = b(cd)$ holds,

(ii) for all $b \in \mathcal{G}$, the pairs (b, b^{-1}) and (b^{-1}, b) belong to $\mathcal{G}^{(2)}$ and, if $(b, c) \in \mathcal{G}^{(2)}$, then $b^{-1}(bc) = c$ and $(bc)c^{-1} = b$.

There are two maps associated to a groupoid, called *range* and *source*, which are defined from $\mathcal{G}$ to itself by

$$\mathfrak{r}(b) = bb^{-1} \quad \text{and} \quad \mathfrak{s}(b) = b^{-1}b,$$

respectively. We call the common image of $\mathfrak{r}$ and $\mathfrak{s}$ the *unit space* of $\mathcal{G}$ and denote it by $\mathcal{G}^{(0)}$. The set $\mathcal{G}^{(2)}$ is called the set of *composable pairs* of the groupoid. The product bc of elements b, c of $\mathcal{G}$ is defined if, and only if, $\mathfrak{s}(b) = \mathfrak{r}(c)$.

Basic facts that follow immediately are:

- $b\mathfrak{s}(b) = b = \mathfrak{r}(b)b$, for all $b \in \mathcal{G}$,
- $\mathfrak{s}(b^{-1}) = \mathfrak{r}(b)$ and $\mathfrak{r}(b^{-1}) = \mathfrak{s}(b)$, for all $b \in \mathcal{G}$,
- if $u \in \mathcal{G}^{(0)}$ then $u^{-1} = u$,

- if $u, v \in \mathcal{G}^{(0)}$ then $(u, v) \in \mathcal{G}^{(2)}$ if, and only if, $u = v$,
- if $bc = d \in \mathcal{G}$, then $(c^{-1}, b^{-1}) \in \mathcal{G}^{(0)}$, $d^{-1} = c^{-1}b^{-1}$, $\mathfrak{r}(d) = \mathfrak{r}(b)$, and $\mathfrak{s}(d) = \mathfrak{s}(c)$,
- $\mathcal{G}^{(0)} = \{b \in \mathcal{G} \mid (b, b) \in \mathcal{G}^{(2)} \text{ and } b^2 = b\}$.

An elegant way to specify a groupoid is to define it as a *small category with inverses*. Let $\mathcal{G}$ be such a category. Since the category is "small", its objects form a set $\mathcal{G}^{(0)}$ of $\mathcal{G}$. The groupoid $\mathcal{G}$ is then identified with its set of morphisms, whose elements are "arrows" b from one object $\mathfrak{s}(b)$ to another $\mathfrak{r}(b)$.

$$\begin{tikzcd} \mathfrak{s}(b) \bullet \arrow[rr, "b"] & & \bullet\, \mathfrak{r}(b) \end{tikzcd} \qquad \begin{tikzcd} \underset{\mathfrak{s}(b) = \mathfrak{r}(b)}{\bullet} \arrow[loop above, "b"] \end{tikzcd}$$

The product operation is simply the composition of arrows. Since the category $\mathcal{G}$ has inverses, every member b of $\mathcal{G}$ has an inverse b^{-1}.

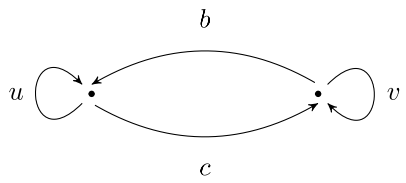
$$\begin{tikzcd} \mathfrak{s}(b) \bullet \arrow[rr, "b"] & & \bullet\, \mathfrak{r}(b) \arrow[ll, bend left, "b^{-1}"] \end{tikzcd}$$

By identifying objects with their respective identity morphisms, we see that the source and range maps act from $\mathcal{G}$ onto $\mathcal{G}^{(0)}$.

Obviously, a product bc of elements b, c of $\mathcal{G}$ makes sense if, and only if, $\mathfrak{s}(b) = \mathfrak{r}(c)$, since the equality just says that the range of c is the same as the source of b, so that the morphisms b, c can be composed.

$$\begin{tikzcd} \mathfrak{s}(c)\ \bullet \arrow[r, "c"] \arrow[rr, bend right, "bc"'] & \underset{\mathfrak{r}(c) = \mathfrak{s}(b)}{\bullet} \arrow[r, "b"] & \bullet\ \mathfrak{r}(b) \end{tikzcd}$$

Example 1.2.1. The diagram below represents the groupoid $\mathcal{G} = \{u, v, b, c\}$ with unit space $\mathcal{G}^{(0)} = \{u, v\}$.



This diagram uniquely determines the product and inverse operation of $\mathcal{G}$. Notice that $u^{-1} = u,\ v^{-1} = v,\ b^{-1} = c$ and $c^{-1} = b$.

Example 1.2.2. Every group is a groupoid in which all pairs are composable and the unit space has only one element, which is the unit of the group.

Example 1.2.3. Let X be a set. Then an equivalence relation R on X has a groupoid structure as follows: $(x, y), (y', z) \in R$ are composable if $y = y'$ and $(x, y)(y', z) = (x, z)$, and $(x, y)^{-1} = (y, x)$. Then, $\mathfrak{r}(x, y) = (x, x)$ and $\mathfrak{s}(x, y) = (y, y)$. The unit space of R is the diagonal $\{(x, x) \mid x \in X\}$ and may be identified with X.

Example 1.2.4. Suppose that the group G acts on the set X. The image of the point $x \in S$ by the transformation associated to $g \in G$ is denoted gx. We let $\mathcal{G} = G \times X$ and define the following groupoid structure: (g, x) and (h, y) are composable if, and only if, $x = hy$ and $(g, x)(h, y) = (gh, y)$, and $(g, x)^{-1} = (g^{-1}, gx)$. Then $\mathfrak{r}(g, x) = (1, gx)$ and $\mathfrak{s}(g, x) = (1, x)$. The map $(1, x) \mapsto x$ identifies $\mathcal{G}^{(0)}$ with X.

Example 1.2.5. [48, Proposition 3.1.4] An inverse semigroup S is a groupoid $\mathcal{G}(S)$ when its product is sutiably restricted. More precisely, as a set, $\mathcal{G}(S)$ is just the inverse semigroup S. The set of composable pairs $\mathcal{G}(S)^{(2)}$ is the set $\{(s, t) \in S \times S \mid s^*s = tt^*\}$. The product map on $\mathcal{G}(S)^{(2)}$ is just the (restricted) product in S and, for $s \in S$, we take $s^{-1} = s^*$. Then $\mathfrak{s}(s) = s^*s$, $\mathfrak{r}(s) = ss^*$, and $\mathcal{G}(S)^{(0)} = \{s^*s \mid s \in S\} = E(S)$. We call $\mathcal{G}(S)$ the *associated groupoid* or *restricted product groupoid* of S.

Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids. A map $\varphi : \mathcal{G} \to \mathcal{H}$ is a *groupoid homomorphism* if $(b, c) \in \mathcal{G}^{(2)}$ implies $(\varphi(b), \varphi(c)) \in \mathcal{H}^{(2)}$ and

$$\varphi(b)\varphi(c) = \varphi(bc).$$

A homomorphism $\varphi : \mathcal{G} \to \mathcal{H}$ between groupoids satisfies the following assertions:

- $\varphi(b^{-1}) = \varphi(b)^{-1}$, for all $b \in \mathcal{G}$,
- if $u \in \mathcal{G}^{(0)}$, then $\varphi(u) \in \mathcal{G}^{(0)}$,
- $\varphi(\mathfrak{s}(b)) = \mathfrak{s}(\varphi(b))$ and $\varphi(\mathfrak{r}(b)) = \mathfrak{r}(\varphi(b))$, for all $b \in \mathcal{G}$,
- if φ is invertible, then φ^{-1} is also a groupoid homorphism. In this case we say φ is an isomorphism.

For each unit $u \in \mathcal{G}^{(0)}$, we define

$$\mathcal{G}_u = \{b \in \mathcal{G} \ : \ \mathfrak{s}(b) = u\} \ \text{ and } \ \mathcal{G}^u = \{b \in \mathcal{G} \ : \ \mathfrak{r}(b) = u\}.$$

It is easy to see that $\mathcal{G}_u^u := \mathcal{G}_u \cap \mathcal{G}^u$ is a group when endowed with the product operation coming from $\mathcal{G}$, and it is called the *isotropy group* at u. We say that a unit u in $\mathcal{G}^{(0)}$ has *trivial isotropy* if $\mathcal{G}_u^u = \{u\}$.

Notice that the set

$$\mathrm{Iso}(\mathcal{G}) = \{b \in \mathcal{G} \mid \mathfrak{s}(b) = \mathfrak{r}(b)\} = \bigcup_{u \in \mathcal{G}^{(0)}} \mathcal{G}_u^u,$$

is a groupoid, where two elements may be composed if, and only if, they lie in the same isotropy group $\mathcal{G}_u^u$. The groupoid $\mathrm{Iso}(\mathcal{G})$ is called the *isotropy subgroupoid* of $\mathcal{G}$.

A groupoid is *principal* if $\mathrm{Iso}(\mathcal{G}) = \mathcal{G}^{(0)}$. Every equivalence relation R on set X is a principal groupoid. Conversely, if $\mathcal{G}$ is a principal groupoid, then the image of the map $(\mathfrak{r}, \mathfrak{s}) : \mathcal{G} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$, defined by $(\mathfrak{r}, \mathfrak{s})(b) = (\mathfrak{r}(b), \mathfrak{s}(b))$, is an equivalence relation on $\mathcal{G}^{(0)}$, and $(\mathfrak{r}, \mathfrak{s})$ is a isomorphism of $\mathcal{G}$ on its image.

1.2.1 Topological groupoids

A *topological groupoid* is a groupoid $\mathcal{G}$ endowed with a topology such that composition and inversion are continuous, where we consider the product topology on $\mathcal{G}^{(2)}$.

Notice that if $\mathcal{G}$ is a topological groupoid, then the maps $\mathfrak{r}$, $\mathfrak{s}$ are continuous. Moreover, if we assume that $\mathcal{G}$ is Hausdorff, then the unit space $\mathcal{G}^{(0)}$ is closed in $\mathcal{G}$.

An *étale groupoid* is a topological groupoid $\mathcal{G}$ such that $\mathcal{G}^{(0)}$ is locally compact and Hausdorff in the relative topology, and its range map is a local homeomorphism from $\mathcal{G}$ to $\mathcal{G}^{(0)}$ (the source map will consequently share such property).

For any étale groupoid $\mathcal{G}$, the unit space $\mathcal{G}^{(0)}$ is an open subset of $\mathcal{G}$. Indeed, let $x_0 \in \mathcal{G}^{(0)}$. There is an open subset B of $\mathcal{G}$ containing x_0, such that $\mathfrak{r}(B)$ is open in $\mathcal{G}^{(0)}$ and $\mathfrak{r} : B \to \mathfrak{r}(B)$ is a homeomorphism. Notice that $x_0 = \mathfrak{r}(x_0) \in B$. Let $U = \mathfrak{r}^{-1}(B) \cap B$. Then U is an open subset of $\mathcal{G}$ containing x_0, and we claim that $U \subseteq \mathcal{G}^{(0)}$. Given $y \in U$, we have $y \in B$, and $\mathfrak{r}(y) \in B$. Since $\mathfrak{r}(y) = \mathfrak{r}(\mathfrak{r}(y))$ and $\mathfrak{r}$ is injective on B then $y = \mathfrak{r}(y) \in \mathcal{G}^{(0)}$.

In the next steps, we will describe an inverse semigroup which is intrinsic to an étale groupoid, and which will be of fundamental importance for this thesis.

We define the product and inverses for subsets of a groupoid $\mathcal{G}$ as follows: If B and C are subsets of a groupoid $\mathcal{G}$, one may form the following subsets of $\mathcal{G}$:

$$BC = \{bc \in \mathcal{G} \mid b \in B,\ c \in C \text{ and } \mathfrak{s}(b) = \mathfrak{r}(c)\},$$

and

$$B^{-1} = \{b^{-1} \mid b \in B\}.$$

A subset B of a groupoid $\mathcal{G}$ is called a *bisection* if the restrictions of $\mathfrak{r}$ and $\mathfrak{s}$ to B are both injective. Equivalently, B is a bisection if, and only if, BB^{-1} and $B^{-1}B$ are contained in $\mathcal{G}^{(0)}$.

Since the source and range maps are local homeomorphisms from an étale groupoid $\mathcal{G}$ to its unit space $\mathcal{G}^{(0)}$, for every open bisection B

of $\mathcal{G}$, we have that $\mathfrak{s}$ and $\mathfrak{r}$ are homeomorphisms from B onto $\mathfrak{s}(B)$ and $\mathfrak{r}(B)$. Moreover, the collection consisting of all open bisections of $\mathcal{G}$, denoted by $\mathcal{G}^{op}$, forms a basis for the topology of $\mathcal{G}$ (see [28, Proposition 3.5]).

Proposition 1.2.6. *[60, Proposition 2.2.3 and 2.2.4] Let $\mathcal{G}$ be an étale groupoid. Then $\mathcal{G}^{op}$ is an inverse semigroup under set product and with set inversion as involution. Moreover,*

(a) $\mathcal{G}^{(0)}$ is the unit of $\mathcal{G}^{op}$,

(b) $\varnothing$ is the zero for $\mathcal{G}^{op}$,

(c) $E(\mathcal{G}^{op})$ is the family of open subsets of $\mathcal{G}^{(0)}$,

(d) the natural order of the inverse semigroup $\mathcal{G}^{op}$ is set inclusion.

Using the notation of [60], we denote by $\mathcal{G}^{a}$ the set of all bisections of $\mathcal{G}$ which are simultaneously compact and open (compact-open).

Definition 1.2.7. An étale groupoid $\mathcal{G}$ is *ample* if it admits a basis of compact-open bisections.

Proposition 1.2.8. *[49, Proposition 2.18(7)] Let $\mathcal{G}$ be an ample groupoid. Then $\mathcal{G}^{a}$ is an inverse subsemigroup of $\mathcal{G}^{(op)}$. Moreover, $\mathcal{G}^{(0)}$ is compact if, and only if, the inverse semigroup $\mathcal{G}^{a}$ has a unit.*

Proof. We have already seen that the product of two open bisections is an open bisection, then, only the compactness remains to be checked. Since $\mathcal{G}^{(0)}$ is Hausdorff, then

$$\mathcal{G}^{(2)} = \{(a,b) \in \mathcal{G} \times \mathcal{G} \mid \mathfrak{s}(a) = \mathfrak{r}(b)\} = (\mathfrak{s} \times \mathfrak{r})^{-1}\left(\left\{(u,u) \mid u \in \mathcal{G}^{(0)}\right\}\right)$$

is closed in $\mathcal{G} \times \mathcal{G}$.

If $A, B \in \mathcal{G}^{a}$, then $A \times B$ is a compact subset of $\mathcal{G} \times \mathcal{G}$, and thus $(A \times B) \cap \mathcal{G}^{(2)}$ is compact as well. Denoting by $m : \mathcal{G}^{(2)} \to \mathcal{G}$ the product map, we have

$$AB = m\left((A \times B) \cap \mathcal{G}^{(2)}\right),$$

which is then image of a compact set under a continuous function, hence compact. $\square$

Recall that a topological space X is said to be *zero-dimensional* if the topology of X admits a basis consisting of clopen sets. If, in addition, X is locally compact, it is easy to see that X also admits a basis formed by compact-open sets.

Proposition 1.2.9. *[29, Proposition 4.1] Let $\mathcal{G}$ be an étale groupoid. Then $\mathcal{G}^{(0)}$ is zero-dimensional if, and only if, $\mathcal{G}$ is ample.*

Proof. Suppose that $\mathcal{G}^{(0)}$ is zero-dimensional and that $c \in U \subseteq \mathcal{G}$, with U open. Choose an open bisection B such that $c \in B \subseteq U$. Thus, as $\mathfrak{s}(B)$ is open and $\mathcal{G}^{(0)}$ is zero-dimensional, there is a compact-open subset T of $\mathcal{G}^{(0)}$ such that $\mathfrak{s}(c) \in T \subseteq \mathfrak{s}(B)$. Since $\mathfrak{s}$ is a homeomorphism from B to $s(B)$, we have that the set $K = B \cap \mathfrak{s}^{-1}(T)$ is homeomorphic to T and, therefore, K is a compact-open bisection as required.

The converse follows immediately from the source map $\mathfrak{s} : \mathcal{G} \to \mathcal{G}^{(0)}$ being open and Hausdorff. With more details: If $\mathcal{G}$ is ample, then let U be an open subset of $\mathcal{G}^{(0)}$. Since $\mathcal{G}^{(0)}$ is open in $\mathcal{G}$, then U is open in $\mathcal{G}$ is well, so we may write it as a union $U = \bigcup_{i \in I} C_i$, where each C_i is compact-open. Since $\mathcal{G}^{(0)}$ is Hausdorff, then each C_i is clopen in $\mathcal{G}^{(0)}$. This proves that $\mathcal{G}^{(0)}$ is zero-dimensional. $\square$

Example 1.2.10. A topological group G is étale if, and only if, G is ample if, and only if, G is a discrete group.

Example 1.2.11. A Hausdorff, locally compact and zero-dimensional topological space X seen as a unit groupoid (this means that $X = X^{(0)}$) is a Hausdorff ample groupoid.

Example 1.2.12. If G is a discrete group acting continuously on a Hausdorff, locally compact and zero-dimensional topological space X, then the transformation groupoid $G \ltimes X$ endowed with the product topology is a Hausdorff ample groupoid.

Example 1.2.13. Let X be a discrete topological space. Then the equivalence relation $R = X \times X$ endowed with the product topology is an ample Hausdorff groupoid.

Example 1.2.14. It is not necessarily true that an ample groupoid is Hausdorff. For example, let $S = \mathbb{N} \cup \{\infty, z\}$ as in the Example 1.1.5, and let $\mathcal{G}(S) = \mathbb{N} \cup \{\infty, z\}$ be the restricted product groupoid (see Example 1.2.5). We see $\mathbb{N} \cup \{\infty, z\}$ as a topological groupoid with the topology whose open sets are either cofinite or contained in $\mathbb{N}$. Notice that $\mathbb{N} \cup \{\infty, z\}$ is ample groupoid, but in this case, there are no open subsets of X which separate the elements ∞ and z.

Example 1.2.15. (*Deaconu–Renault groupoids*) Let X be a Hausdorff locally compact topological space and let φ be a local homeomorphism from an open subset $\mathrm{dom}(\varphi)$ of X onto an open subset $\mathrm{im}(\varphi)$ of X. The *Deaconu-Renault groupoid* associated to φ is

$$\mathcal{G}(X, \varphi) = \{(x, m-n, y) \in X \times \mathbb{Z} \times X \mid \varphi^m(x) = \varphi^n(y)\}.$$

Two elements (x, m, y) and (z, n, w) of $\mathcal{G}$ are composable if, and only if, $y = z$ and, in this case, their product is

$$(x, m, y)(z, n, w) = (x, m+n, w).$$

The inverse map on $\mathcal{G}$ is defined by

$$(x, n, y)^{-1} = (y, -n, x).$$

Thus the range and source maps are given by

$$\mathfrak{r}(x, n, y) = (x, 0, x) \quad \text{and} \quad \mathfrak{s}(x, n, y) = (y, 0, y).$$

Hence the unit space $\mathcal{G}^{(0)}$ may be identified with X via the map

$$(x, 0, x) \to x.$$

We endow $\mathcal{G}$ with the topology generated by the basis consisting of sets of the form

$$\mathcal{G}(U, m, n, V) = \{(x, m-n, y) \mid \varphi^m(x) = \varphi^n(y),\ x \in U, y \in V\},$$

where m and $n \in \mathbb{N}$, U and V are open subsets of X such that both $\varphi^m|_U$ and $\varphi^n|_V$ are injective and $\varphi^m(U) = \varphi^n(V)$. Notice that the range map $\mathfrak{r}$ induces a homeomorphism $\mathcal{G}(U, m, n, V) \simeq U$. Hence, with this topology $\mathcal{G}$ is an étale groupoid. Moreover, if X is zero-dimensional, then $\mathcal{G}(X, \varphi)$ is ample. It easy see that the unit space $\mathcal{G}^{(0)}$ is closed, then we can also conclude that the groupoid $\mathcal{G}(X, \varphi)$ is Hausdorff (see [20], [65].)

Example 1.2.16. (*Boundary path groupoid*) Let $E = (E^0, E^1, r, s)$ be a directed graph with vertex set E^0 and edge set E^1. For each edge e, $s(e)$ is the initial vertex (*source*) of e and $r(e)$ is terminal vertex (*range*) of e. A *countable graph* is one where E^0 and E^1 are countable sets.

A *finite path* is a sequence μ of edges $\mu_1, \ldots, \mu_k$, where $s(\mu_{i+1}) = r(\mu_i)$ for $1 \leq i \leq k-1$. We write $\mu = \mu_1\mu_2 \cdots \mu_k$. The length $|\mu|$ of μ is just k. Each vertex v is regarded as a finite path of length 0. The set of finite paths in E is denoted by $E^\star$. We define $r(\mu) = r(\mu_k)$ and $s(\mu) = s(\mu_1)$. For $v \in V$, we set $r(v) = v = s(v)$.

A vertex v is called a *sink* if $s^{-1}(v) = \varnothing$ and it is called an *infinite emitter* if $|s^{-1}(v)| = \infty$. If $v \in E^0$ is either a sink or an infinite emitter then it is called *singular*. An *infinite path* is an infinite sequence $x = (x_i)_{i \in \mathbb{N}}$, where $x_i \in E^1$ and $r(x_i) = s(x_{i+1})$, for all $i \in \mathbb{N}$. We denote by E^∞ the collection of all infinite paths in E.

Paths can be concatenated if their ranges and sources agree: if $\mu, \nu \in E^\star$ with $r(\mu) = s(\nu)$, then $\mu\nu = \mu_1 \cdots \mu_{|\mu|}\nu_1 \cdots \nu_{|\nu|} \in E^\star$ and, if $x \in E^\infty$ with $r(\mu) = s(x)$, then $\mu x = \mu_1 \cdots \mu_{|\mu|} x_1 x_2 \cdots \in E^\infty$. For any finite path μ, we specify that $s(\mu)\mu = \mu = \mu r(\mu)$.

We define the *boundary path space of E* as

$$\partial E := E^\infty \cup \{\mu \in E^\star \mid r(\mu) \text{ is singular}\}.$$

For a finite path $\mu \in E^\star$, we define the *cylinder set*

$$Z(\mu) = \{\mu x \mid x \in \partial E \text{ and } r(\mu) = s(x)\} \subseteq \partial E,$$

and for a finite set $F \subseteq s^{-1}(r(\mu))$ (possibly empty), we define a neighbourhood base for $\mu \in E^\star$ consisting of *generalised cylinder sets*

$$Z(\mu, F) = Z(\mu) \setminus \bigcup_{e \in F} Z(\mu e) = \{\mu x \mid x \in \partial E, x_1 \notin F \text{ and } r(\mu) = s(x)\}.$$

The generalised cylinder sets provide a basis of compact-open sets for a Hausdorff topology on ∂E (see [77, Theorems 2.1 and Theorem 2.2]).

For $n \in \mathbb{N}$, let $\partial E^{\geq n} = \{x \in \partial E \mid |x| \geq n\}$. Then $\partial E^{\geq n} = \bigcup_{\mu \in E^n} Z(\mu)$ is an open subset of ∂E. We define the *one-sided shift* map $\sigma : \partial E^{\geq 1} \to \partial E$ as follows:

$$\sigma(x) = \begin{cases} r(x), & \text{if } x \in E^\star \cap \partial E \text{ and } |x| = 1 \\ x_2 \cdots x_{|x|}, & \text{if } x \in E^\star \cap \partial E \text{ and } |x| \geq 2 \\ x_2 x_3 \cdots, & \text{if } x \in E^\infty. \end{cases} \tag{1.1}$$

The n-fold composition σ^n is defined on $\partial E^{\geq n}$ and we understand $\sigma^0 : \partial E \to \partial E$ as the identity map. We define the *boundary path groupoid*

$$\begin{aligned} \mathcal{G}_E &= \{(x, m-n, y) \in \partial E \times \mathbb{Z} \times \partial E \mid \sigma^m(x) = \sigma^n(y)\} \\ &= \{(\mu x, |\mu| - |\nu|, \nu x) \mid \mu, \nu \in E^\star,\ x \in \partial E,\ r(\mu) = r(\nu) = s(x)\}, \end{aligned}$$

and view $(x, k, y) \in \mathcal{G}_E$ as a morphism with domain y and codomain x. The composition of morphisms and their inverses are defined by the formulae

$$(x, k, y)(y, l, z) = (x, k+l, z) \quad \text{and} \quad (x, k, y)^{-1} = (y, -k, x).$$

$\mathcal{G}_E$ is a groupoid with unit space $\mathcal{G}_E^{(0)} = \{(x, 0, x) : x \in \partial E\}$, which we identify with ∂E. To put a topology on $\mathcal{G}_E$, we consider finite paths $\mu, \nu \in E^\star$ with $r(\mu) = r(\nu)$, and a finite set of edges $F \subseteq s^{-1}(r(\mu))$. Then we define the open sets

$$Z(\mu, \nu) := \{(\mu x, |\mu| - |\nu|, \nu x) \mid x \in \partial E, r(\mu) = s(x)\}$$

and

$$\begin{aligned} Z(\mu, \nu, F) &:= Z(\mu, \nu) \setminus \bigcup_{e \in F} Z(\mu e, \nu e) \\ &= \{(\mu x, |\mu| - |\nu|, \nu x) \mid x \in \partial E, x_1 \notin F \text{ and } r(\mu) = s(x)\}. \end{aligned}$$

The collection of these sets provides a basis of compact-open bisections for a Hausdorff topology on $\mathcal{G}_E$ (see [47, Proposition 2.6] for more details in the case of row-finite graphs and [61, Section 3] for the general case).

Definition 1.2.17. Let $\mathcal{G}$ be a topological groupoid. We say that

1. $\mathcal{G}$ is *topologically principal* if the set of points in $\mathcal{G}^{(0)}$ with trivial isotropy is dense in $\mathcal{G}^{(0)}$,

2. $\mathcal{G}$ is *effective* if the interior of $\mathrm{Iso}(\mathcal{G})$ is just $\mathcal{G}^{(0)}$.

Notice that if $\mathcal{G}$ is étale, then $\mathcal{G}^{(0)}$ is open, and we automatically already have the inclusion $\mathcal{G}^{(0)} \subseteq \mathrm{int}(\mathrm{Iso}(\mathcal{G}))$.

Proposition 1.2.18. *[66, Proposition 3.6] Let $\mathcal{G}$ be an étale groupoid. Then the following assertions hold:*

(a) If $\mathcal{G}$ is Hausdorff and topologically principal, then $\mathcal{G}$ is effective.

(b) If $\mathcal{G}$ is effective, second countable, then $\mathcal{G}$ is topologically principal.

Example 1.2.19. Consider the groupoid $\mathcal{G} = \mathbb{N} \cup \{\infty, z\}$ as in Example 1.2.14. Notice that $\mathcal{G}^{(0)} = \mathbb{N} \cup \{\infty\}$ and

$$\{x \in \mathcal{G}^{(0)} \mid \mathcal{G}_x^x = \{x\}\} = \mathbb{N},$$

which is dense $\mathbb{N} \cup \{\infty\} = \mathcal{G}^{(0)}$, and therefore, $\mathcal{G}$ is topologically principal. However, $\mathcal{G}$ is not effective since

$$\mathcal{G}^{(0)} = \mathbb{N} \cup \{\infty\} \subsetneq \mathbb{N} \cup \{\infty, z\} = \mathrm{int}(\mathrm{Iso}(\mathcal{G})).$$

Example 1.2.20. Let $E = (E^0, E^1, r, s)$ be a directed graph (see Example 1.2.16). A *cycle* in E is a path $\mu \in E^\star$ such that $|\mu| \geq 1$ and $s(\mu) = r(\mu)$. An edge e is an exit to the cycle μ if there exists i such that $s(e) = s(\mu_i)$ and $e \neq \mu_i$. A graph is said to satisfy *condition (L)* if every cycle has an exit.

By [71, Proposition 5.2], the boundary path groupoid $\mathcal{G}_E$ (see Example 1.2.16) is effective if, and only if, the graph E satisfies condition (L). In particular, if E is a countable graph, $\mathcal{G}_E$ is topologically principal if, and only if, the graph E satisfies condition (L) (see [9, Proposition 2.3]).

A subset U of the unit space $\mathcal{G}^{(0)}$ of $\mathcal{G}$ is *invariant* if any $b \in \mathcal{G}$ such that $\mathfrak{s}(b) \in U$ then $\mathfrak{r}(b) \in U$, or equivalently, $\mathfrak{r}(\mathfrak{s}^{-1}(U)) = U$. We say that $\mathcal{G}$ is *minimal* if $\mathcal{G}^{(0)}$ has no nontrivial open invariant subset.

Example 1.2.21. Let X be a discrete topological space. Then the equivalence relation $R = X \times X$ with the product topology is a minimal groupoid.

1.3 Steinberg algebras

To an ample groupoid we can associate an algebra, known as *Steinberg algebra*. Such algebras were independently introduced by Steinberg in [73] and by Clark et al. in [16]. They are the "algebraisation" of Renault's C*-algebras of groupoids. The development of the theory of Steinberg algebras has attracted a lot of attention lately. In particular, Steinberg algebras include the Kumjian–Pask algebras of higher-rank graphs introduced in [62], which in turn include Leavitt path algebras. See [74], [13], [17], [12], [75] and [15] for a few examples of the development of the theory.

Let R be a unital commutative ring and let $\mathcal{G}$ be an ample groupoid. Consider $A_R(\mathcal{G})$ the R-module of R-valued functions on $\mathcal{G}$ generated by the characteristic functions of the compact-open bisections of $\mathcal{G}$, that is,

$$A_R(\mathcal{G}) = \mathrm{Span}_R\{1_B \mid B \in \mathcal{G}^a\},$$

where 1_B denotes the characteristic function of B.

Let $f, g \in A_R(\mathcal{G})$. Then their convolution product $f * g$ is defined by

$$(f * g)(x) = \sum_{bc=x} f(b)g(c), \qquad \text{for all } x \in \mathcal{G}. \tag{1.2}$$

Of course, one must show that this sum is really finite and $f * g$ belongs to $A_R(\mathcal{G})$. Since the functions of $A_R(\mathcal{G})$ are linear combinations of characteristic functions of the compact-open bisections it is enough to prove that, for every $B, C, D \in \mathcal{G}^a$,

$$(1_B + 1_C) * 1_D = 1_B * 1_D + 1_C * 1_D, \tag{1.3}$$

$$1_D * (1_B + 1_C) = 1_D * 1_B + 1_D * 1_C, \tag{1.4}$$

and

$$1_B * 1_C = 1_{BC}. \tag{1.5}$$

Equations (1.3) and (1.4) follow directly from the convolution product definition in (1.2). In order to prove (1.5), first suppose $x \in BC$. Then we can find $b \in B$ and $c \in C$ so that $x = bc$. Notice that $\mathfrak{r}(b) = \mathfrak{r}(x)$ and $\mathfrak{s}(c) = \mathfrak{s}(x)$. Since B and C are bisections, we get that b and c are the unique elements of B and C, respectively, with these properties. Then

$$1_B * 1_C(x) = 1 = 1_{BC}(x).$$

Clearly, if $x \notin BC$ then

$$1_B * 1_C(x) = 0 = 1_{BC}(x).$$

Therefore, the equality (1.5) holds, as required.

Notice that the convolution product gives us that, for $f, g \in A_R(\mathcal{G})$,

$$\operatorname{supp}(f*g) \subseteq m\left((\operatorname{supp}(f) \times \operatorname{supp}(g)) \cap \mathcal{G}^{(2)}\right) = \operatorname{supp}(f)\operatorname{supp}(g), \tag{1.6}$$

where $m : \mathcal{G}^{(2)} \to \mathcal{G}$ is the product map in $\mathcal{G}$. Moreover, it is easy to see that with appropriate change of variables, we obtain

$$(f * g)(x) = \sum_{\mathfrak{s}(c)=\mathfrak{s}(x)} f(xc^{-1})g(c) = \sum_{\mathfrak{r}(b)=\mathfrak{r}(x)} f(b)g(b^{-1}x).$$

Proposition 1.3.1. *Let R be a unital commutative ring and let $\mathcal{G}$ be an ample groupoid. Then the R-module $A_R(\mathcal{G})$ is an associative R-algebra with the convolution product defined in (1.2).*

Proof. The distributivity in $A_R(\mathcal{G})$ follows from (1.3) and (1.5). For associativity, take B, C and D in $\mathcal{G}^a$. Then

$$(1_B * 1_C) * 1_D \overset{(1.5)}{=} 1_{BC} * 1_D.$$

By Proposition 1.2.8, BC is also a compact-open bisection. Again, using (1.5), we obtain

$$(1_B * 1_C) * 1_D = 1_{B(CD)} \overset{1.2.8}{=} 1_{BCD}.$$

For the same reasons, $1_B * (1_C * 1_D) = 1_{BCD}$, that is,

$$(1_B * 1_C) * 1_D = 1_B * (1_C * 1_D). \tag{1.7}$$

Therefore, the associativity in $A_R(\mathcal{G})$ follows from the facts that any function $f \in A_R(\mathcal{G})$ is a linear combination of characteristic functions of the compact-open bisections and by equalities (1.3), (1.4) and (1.7). $\square$

Definition 1.3.2. Let R be a unital commutative ring and let $\mathcal{G}$ be an ample groupoid. The algebra $A_R(\mathcal{G})$, with the structure of the Proposition 1.3.1, is called the *Steinberg algebra* associated to $\mathcal{G}$.

For $f \in A_R(\mathcal{G})$, we define the *support* of f by

$$\operatorname{supp}(f) = \{b \in \mathcal{G} \mid f(b) \neq 0\}.$$

We say that an R-valeud function $f : \mathcal{G} \to R$ is *locally constant* if, for every $b \in \mathcal{G}$, there is an open subset U of $\mathcal{G}$ such that $f|_U$ is constant. Notice that a function $f : \mathcal{G} \to R$ is locally constant if, and only if, it is continuous once we equip R with the discrete topology.

Remark 1.3.3. We are more interested in Steinberg algebras $A_R(\mathcal{G})$ in the case where the groupoid $\mathcal{G}$ is **Hausdorff**.

Proposition 1.3.4. *Let R be a unital commutative (discrete) ring and let $\mathcal{G}$ be an ample Hausdorff groupoid. Then, the Steinberg algebra $A_R(\mathcal{G})$ associated to $\mathcal{G}$ consists of all functions from $\mathcal{G}$ to R that are locally constant and have compact support. In particular, $\operatorname{supp}(f)$ is clopen, for all $f \in A_R(\mathcal{G})$.*

Proof. If B is a compact-open bisection of $\mathcal{G}$, then its characteristic function 1_B is locally constant, because B is closed (since $\mathcal{G}$ is Hausdorff), and has compact support (namely, B). As $f \in A_R(\mathcal{G})$ is a linear combination of functions 1_{B_i}, where $B_1, \ldots, B_1 \in \mathcal{G}^a$, then f is also locally constant. From the above comments, we can conclude that f is continuous.

Now, let us prove that $\mathrm{supp}(f)$ is clopen. Indeed, since f is locally constant, for any $b \in \mathcal{G}$ with $f(b) = 0$, there is an open subset U_b containing b such that $f|_{U_b}$ is constant equal to 0. Hence,

$$\{b \in \mathcal{G} \mid f(b) = 0\} = \bigcup_{f(b)=0} U_b$$

is open. By continuity of f, we get that $\{b \in \mathcal{G} \mid f(b) = 0\}$ is closed. Therefore, $\mathrm{supp}(f)$ is clopen.

As $f \in A_R(\mathcal{G})$ is a linear combination of functions 1_{B_i}, where $B_1, \cdots, B_1 \in \mathcal{G}^a$, then the $\mathrm{supp}(f)$ is a clopen subset of the compact $\bigcup_{i=1}^n B_i$, hence itself compact. Thus every function in $A_R(\mathcal{G})$ is locally constant and compactly supported.

Conversely, suppose that $f : \mathcal{G} \to R$ is a locally constant and compactly supported function. Since $\mathcal{G}$ has a basis of compact-open bisections and $\mathrm{supp}(f)$ is compact, we may find finitely many compact-open bisections $B_1, \cdots, B_n$ such that $\mathrm{supp}(f) = \bigcup_{i=1}^n B_i$. As f is locally constant, we can suppose that $f|_{B_i}$ is constant equal to $r_i \in R$, for all $i = 1, \cdots, n$. By Hausdorffness of $\mathcal{G}$ the following pairwise disjoint subsets

$$D_1 = B_1 \quad \text{and} \quad D_j = B_j \setminus \cup_{i=1}^{j-1} B_i$$

are also compact-open bisections, and so,

$$f = \sum_{i=1}^{n} r_i 1_{D_i} \in A_R(\mathcal{G}). \qquad \square$$

Remark 1.3.5. The last part of the proof of the above proposition, show that any $f \in A_R(\mathcal{G})$ can be written as a linear combination of characteristic functions of pairwise disjoint compact-open bisections.

Definition 1.3.6. [16, Definition 3.10] Let $\mathcal{G}$ be an ample Hausdorff groupoid and let R be a unital commutative ring. A *representation*[1] of $\mathcal{G}^a$ in an R-algebra B is a family $\{t_D \mid D \in \mathcal{G}^a\} \subseteq B$ satisfying

(i) $t_\varnothing = 0$,

(ii) $t_C t_D = t_{CD}$, for all $B, C \in \mathcal{G}^a$,

(iii) $t_C + t_D = t_{C\cup D}$, whenever C and D are disjoint elements of for $\mathcal{G}^a$.

Theorem 1.3.7. *[16, Theorem 3.11] and [19, Theorem 4.4.8] Let $\mathcal{G}$ be an ample Hausdorff groupoid and let R be a unital commutative ring. Then $\{1_D \mid D \in \mathcal{G}^a\} \subseteq A_R(\mathcal{G})$ is a representation of $\mathcal{G}^a$ in $A_R(\mathcal{G})$. Moreover, $A_R(\mathcal{G})$ is universal for representations of $\mathcal{G}^a$ in the sense that, for every representation $\{t_D \mid D \in \mathcal{G}^a\}$ of $\mathcal{G}^a$ in an R-algebra B, there is a unique R-homomorphism $\Phi : A_R(\mathcal{G}) \to B$ such that $\Phi(1_D) = t_D$, for all $D \in \mathcal{G}^a$.*

Proof. Clearly the family $\{1_B \mid B \in \mathcal{G}^a\}$ satisfies (i) and (ii), and it satisfies (iii) by Equation 1.5.

Given $f \in A_R(\mathcal{G})$, we can write f as a linear combination $f = \sum_{i=1}^n r_i 1_{C_i}$, for certain $r_1, \cdots, r_n \in R\backslash\{0\}$ and pairwise disjoint compact-open bisections $C_1, \cdots, C_n$ of $\mathcal{G}$. Define $\Phi : A_R(\mathcal{G}) \to \mathcal{G}^a$ by

$$\Phi(f) = \Phi\left(\sum_{i=1}^n r_i 1_{C_i}\right) = \sum_{i=1}^n r_i t_{C_i}.$$

We need to prove that Φ is well-defined. Suppose that

$$\sum_{i=1}^n r_i 1_{C_i} = \sum_{j=1}^m s_j 1_{D_j},$$

where $s_1 \cdots, s_m \in R\backslash\{0\}$ and $C_1, \cdots, C_m$ are pairwise disjoint compact-open bisections in $\mathcal{G}^a$. Let us first to show that, for every pair i, j, we have

$$r_i t_{(U_i\cap V_j)} = s_j t_{(U_i\cap V_j)}. \tag{1.8}$$

[1] It is in fact a Boolean inverse monoid representation of S, for more details see [19] and also [69].

There are two possibilities:

- If $U_i \cap V_j = \varnothing$, then $t_{U_i \cap V_i} = 0$, and so Equation (1.8) is valid.
- If $U_i \cap V_j \neq \varnothing$, choose $a \in U_i \cap V_j$. Then $r_i = f(a) = s_j$, and so the Equation (1.8) is valid too.

Since $\bigcup_{i=1}^{n} C_i = \operatorname{supp}(f) = \bigcup_{j=1}^{m} D_j$, we can partition each C_i as $C_i = \bigcup_{j=1}^{m} C_i \cap D_j$, and similarly we can partition each $D_j = \bigcup_{i=1}^{n} C_i \cap D_j$. Then using the Equation (1.8) and the proprieties of the representation of $\mathcal{G}^a$ in B, we get

$$\sum_i r_i t_{U_i} = \sum_{i,j} r_i t_{U_i \cap V_j} = \sum_{i,j} s_j t_{U_i \cap V_j} = \sum_j s_j t_{U_i \cap V_j},$$

proving that Φ is well-defined.

In order to prove that Φ is a R-homomorphism, we apply an argument similar to the one above: If $f, g \in A_R(\mathcal{G})$ have representations $f = \sum_{i=1}^{n} r_i 1_{C_i}$ and $\sum_{j=1}^{m} s_j 1_{D_j}$, we can, if necessary, add terms of the form $0 \cdot 1_{D_j \setminus \operatorname{supp}(f)}$ to the representation of f, and similarly for g, add terms of the form $0 \cdot 1_{C_i \setminus \operatorname{supp}(g)}$, and assume that $\bigcup_{i=1}^{n} C_i = \bigcup_{j=1}^{m} D_j$. Therefore we may rewrite

$$f = \sum_{i,j} r_i 1_{C_i \cap D_j} \quad \text{and} \quad g = \sum_{i,j} s_j 1_{C_i \cap D_j},$$

hence, for all $\lambda \in R$,

$$f + \lambda g = \sum_{i,j} (r_i + \lambda s_j) 1_{C_i \cap D_j},$$

and the definition of Φ readily implies $\Phi(f + \lambda g) = \Phi(f) + \lambda \Phi(g)$. If $C, D \in \mathcal{G}^a$, then

$$\Phi(1_C 1_D) \overset{(1.5)}{=} \Phi(1_{CD}) = t_{CD} = t_C t_D = \Phi(1_C)\Phi(1_D),$$

and since $\{1_D \mid D \in \mathcal{G}^a\}$ generates $A_R(\mathcal{G})$, then Φ is an R-homomorphim as required.

Uniqueness of such Φ is immediate as $A_R(\mathcal{G})$ is generated by the family $\{1_D \mid D \in \mathcal{G}^a\}$. $\square$

Definition 1.3.8. Let $\mathcal{G}$ be an ample Hausdorff groupoid. We define the *diagonal* of $A_R(\mathcal{G})$, denoted by $D_R(\mathcal{G})$, as the R-subalgebra of $A_R(\mathcal{G})$ generated by all characteristic functions of compact-open subsets of the unit space $\mathcal{G}^{(0)}$.

Notice that if U, V are compact-open subsets of $\mathcal{G}^{(0)}$, then U, V are compact-open bisections of $\mathcal{G}$ and $1_U * 1_V = 1_{UV} = 1_{U\cap V}$, and therefore,

$$D_R(\mathcal{G}) = \mathrm{Span}_R\{1_U \mid U \text{ is compact-open subset of } \mathcal{G}^{(0)}\}.$$

Moreover, if $f \in D_R(\mathcal{G})$ then clearly $\mathrm{supp}(f) \subseteq \mathcal{G}^{(0)}$. The converse is also true. In fact, if $f = \sum_{i=1}^n r_i 1_{D_i} \in A_R(\mathcal{G})$ with $\mathrm{supp}(f) \subseteq \mathcal{G}^{(0)}$ then

$$f = f*1_{\mathrm{supp}(f)} = \sum_{i=1}^n r_i \left(1_{D_i} * 1_{\mathrm{supp}(f)}\right) = \sum_{i=1}^n r_1 1_{D_i\cap \mathrm{supp}(f)} \in D_R(\mathcal{G}^{(0)}).$$

Therefore,

$$D_R(\mathcal{G}) = \{f \in A_R(\mathcal{G}) \mid \mathrm{supp}(f) \subseteq \mathcal{G}^{(0)}\}.$$

The convolution product on $D_R(\mathcal{G})$ coincides with the pointwise product: given $f, g \in D_R(\mathcal{G})$ and $x \in \mathcal{G}^{(0)}$, we get that $\mathfrak{s}^{-1}(x) \cap \mathcal{G}^{(0)} = \{x\}$ and hence

$$\begin{aligned} f * g(x) &= \sum_{\mathfrak{s}(c)=\mathfrak{s}(x)} f(xc^{-1})g(c) = \sum_{c\in \mathfrak{s}^{-1}(x)\cap\mathcal{G}^{(0)}} f(xc^{-1})g(c) \\ &= f(xx^{-1})g(x) = f(x)g(x). \end{aligned}$$

Since R is a commutative ring, we can conclude that $D_R(\mathcal{G})$ is a commutative subalgebra of $A_R(\mathcal{G})$.

Since $\mathcal{G}^{(0)}$ is clopen, there is an embedding

$$\iota : A_R(\mathcal{G}^{(0)}) \to A_R(\mathcal{G}) \tag{1.9}$$

such that $\iota(f)|_{\mathcal{G}\setminus\mathcal{G}^{(0)}} = 0$. With this embedding, $A_R(\mathcal{G}^{(0)})$ is isomorphic to $D_R(\mathcal{G})$ as commutative subalgebra of $A_R(\mathcal{G})$.

Definition 1.3.9. For ample Hausdorff groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$, we say that an isomorphism $\phi : A_R(\mathcal{G}_1) \to A_R(\mathcal{G}_2)$ is *diagonal-preserving* if $\phi(D_R(\mathcal{G}_1)) = D_R(\mathcal{G}_2)$.

Example 1.3.10. Let R be a unital commutative ring and let X be a Hausdorff, locally compact and zero-dimensional topological space seen as a unit groupoid (i.e., $X = X^{(0)}$). Then the Steinberg algebra $A_R(X)$ is the R-algebra of all locally constant, compactly supported R-valued functions on X, with pointwise operations (since $A_R(X) = D_R(X)$).

Example 1.3.11. Recall that the *Leavitt path algebra* $L_R(E)$ of a directed graph E with coefficients in a unital commutative ring R is the R-algebra generated by a set $\{v \in E^0\}$ of pairwise orthogonal idempotents and a set of variables $\{e, e^* \mid e \in E^1\}$ satisfying the relations:

(i) $s(e)e = e = er(e)$ for all $e \in E^1$;

(ii) $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$;

(iii) $e^*f = \delta_{e,f}r(e)$ for all $e, f \in E^1$;

(iv) $v = \sum_{e \in s^{-1}(v)} ee^*$ whenever v is not a sink and not an infinite emitter.

Let $E = (E^0, E^1, r, s)$ be a directed graph. The boundary path groupoid $\mathcal{G}_E$, as in Example 1.2.16, comes with a canonical $\mathbb{Z}$-grading given by the continuous functor $\varphi : (x, k, y) \to k$. The Steinberg algebra $A_R(\mathcal{G}_E)$ associated to the boundary path groupoid $\mathcal{G}_E$ is isomorphic to the Leavitt path algebra $L_R(E)$, and such graded isomorphism $\pi_E : L_R(E) \to A_R(\mathcal{G}_E)$ is given by

$$\pi_E(\mu\nu^* - \sum_{e \in F} \mu ee^*\mu^*) = 1_{Z(\mu,\nu,F)}$$

(see [17, Example 3.2]).

1.4 Inverse semigroup actions

Definition 1.4.1. An *action* of an inverse semigroup S on a set X is a semigroup homomorphism

$$\begin{aligned} \theta : \quad S &\longrightarrow \mathcal{I}(X) \\ s &\longmapsto \theta_s. \end{aligned}$$

If S has a zero element 0, then one assumes that θ_0 is the empty bijection $\varnothing \to \varnothing$.

It follows immediately from Definition 1.4.1 that, for every $s \in S$,

$$\theta_s : \mathrm{dom}(\theta_s) \to \mathrm{im}(\theta_s)$$

is a bijection between subsets of X, and

$$\theta_{s^*} = \theta_s^* = \theta_s^{-1}.$$

By this reason, we will denote by X_{s^*} and X_s, respectively, the domain and range of θ_s.

It also follows from Definition 1.4.1 that, for every $s, t \in S$,

$$\theta_{st} = \theta_s \circ \theta_t,$$

that is,

$$X_{(st)^*} = \mathrm{dom}(\theta_{st}) = \theta_t^{-1}\left(\mathrm{im}(\theta_t) \cap \mathrm{dom}(\theta_s)\right) = \theta_t^{-1}(X_t \cap X_{s^*}),$$

$$X_{st} = \mathrm{im}(\theta_{st}) = \theta_s(\mathrm{im}(\theta_t) \cap \mathrm{dom}(\theta_s)) = \theta_s(X_t \cap X_{s^*}),$$

and

$$\theta_{st}(x) = \theta_s(\theta_t(x)),$$

for all $x \in \mathrm{dom}(\theta_{st}) = X_{(st)^*}$.

Since θ is an inverse semigroup homomorphism from S to $\mathcal{I}(X)$, we easily get the following proprieties:

- $X_{(st)^*} = \theta_{t^*}(X_t \cap X_{s^*})$, for all $s, t \in S$,
- $X_{s^*s} = X_{s^*}$, for all $s \in S$,
- $\theta_e = \mathrm{id}_{X_e}$, for all $e \in E(S)$,
- if $s, t \in S$ and $s \leq t$, then $X_s \subseteq X_t$ and $\theta_s(x) = \theta_t(x)$, for all $x \in X_{s^*}$.

From the comments above, we obtain the following characterization of an inverse semigroup action:

Proposition 1.4.2. *Let S be an inverse semigroup, let X be a set, and let $\theta : S \to \mathcal{I}(X)$ be a map. For each $s \in S$, let X_{s^*} and X_s be the domain and the image of θ_s, respectively. Then θ is an action of S on X if, and only if, for all $s, t \in S$, the following holds:*

(i) $\theta_{s^*} = \theta_s^{-1}$,

(ii) $X_{(st)^*} = \theta_{t^*}(X_t \cap X_{s^*})$,

(iii) $\theta_s(\theta_t(x)) = \theta_{st}(x)$, *for all* $x \in X_{(st)^*}$.

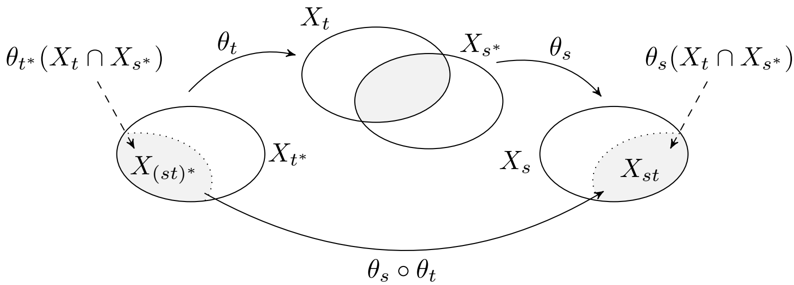


Figure 1 – Inverse semigroup action.

When it is necessary to make explicit each θ_s with its domain and image we will describe an action θ of S on X by

$$\theta = (\{X_s\}_{s \in S}, \{\theta_s\}_{s \in S}).$$

By a *dynamical system* we shall mean a quadruple

$$(X, S, \{X_s\}_{s \in S}, \{\theta_s\}_{s \in S}),$$

where X is a set, S is an inverse semigroup, and $(\{X_s\}_{s \in S}, \{\theta_s\}_{s \in S})$ is an action of S on X.

Remark 1.4.3. If θ is an action of an inverse semigroup S on a set X, we do not necessarily have that $X_s = X$, for all $s \in S$ (as in the case of a group action).

Example 1.4.4. If $\mathcal{G}$ is a group, then an action of G, regarded as an inverse semigroup, is the same as an action of G on a set X.

Example 1.4.5. Let X be any set. We have a natural action θ of the inverse semigroup of partial isometries $\mathcal{I}(X)$ on the set X, given by, for every $f \in \mathcal{I}(X)$,

$$\begin{aligned} \theta_f : \ X_{f^{-1}} &\longrightarrow X_f \\ x &\longmapsto f(x), \end{aligned}$$

where $X_{f^{-1}} = \mathrm{dom}(f)$, $X_f = \mathrm{im}(f)$.

Example 1.4.6. Let $\theta : S \to \mathcal{I}(X)$ be an action of an inverse semigroup S on set X and let T be an inverse subsemigroup of S. Then, the restriction $\theta|_T : T \to \mathcal{I}(X)$ is an action of T on X.

Example 1.4.7. Let X be a non-empty set, let f be a bijection of X, and let $[[f]]$ be the inverse semigroup defined in Example 1.1.6. Then $[[f]]$ acts naturally on X, as in the Example 1.4.5.

Example 1.4.8. There is a natural action θ of any inverse semigroup S on its idempotent semilattice $E(S)$ given as follows: for every $s \in S$, set

$$X_s = \{e \in E(S) \mid e \leq ss^*\},$$

and $\theta_s(e) = ses^*$, for all $e \in X_{s^*}$. This action is known as the *Munn representation of* S (see [58]).

Example 1.4.9. Given an action $\theta = (\{X_s\}_{s \in S}, \{\theta_s\}_{s \in S})$ of S on X, we say that a subset $Y \subseteq X$ is *invariant* under θ if

$$\theta_s(Y \cap X_{s^*}) \subseteq Y, \quad \text{for all } s \in S.$$

Given $Y \subseteq X$ an invariant subset, let

$$Y_s := Y \cap X_s, \quad \text{for all } s \in S,$$

and let γ_s be the restriction of θ_s to Y_{s^*}. Then

$$\gamma = (\{Y_s\}_{s \in S}, \{\gamma_s\}_{s \in S})$$

is an action of S on Y. This is simply the compression of θ to id_Y, the identity of Y (see Example 1.1.12).

1.5 Inverse semigroup partial actions

In this section we will be concerned with partial actions of inverse semigroups, defined in [10], which are a common generalization of both partial actions of groups and actions of inverse semigroups. We restrict our study to partial inverse semigroup actions on topological spaces, algebras and rings, although the same theory can be developed with appropriate modifications – if any at all – to other classes of algebraic or topological structures. This already leads to an immediate generalization of crossed products to so-called "partial skew inverse semigroup algebras" (or rings).

Definition 1.5.1. A *partial action* of the inverse semigroup S on the set X is a partial homomorphism

$$\begin{array}{rccc} \theta : & S & \longrightarrow & I(X) \\ & s & \longmapsto & \theta_s \end{array}.$$

If S has a zero element 0, we assume that θ_0 is the empty bijection $\varnothing \to \varnothing$.

It follows immediately from Definition 1.5.1 and Proposition 1.1.10 that, for each $s \in S$, θ_s is a bijection and $\theta_{s^*} = \theta_s^{-1}$. In the same way as for inverse semigroup actions, we denote by X_{s^*} and X_s, respectively, the domain and range of θ_s.

It also follows from Definition 1.5.1 and Proposition 1.1.8 that, for each pair $s, t \in S$, $\theta_s \circ \theta_t$ is a restriction of θ_{st}, that is,

$$\mathrm{dom}(\theta_s \circ \theta_t) \subseteq \mathrm{dom}(\theta_{st}),$$

and

$$\theta_s \circ \theta_t(x) = \theta_{st}(x), \quad \text{for all } x \in \mathrm{dom}(\theta_s \circ \theta_t).$$

Hence, for every $s, t \in S$,

$$\theta_{t^*}(X_t \cap X_{s^*}) = \theta_t^{-1}(X_t \cap X_{s^*}) = \mathrm{dom}(\theta_s \circ \theta_t) \subseteq \mathrm{dom}(\theta_{st}) = X_{(st)^*},$$

Furthermore,

$$\theta_{t^*}(X_t \cap X_{s^*}) = \mathrm{dom}(\theta_s \circ \theta_t) \subseteq \mathrm{dom}(\theta_t) = X_{t^*}.$$

Therefore,

$$\theta_{t^*}(X_t \cap X_{s^*}) \subseteq X_{t^*} \cap X_{(st)^*}. \tag{1.10}$$

In fact, (1.10) is an equality of sets. In order to prove it, apply θ_t to both sides of (1.10)

$$X_t \cap X_{s^*} \subseteq \theta_t(X_{t^*} \cap X_{(st)^*}). \tag{1.11}$$

Notice that the suitable change of variables, $t \leftrightarrow t^*$ and $s \leftrightarrow st$, in (1.11) yields

$$X_{t^*} \cap X_{(st)^*} \subseteq \theta_{t^*}(X_t \cap X_{s^*}).$$

which happens to be precisely the converse of the inclusion in (1.10). Therefore

$$\theta_{t^*}(X_t \cap X_{s^*}) = X_{t^*} \cap X_{(st)^*},$$

and $\theta_s(\theta_t(x)) = \theta_{st}(x)$, for all $x \in X_{t^*} \cap X_{(ts)^*}$.

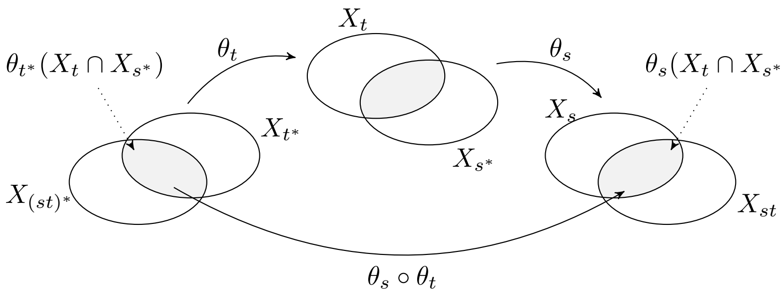


Figure 2 – Inverse semigroup partial actions

Proposition 1.5.2. *[10, Proposition 3.4] Let S be an inverse semigroup, let X be a set and let $\theta : S \to \mathcal{I}(X)$ be a map. For each $s \in S$, let X_{s^*} and X_s be the domain and image of θ_s, respectively. Then θ is a partial action of S on X if, and only if, for all $s, t \in S$, the following holds:*

(a) $\theta_{s^*} = \theta_s^{-1}$,

(b) $\theta_{t^*}(X_t \cap X_{s^*}) = X_{t^*} \cap X_{(st)^*}$,

(c) $\theta_s(\theta_t(x)) = \theta_{st}(x)$, *for all* $x \in X_{t^*} \cap X_{(st)^*}$.

Proof. It follows from previous observations that if $\theta : S \to \mathcal{I}(X)$ is a partial action of S on X, then the three axioms *(a)* - *(c)* are satisfied.

Now, we assume that $\theta : S \to \mathcal{I}(X)$ is a map and that the axioms *(a)* - *(c)* hold. Let us prove that θ satisfies the three axioms *(i′)* - *(iii′)* of the Proposition 1.1.10.

Proposition 1.1.10 *(iii′)* immediately follows from *(a)*.

To show that the Proposition 1.1.10 *(i′)* is satisfied, take s, t in S. Notice that, by *(a)* and *(b)*, the domain of $\theta_s \circ \theta_t$ coincides with

$$\theta_{t^*}(X_t \cap X_{s^*}) = X_{t^*} \cap X_{(st)^*}.$$

Evidently this is contained in $X_{(st)^*}$, which is the domain of θ_{st}. By *(c)*, we see that $\theta_s \circ \theta_t$ coincides with θ_{st} on the domain of the former set, which means that

$$\theta_s \circ \theta_t \leq \theta_{st},$$

as desired.

Let $e \in S$, and take any $x \in X_e = X_{e^*} \cap X_{(ee)^*}$. By *(c)* and *(a)*, we have that

$$x = \theta_e^{-1}(\theta_e(x)) = \theta_e(\theta_e(x)) = \theta_{e^2}(x) = \theta_e(x),$$

which shows that $\theta_e = \mathrm{id}_{X_e}$.

To finish, we take $s, t \in S$ with $s \leq t$, so that $s = ts^*s$. As seen above θ_{s^*s} is the identity on X_{s^*s}, so

$$X_{s^*s} \cap X_{t^*} = \theta_{s^*s}(X_{s^*s} \cap X_{t^*}) \overset{(b)}{=} X_{s^*s} \cap X_{s^*st^*} = X_{s^*s} \cap X_{s^*}$$
$$\overset{(b)}{=} \theta_{s^*}(X_s \cap X_s) = \theta_{s^*}(X_s) = X_{s^*}.$$

This implies that $X_{s^*} \subseteq X_{t^*}$, and, for every $x \in X_{s^*}$, we have $x \in X_{s^*s} \cap X_{s^*st^*}$, hence by *(c)*.

$$\theta_s(x) = \theta_{ts^*s}(x) = \theta_t \circ \theta_{s^*s}(x) = \theta_t(x),$$

proving that $\theta_s \leq \theta_t$, that is, Proposition 1.1.10 *(ii′)*. □

Notice that in the case of partial actions we only have the inclusion $X_{s^*} \subseteq X_{s^*s}$, and no longer the equality as in the case of actions. This happens because $\theta_{s^*} \circ \theta_s$ is only a restriction of θ_{s^*s}, and so the domain of $\theta_{s^*} \circ \theta_s$ is only contained in the domain of θ_{s^*s}.

Remark 1.5.3. The definition of a partial action of an inverse semigroup S on a set X can be reformulated in several ways. For example, another way can be found in [10], which requires minimal effort to check if a map is a partial action. More precisely, by [10, Proposition 3.4] one can replace equality in Proposition 1.5.2 *(b)* by the inclusion

$$\theta_{t^*}(X_t \cap X_{s^*}) \subseteq X_{t^*} \cap X_{(st)^*},$$

and add the following condition

$$X_s \subseteq X_t, \quad \text{whenever} \quad s \leq t.$$

Again, if we need to make explicit each bijection θ_s with its domain we will denote a partial action by

$$\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S}).$$

By a *partial dynamical system* we shall mean a quadruple

$$(X, S, \{X_s\}_{s\in S}, \{\theta_s\}_{s\in S}),$$

where X is a set, S is a group, and $(\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ is a partial action of S on X.

Example 1.5.4. In particular, any inverse semigroup action is an inverse semigroup partial action.

Example 1.5.5. Any partial group action (see [24, Definition 1.1. (i)]), where the group is regarded as an inverse semigroup, is the same as an inverse semigroup partial action.

Example 1.5.6. Suppose we are given an action $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ of an inverse semigroup S on a set X. Suppose further that Y is a given subset of X which is not necessarily θ-invariant (this means that there may be a $s \in S$ such that $\theta_s(X_{s^*} \cap Y) \not\subseteq Y$). By setting

$$Y_s = \theta_s(X_{s^*} \cap Y) \cap Y,$$

we may let

$$\gamma_s : Y_{s^*} \longrightarrow Y_s$$

be the restriction of θ_s to Y_{s^*}, for each s in S. Since

$$\begin{aligned}\theta_s(Y_{s^*}) &= \theta_s(\theta_{s^*}(X_s \cap Y) \cap Y) = \theta_s(\theta_{s^*}(X_s \cap Y) \cap (X_s \cap Y)) \\ &= (X_s \cap Y) \cap \theta_{s^*}(X_s \cap Y) = Y \cap \theta_{s^*}(X_s \cap Y) = Y_s,\end{aligned}$$

γ_s is a bijection, for all $s \in S$. We get that $\gamma = (\{Y_s\}_{s\in S}, \{\gamma_s\}_{s\in S})$ is a partial action of S on Y.

Notice that id_Y is an idempotent element of $\mathcal{I}(X)$ and that $\gamma : S \to \mathcal{I}(Y)$ is exactly the partial homomorphism given by

$$\gamma_s = \mathrm{id}_Y \circ \theta_s \circ \mathrm{id}_Y,$$

for all $s \in S$, as in the Example 1.1.12.

An inverse semigroup S may act partially on a topological space, an algebra, a C*-algebra, among other objects. For each situation we add some conditions in the definition of a partial action to be in accordance with the characteristics of each of these objects.

1.5.1 Topological partial actions

Definition 1.5.7. A (non-degenerate) *topological partial action* of an inverse semigroup S is a partial action $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in X_s})$ of S on a topological space X which satisfies:

(i) X_s is an open subset of X and $\theta_s : X_{s^*} \to X_s$ is a homeomorphism, for all $s \in S$,

(ii) $X = \bigcup_{e \in E(S)} X_e$.

Remark 1.5.8. Condition (ii) is usually called "non-degeneracy", and we will assume this condition in all partial actions that appear in our work. If just the condition (i) is satisfies, we can change X to $\bigcup_{e \in E(S)} X_e$.

Notice that if S has unit 1, then $e \leq 1$ for every idempotent e of S. Thus $X_e \subseteq X_1$, $X = \bigcup_{e \in E(S)} X_e = X_1$ and $\theta_1 = \mathrm{id}_X$, what is exactly the required condition in partial group actions of [24, Definition 1.1. (i)]

Example 1.5.9. Let $\mathcal{G}$ be an étale groupoid and $\mathcal{G}^{op}$ the semigroup of all open bisections of $\mathcal{G}$. There is a canonical action of $\mathcal{G}^{op}$ on the locally compact Hausdorff space $\mathcal{G}^{(0)}$ as follows: Given an open bisection B, we have that $\mathfrak{s}(B)$ and $\mathfrak{r}(B)$ are open subsets of $\mathcal{G}^{(0)}$, and moreover the maps

$$\mathfrak{s}_B : B \to \mathfrak{s}(B) \quad \text{and} \quad \mathfrak{r}_B : B \to \mathfrak{r}(B),$$

obtained by restricting $\mathfrak{s}$ and $\mathfrak{r}$, respectively, are homeomorphisms. Given $u \in \mathfrak{s}(B)$ we define

$$\theta_B(u) = \mathfrak{r}_B(\mathfrak{s}_B^{-1}(u)), \tag{1.12}$$

that is, $\theta_B(\mathfrak{s}(b)) = \mathfrak{r}(b)$, for all $b \in \mathcal{G}$ with $\mathfrak{s}(b) \in \mathfrak{s}(B)$.

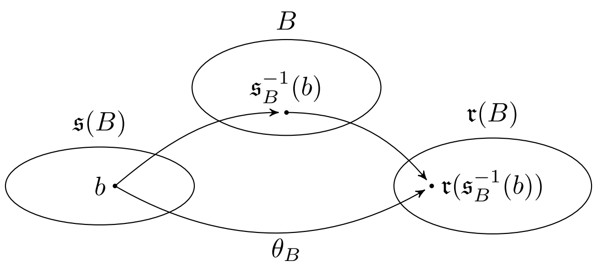


Figure 3 – Canonical action of the bisections

Clearly θ_B is a homeomorphism from $\mathfrak{s}(B)$ to $\mathfrak{r}(B)$. The collection $\theta = \left(\{\mathfrak{r}(B)\}_{B\in\mathcal{G}^{op}}, \{\theta_B\}_{B\in\mathcal{G}^{op}}\right)$ is an action of $\mathcal{G}^{op}$ on the unit space $\mathcal{G}^{(0)}$.

As $\mathcal{G}^{(0)} = \mathfrak{r}(\mathcal{G}^{(0)})$, θ is non-degenerate. Then it remains only to prove that, for every $B, C \in \mathcal{G}^{op}$,

$$\theta_C \circ \theta_B = \theta_{CB},$$

that is, $\theta_{B^*}(\mathfrak{r}(B)\cap\mathfrak{s}(C)) = \mathfrak{s}(CB)$ and $\theta_C \circ \theta_B(u) = \theta_{CB}(u)$, for all $u \in \mathfrak{s}(CB)$. We have that

$$\begin{aligned}
\theta_{B^*}(\mathfrak{r}(B)\cap\mathfrak{s}(C)) &= \mathfrak{s}\left(\mathfrak{r}_B^{-1}\left(\mathfrak{r}(B)\cap\mathfrak{s}(C)\right)\right)\\
&= \mathfrak{s}\left(\{b\in B \mid \mathfrak{r}(b)\in\mathfrak{s}(C)\}\right)\\
&= \{\mathfrak{s}(b)\in B \mid \mathfrak{r}(b)\in\mathfrak{s}(C)\}\\
&= \{\mathfrak{s}(cb)\in B \mid b\in B, c\in C \text{ and } \mathfrak{r}(b)=\mathfrak{s}(c)\}\\
&= \mathfrak{s}(CB).
\end{aligned}$$

Suppose that $u \in \mathfrak{s}(CB)$. Take $c \in C$ and $b \in B$ such that $u = \mathfrak{s}(cb)$. Thus

$$\begin{aligned}
\theta_{BC}(u) = \theta_{BC}(\mathfrak{s}(cb)) = \mathfrak{r}(cb) = \mathfrak{r}(c) = \theta_C(\mathfrak{s}(c)) = \theta_c(\mathfrak{r}(b))\\
= \theta_C(\theta_B(\mathfrak{s}(b)) = \theta_C(\theta_B(\mathfrak{s}(cb)) = \theta_B(\theta_C(u)),
\end{aligned}$$

as required.

We can replace $\mathcal{G}^{op}$ by the inverse semigroup $\mathcal{G}^a$ of the compact-open bisections and obtain, likewise, an action of $\mathcal{G}^a$ on $\mathcal{G}^{(0)}$.

Example 1.5.10. Let $\mathcal{G}$ be an étale groupoid. There is an action of the inverse semigroup $\mathcal{G}^{op}$ on the locally compact Hausdorff space $\mathrm{Iso}(\mathcal{G}^{(0)})$ as follows: Given $B \in \mathcal{G}^{op}$, we define

$$X_B = \{x \in \mathrm{Iso}(\mathcal{G}^{(0)}) \mid \mathfrak{s}(x) \in \mathfrak{r}(B)\} = \mathfrak{s}^{-1}(\mathfrak{r}(B)) \cap \mathrm{Iso}(\mathcal{G})$$

and

$$X_{B^*} = \{x \in \mathrm{Iso}(\mathcal{G}^{(0)}) \mid \mathfrak{s}(x) \in \mathfrak{s}(B)\} = \mathfrak{s}^{-1}(\mathfrak{s}(B)) \cap \mathrm{Iso}(\mathcal{G}),$$

which are open subsets of $\mathrm{Iso}(\mathcal{G}^{(0)})$. Notice that, for each $x \in X_{B^*}$, BxB^{-1} is a subset of X_B with only one element. In fact, BxB^{-1} is only defined when $\mathfrak{s}(x) \in \mathfrak{r}(B^{-1}) = \mathfrak{s}(B)$, and in this case, $y \in BxB^{-1}$ if there are $b, c \in B$ such that $\mathfrak{s}(b) = \mathfrak{r}(x) = \mathfrak{s}(x) = \mathfrak{r}(c^{-1}) = \mathfrak{s}(c)$ and $y = bxc^{-1}$. Since B is an open bisection, $b = c$ and b is unique element of B with this proprieties, and so, $BxB^{-1} = \{bxb^{-1}\}$.

For each $B \in \mathcal{G}^{op}$, we define $\theta_B : X_{B^*} \to X_B$ by

$$\theta_B(x) = BxB^{-1}.$$

Then, $\theta = (\{X_B\}_{B\in\mathcal{G}^{op}}, \{\theta_B\}_{B\in\mathcal{G}^{op}})$ is an action of $\mathcal{G}^{op}$ on $\mathrm{Iso}(\mathcal{G}^{(0)})$.

Moreover, the open subsets $\mathrm{int}(\mathrm{Iso}(\mathcal{G}))$ and $\mathcal{G}^{(0)}$ are invariant under θ. Indeed, for the first open subset, take $B \in \mathcal{G}^{op}$ and $x \in \mathrm{int}(\mathrm{Iso}(\mathcal{G})) \cap X_{B^*}$. Then, there is an open subset U of $\mathrm{Iso}(\mathcal{G}) \cap X_{B^*}$ containing x. As BUB^{-1} is an open subset of $\mathrm{Iso}(\mathcal{G}) \cap X_B$, we get that

$$\theta_B(x) \in BUB^{-1} \subseteq \mathrm{int}(\mathrm{Iso}(\mathcal{G})) \cap X_B,$$

proving that $\mathrm{int}(\mathrm{Iso}(\mathcal{G}))$ is θ-invariant.

In the case of the unit space $\mathcal{G}^{(0)}$, notice that, for each $B \in \mathcal{G}^{op}$, $X_B = \mathfrak{s}(B)$, $X_{B^*} = \mathfrak{r}(B)$ and θ_B is exactly the homeomorphism of the previous example defined in (1.12).

Example 1.5.11. Let X be a topological space and let $\varphi : X \to X$ be a homeomorphism. Consider $[[\psi]]$ the subset of $\mathcal{I}(X)$ consisting of all partial homeomorphisms φ of X for which there are a finite partition $X_1, \cdots, X_k$ of $\mathrm{dom}(\varphi)$ and a set of integers $\{n_1, \cdots, n_k\}$ such that $\varphi|_{X_i} = \psi^{n_i}|_{X_i}$, for all $i \in \{1, \cdots, k\}$. Then $[[\psi]]$ is an inverse subsemigroup of $\mathcal{I}(X)$, similarly as in Example 1.1.6. In particular $[[\psi]]$ acts naturally on X as in Example 1.4.5.

Example 1.5.12. Using the previous example we can get a partial topological inverse semigroup action of the inverse semigroup $[[\psi]]$ on an open subset Y of X by restriction, as in Example 1.5.6.

Example 1.5.13. To the directed graph $E = (E^0, E^1, r, s)$ (see Example 1.2.16) we can associate an inverse semigroup. Let

$$\mathcal{S}_E = \{(\mu, \nu) \mid \mu, \nu \in E^\star \text{ and } r(\mu) = r(\nu)\} \cup \{0\}.$$

The product is defined by

$$(\mu, \nu)(\zeta, \eta) = \begin{cases} (\mu, \eta\gamma), & \text{if } \nu = \zeta\gamma \text{ for some } \gamma \in E^\star \\ (\mu\gamma, \eta), & \text{if } \zeta = \nu\gamma \text{ for some } \gamma \in E^\star \\ 0, & \text{otherwise} \end{cases}$$

and involution on $\mathcal{S}_E$ is defined by $(\mu, \nu)^* = (\nu, \mu)$ and $0^* = 0$. By [61, Proposition 1], the set $\mathcal{S}_E$ with the operations above is an inverse semigroup.

Notice that the product of two pairs $(\mu, \nu), (\zeta, \eta)$ is non-zero if, and only if, ν is an initial segment of ζ or conversely. In this case, we say that ν, ζ are *comparable*. Moreover, if $\mu\alpha, \zeta\beta$ are comparable, then so are μ, ζ. It is easy see that if $(\mu, \nu) \leq (\zeta, \eta)$ then there is $\gamma \in E^\star$ such that $(\mu, \nu) = (\zeta\gamma, \eta\gamma)$ or $(\mu, \nu) = 0$. The set $E(\mathcal{S}_E)$ of all idempotents of $\mathcal{S}_E$ is just the set of pairs (μ, μ), where $\mu \in E^\star$.

Now, we will associate to a graph E an action of the inverse semigroup $\mathcal{S}_E$ on the boundary path space ∂E. As in Example 1.2.16,

$$\partial E = E^\infty \cup \{\mu \in E^\star \mid r(\mu) \text{ is singular}\},$$

for a finite path $\mu \in E^\star$,

$$Z(\mu) = \{\mu x \mid x \in \partial E \text{ and } r(\mu) = s(x)\} \subseteq \partial E,$$

and for a finite set $F \subseteq s^{-1}(r(\mu))$,

$$Z(\mu, F) = Z(\mu) \setminus \bigcup_{e \in F} Z(\mu e).$$

Recall that the collection of all sets of the form $Z(\mu, F)$ is basis for a topology on ∂E, and this topology makes ∂E a locally compact, Hausdorff and zero-dimensional space.

Given $(\mu,\nu) \in \mathcal{S}_E \setminus \{0\}$ we let

$$\begin{array}{rcl}\theta_{(\mu,\nu)} : Z(\nu) & \to & Z(\mu) \\ \nu x & \mapsto & \mu x\end{array}, \tag{1.13}$$

and $\theta_0 : \varnothing \to \varnothing$ is empty map. Let us prove that the collection

$$\theta = \left(\{Z(\mu)\}_{(\mu,\nu)\in\mathcal{S}_E}, \{\theta_{(\mu,\nu)}\}_{(\mu,\nu)\in S}\right)$$

is an action of the inverse semigroup $\mathcal{S}_E$ on the boundary path space ∂E.

It is immediate that each map $\theta_{(\mu,\nu)} : Z(\nu) \to Z(\mu)$ is a bijection between the compact-open subsets $Z(\mu)$ and $Z(\nu)$ of ∂E. By symmetry, it suffices to show that $\theta_{(\mu,\nu)}$ is an open map to conclude that this map is a homeomorphism. Since the image of a basic open $Z(\mu, F)$ by $\theta_{(\mu,\nu)}$ is also a basic open $Z(\nu, F)$, it follows that map $\theta_{(\mu,\nu)}$ is open.

The non-degeneracy of θ follows from the fact that $\bigcup_{v\in E^0} Z(v) = \partial E$. Let us verify that θ satisfies the conditions of Proposition 1.4.2:

(i) It is easy to see that $\theta_{(\mu,\nu)^*} = \theta_{(\nu,\mu)} = \theta^{-1}_{(\mu,\nu)}$.

(ii) We need prove that, for every $s, t \in \mathcal{S}_E$, $\theta_t^{-1}(X_t \cap X_{s^*}) = X_{(st)^*}$. There are six cases to consider

If $s = t = 0$: It is obvious.

If $s = 0$ and $t = (\zeta, \eta)$: We have that $X_{s^*} = \varnothing, X_t = Z(\zeta)$ and $X_{(st)^*} = \varnothing$. Thus,

$$\theta_{t^*}(X_t \cap X_{s^*}) = \varnothing = X_{(st)^*}.$$

If $s = (\mu, \nu)$ and $t = 0$: It is similar to the previous one.

If $s = (\mu, \nu)$, $t = (\zeta, \eta)$ and $st = 0$: Notice that in this case ν and ζ are not comparable and $Z(\zeta) \cap Z(\nu) = \varnothing$. Since $X_t = Z(\zeta), X_{s^*} = Z(\nu)$ and $X_{(st)^*} = \varnothing$, it follows that

$$\theta_t^{-1}(X_t \cap X_{s^*}) = \theta_{(\eta,\zeta)}(\varnothing) = \varnothing = X_{(st)^*}.$$

If $s = (\mu, \zeta\gamma)$ and $t = (\zeta, \eta)$: We have that $X_t = Z(\zeta), X_{s^*} = Z(\zeta\gamma)$ and $X_{(st)^*} = Z(\eta\gamma)$. Thus

$$\begin{aligned}\theta_t^{-1}(X_t \cap X_{s^*}) &= \theta_{(\eta,\zeta)}(Z(\zeta) \cap Z(\zeta\gamma)) \\ &= \theta_{(\eta,\zeta)}(Z(\zeta\gamma)) = Z(\eta\gamma) = X_{(st)^*}.\end{aligned}$$

If $s = (\mu, \nu)$ and $t = (\nu\gamma, \eta)$: It is similar to the previous one.

(iii) For every $s, t \in \mathcal{S}_E$ we need to prove that $\theta_s \circ \theta_t(x) = \theta_{st}(x)$, whenever that $x \in X_{(st)^*}$.

Clearly, in the first four cases of the previous item, we have $\theta_s \circ \theta_t = \theta_0 = \theta_{st}$. We still have two cases to check:

If $s = (\mu, \zeta\gamma)$ and $t = (\zeta, \eta)$: For any $x = \eta\gamma y \in Z(\eta\gamma) = X_{(st)^*}$, we get

$$\begin{aligned}\theta_s(\theta_t(x)) &= \theta_{(\mu,\zeta\gamma)}(\theta_{(\zeta,\eta)}(\eta\gamma y)) = \theta_{(\mu,\zeta\gamma)}(\zeta\gamma y) \\ &= \mu y = \theta_{(\mu,\eta\gamma)}(\eta\gamma y) = \theta_{st}(x).\end{aligned}$$

If $s = (\mu, \nu)$ and $t = (\nu\gamma, \eta)$: It is similar to the previous one.

1.5.2 From partial actions of groups to partial actions of inverse semigroups and vice-versa

We will now describe how to construct partial actions of groups from actions of inverse semigroups and vice-versa. The class of inverse semigroups which allows us to do this more precisely is the of E-unitary inverse semigroups.

Recall that an inverse semigroup S is *E-unitary* if, whenever $s \in S$, $e \in E(S)$ and $e \leq s$, then $s \in E(S)$.

There are other similar ways to define E-unitary inverse semigroups. For example, S is E-unitary if, whenever $s \in S$, $e \in E(S)$ and $es \in E(S)$ then $s \in E(S)$, or equivalently, if $s \in S$, $e \in E(S)$ and $se \in E(S)$ imply $s \in E(S)$ (see [48, Lemma 2.4.3]).

Example 1.5.14. Every group is E-unitary.

Example 1.5.15. Every $\wedge$-semilattice is E-unitary.

Example 1.5.16. An inverse semigroup with 0 is E-unitary if, and only if, $S = E(S)$, which from our point of view is a rather degenerate inverse semigroup.

In the case when S is an inverse semigroup with 0, we say that S is E^*-unitary (or 0-E-unitary) if, whenever $e \in E(S) \setminus \{0\}$, $s \in S$, and $e \leq s$, then $s \in E(S)$.

Let S be an inverse semigroup. For every $s, t \in S$, the *compatibility relation* is defined by

$$s \approx t \iff st^*,\ s^*t \in E(S). \tag{1.14}$$

It is clear that this relation is reflexive and symmetric, but need not be transitive. A subset T of an inverse semigroup is said to be *compatible* if any pair of elements in T are compatible.

Proposition 1.5.17. *[48, Theorem 2.4.4] Let S be an inverse semigroup. Then the compatibility relation is transitive if, and only if, S is E-unitary.*

For each inverse semigroup S we can naturally associate a group $\mathbf{G}(S)$ in the following manner: we define a relation in S by

$$s \sim t \iff \exists\ u \in S \text{ such that } u \leq s, t. \tag{1.15}$$

Alternatively, $s \sim t$ if, and only if, there is $e \in E(S)$ such that $es = et$.

It is easy to see that $\sim$ is an equivalence relation. Moreover, from (1.15) and the fact that the order of S is preserved under products and inverses, we have that $\sim$ is in fact a congruence. We endow $S/{\sim}$ with the quotient semigroup structure. Given $s \in S$, we denote by $[s]$ the equivalence class of s with respect to the relation (1.15).

Proposition 1.5.18. *[60, Proposition 2.1.2] Given an inverse semigroup S the quotient*

$$\mathbf{G}(S) := S/{\sim}$$

is a group. Furthermore, $\mathbf{G}(S)$ *is the* maximal group homomorphic image *of* S *in the sense that if there is a semigroup homomorphism* ψ *form* S *to a group* G *then* ψ *factors through* $\mathbf{G}(S)$.

Example 1.5.19. If G is a group then $\mathbf{G}(G)$ is isomorphic to G.

Example 1.5.20. If L is a $\wedge$ - semilattice then $\mathbf{G}(L) = \{1\}$ is the trivial group.

Example 1.5.21. If S is an inverse semigroup with a zero, then $\mathbf{G}(S) = \{1\}$ is the trivial group.

Given an inverse semigroup S, we have that any two idempotent elements are related by 1.15. But, in general, non-idempotent elements may be related to idemponents, as in the Example 1.5.21. It is immediate that if the set of idempotents of S forms an equivalence class for the relation (1.15) then S is a E-unitary. More precisely, consider $\sigma : S \to \mathbf{G}(S)$ the quotient map and 1 the unit of $\mathbf{G}(S)$. Then S is E-unitary if, and only if, $\sigma^{-1}(1) = E(S)$.

We can also reword the E-unitary property in terms of compatibility of elements.

Lemma 1.5.22. *[48, Theorem 2.4.6] Let* S *be an inverse semigroup. Then* S *is* E*-unitary if, and only if,* $s, t \in S$ *and* $s \sim t$ *implies that* $s \approx t$.

We will now be interested in relating partial actions of inverse semigroups and partial actions of their maximal group images. A version this proposition has been proven in [72, Lemma 3.8] when considering global actions of inverse semigroups. The next theorem is a specific instance of [46, Lemma 2.2], where the author in fact considers a *strictly weaker* notion of partial action – namely, condition 1.1.8(iii) is not required. Note that this condition is trivial when considering partial actions of groups, and thus we may apply [46, Lemma 2.2] without problems.

Proposition 1.5.23 ([46, Remark 2.3]). *Let $\theta = \left(\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S}\right)$ be a topological partial action of an E-unitary inverse semigroup S on a space X. Then there is a unique partial action*

$$\widetilde{\theta} = \left(\left\{\widetilde{X}_\gamma\right\}_{\gamma\in\mathbf{G}(S)}, \left\{\widetilde{\theta}_\gamma\right\}_{\gamma\in\mathbf{G}(S)}\right)$$

of $\mathbf{G}(S)$ on X such that, for any $s \in S$,

(i) $\widetilde{X}_\gamma = \bigcup_{[s]=\gamma} X_s$, for all $\gamma \in \mathbf{G}(S)$,

(ii) $\widetilde{\theta}_{[s]}(x) = \theta_s(x)$, for all $s \in S$ and $x \in X_{s^*}$.

(in other words, $\widetilde{\theta}_\gamma$ is the join of $\{\theta_s : [s] = \gamma\}$ in $\mathcal{I}(X)$, which is commonly denoted by $\bigvee_{[s]=\gamma} \theta_s$).

Now, we will be interested in the other direction, that is, to each group we want to associate an inverse semigroup (which will be E-unitary). This construction was initially done in [27], where Exel defines the universal inverse semigroup $\mathbf{S}(G)$ of a group G.

Given a group G, let $\mathbf{S}(G)$ be the *universal semigroup* generated by symbols of the form $[g]$, $g \in G$, modulo the relations

(i) $[g^{-1}][g][h] = [g^{-1}][gh]$,

(ii) $[g][h][h^{-1}] = [gh][h^{-1}]$,

(iii) $[g][1] = [g]$,

(iv) $[1][g] = [g]$.

Exel proved that $\mathbf{S}(G)$ is an inverse semigroup with unit $[1]$ (see [27, Theorem 3.4]). We will describe all the necessary properties of $\mathbf{S}(G)$ that we will need. For every $g \in G$, the inverse of $[g]$ is $[g^{-1}]$. Let us denote

$$\epsilon_g = [g][g^{-1}].$$

By [27, Proposition 2.5 and 3.2], for each $\gamma \in \mathbf{S}(G)$, there is a unique $n \geq 0$ and distinct elements $r_1, \dots, r_n, g \in G$ such that

1. $\gamma = \epsilon_{r_1} \cdots \epsilon_{r_n}[g]$, (if $n = 0$, this is simply $[g]$), and

2. $r_i \neq 1$ for all i.

We call such a decomposition $\gamma = \epsilon_{r_1} \cdots \epsilon_{r_n}[g]$ the *standard form* of γ, which is unique up to the order of $r_1, \ldots, r_n$. Moreover, given $g, r \in G$, we have $[g]\epsilon_r = \epsilon_{gr}[g]$. Thus, for $\gamma = \epsilon_{r_1} \cdots \epsilon_{r_n}[g] \in \mathbf{S}(G)$, the inverse of γ is written in standard form as

$$\gamma^* = [g^{-1}]\epsilon_{r_n} \cdots \epsilon_{r_1} = \epsilon_{g^{-1}r_n} \cdots \epsilon_{g^{-1}r_1}[g^{-1}],$$

The idempotents of $\mathbf{S}(G)$ are the elements of the form $\epsilon = \epsilon_{r_1} \cdots \epsilon_{r_n}[1]$.

Example 1.5.24. Take $G = \mathbb{Z}_2 = \{1, g\}$. In this case, we have that $g = g^{-1}$. Consider $1 = [1]$, $s = [g]$ and $e = [g][g^{-1}]$. Then 1 and e are idempotent and since $es = [g][g^{-1}][g] = [g] = s$, $se = [g][g^{-1}][g] = [g] = s$ and $ss = [g][g] = e$, we can conclude that $\mathbf{S}(\mathbb{Z}_2) = \{1, e, s\}$.

For any group G the inverse semigroup associated $\mathbf{S}(G)$ is E-unitary ([27, Remark 3.5]). Indeed, suppose $\gamma \in \mathbf{S}(G)$, $\epsilon \in E(\mathbf{S}(G))$ and $\epsilon \leq \gamma$. Writing γ and ϵ in standard form, we obtain

$$\gamma = \epsilon_{s_1} \cdots \epsilon_{s_n}[s] \qquad \text{and} \qquad \epsilon = \epsilon_{e_1} \cdots \epsilon_{e_m}[1].$$

Since $\epsilon = \epsilon\gamma$ and $[1]$ is a unit of $\mathbf{S}(G)$, we obtain

$$\epsilon_{e_1} \cdots \epsilon_{e_m}[1] = \epsilon = \epsilon\gamma = \epsilon_{e_1} \cdots \epsilon_{e_m}\epsilon_{s_1} \cdots \epsilon_{s_n}[s].$$

From the uniqueness of the standard form of ϵ we conclude that $s = 1$ and γ is an idempotent.

The main result of [27] is the following property of the semigroup $\mathbf{S}(G)$. Although it is proven in principle only for partial on discrete sets, the same proof applies in the topological setting.

Proposition 1.5.25. *[27, Theorem 4.2.] Let $\theta = \left(\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S}\right)$ be a topological partial action of a group G on a space X. Then there is a unique topological action $\overline{\theta}$ of $\mathbf{S}(G)$ on X such that $\overline{\theta}_{[g]} = \theta_g$, for all $g \in G$.*

Proposition 1.5.26. *Let G be a group and $\mathbf{S}(G)$ the universal semigroup of G. Then the map $G \to \mathbf{G}(\mathbf{S}(G))$, $g \mapsto [[g]]$, is an isomorphism.*

Proof. First note that for all $s, t \in G$,

$$[s][t] = [s][t][t^{-1}][t] = [st]\epsilon_t \leq [st],$$

so the map $G \to S(G)$, $g \mapsto [g]$, is a partial homomorphism. Since the map $S(G) \to \mathbf{G}(S(G))$, $\gamma \mapsto [\gamma]$, is a homomorphism, we have that $g \mapsto [[g]]$ is a partial homomorphism between groups, hence a homomorphism.

Given $\gamma \in \mathbf{S}(G)$, since $\gamma = \epsilon_{s_1} \cdots \epsilon_{s_n}[s]$ for certain $s, s_1, \ldots, s_n \in G$, we get $[\gamma] = [[s]]$, so $g \mapsto [[g]]$ is surjective.

If $[[g]] = 1 = [[1]]$, then there is an idempotent $\epsilon = \epsilon_{e_1} \cdots \epsilon_{e_n}[1]$ for which

$$[g]\epsilon_{e_1} \cdots \epsilon_{e_n} = [1]\epsilon_{e_1} \cdots \epsilon_{e_n}$$

and the uniqueness of the standard form implies $g = 1$. □

1.5.3 Algebraic partial actions

Definition 1.5.27. Let S be an inverse semigroup, let R be a unital commutative ring and let A be an associative R-algebra. A (non-degenerate) *algebraic partial action* of S on A is a partial action $\alpha = (\{D_s\}_{s\in S}, \{\alpha_s\}_{s\in S})$ such that

(i) for each $s \in S$, D_s is an ideal of A and $\alpha_s : D_{s^*} \to D_s$ is an R-isomorphism,

(ii) $A = \mathrm{Span}_R\left(\bigcup_{e\in E(S)} D_e\right)$.

Example 1.5.28. We assume that R is a unital commutative ring, and X is a Hausdorff, locally compact, zero-dimensional topological space. Let $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$ be the commutative R-algebra formed by all locally constant, compactly supported, R-valued functions on X, and with pointwise addition and product. Notice that $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$ is exactly the Steinberg algebra of X (see Example 1.3.10). Therefore, every $f \in \mathcal{L}_{\mathrm{c}}(\mathrm{X})$ is a linear combination of the form

$$f = \sum_{i=1}^{n} r_i 1_{K_i},$$

where the K_i are pairwise disjoint compact-open subsets, and each r_i lies in R.

Let $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ be a topological partial action of an inverse semigroup S on a locally compact, Hausdorff, zero-dimensional topological space X. Such an action induces an action in the algebra level, as done in [4] and [25]: For each s in S, consider the ideal

$$D_s = \{f \in \mathcal{L}_{\mathrm{c}}(\mathrm{X}) \mid \mathrm{supp}(f) \subseteq X_s\}$$

in $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$, and define the R-isomorphism $\alpha_s : D_{s^*} \to D_s$ by

$$\alpha_s(f)(x) = \begin{cases} f \circ \theta_{s^*}(x), & \text{if } x \in X_s, \\ 0, & \text{if } x \notin X_s. \end{cases}$$

It is routine to check that $\alpha = (\{D_s\}_{s\in S}, \{\alpha_s\}_{s\in S})$ is a partial action. We wish to convince the reader that α is non-degenerate. Let f be an element of $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$. By non-degeneracy of θ, for any $x \in \mathrm{supp}(f)$ there is a compact-open neighborhood L of x contained in X_e, for some $e \in E(S)$, and such that $f|_L$ is constant. By compactness of $\mathrm{supp}(f)$ we can find finitely many compact-open subsets $L_1, \ldots, L_n$ such that $L_i \subseteq X_{e_i}$, $e_i \in E(S)$, and $\mathrm{supp}(f) \subseteq \bigcup_{i=1}^{n} L_i$. By putting

$$K_1 = L_1 \cap \mathrm{supp}(f) \quad \text{and} \quad K_j = \left(L_j \setminus \bigcup_{i=1}^{j-1} (L_i)\right) \cap \mathrm{supp}(f),$$

for all $j \in \{2, \ldots, n\}$, we get that $\mathrm{supp}(f)$ is equal to the disjoint union of the compact-open subsets $K_1, \ldots, K_n$, and that

$$f = \sum_{i=1}^{n} f 1_{K_i} \in \mathrm{Span}_R \left(\bigcup_{e\in E(S)} D_e \right),$$

because $\mathrm{supp}(f) \subseteq D_{e_i}$.

Again, we will only be interested in non-degenerate algebraic partial actions, and it is more suitable to assume this as a necessary condition on all the partial actions we will work. Similarly as in Remark 1.5.8, we may always replace A by $\mathrm{Span}_R\left(\bigcup_{e\in E(S)} D_e\right)$.

Remark 1.5.29. We can define a partial action of an inverse semigroup on a ring A similarly to Definition 1.5.27, just making a change in item (ii): we assume that for each $s\in S$, α_s is an isomorphism of rings.

1.6 Partial skew inverse semigroup algebras

Given a partial action of an inverse semigroup on an algebra we can construct a new algebra associated to this structure, namely the partial skew inverse semigroup algebra (or algebraic crossed product of inverse semigroup). Partial skew inverse semigroup algebras are defined in the same way as skew inverse semigroups in [33] (for actions of inverse semigroups). They are also a generalization of the partial skew group algebras (see [24]), but their construction needs more steps.

Throughout this section we will assume that R is a unital commutative ring, and $\alpha=(\{D_s\}_{s\in S},\{\alpha_s\}_{s\in S})$ is an algebraic partial action of an inverse semigroup S on a R-algebra A.

Definition 1.6.1. Let $\alpha=(\{D_s\}_{s\in S},\{\alpha_s\}_{s\in S})$ be an algebraic partial action of S on A. Let $\mathcal{L}(\alpha)$ be the R-module of all finite formal sums

$$\sum_{s\in S} a_s\delta_s,$$

where $a_s\in D_s$ and δ_s is a formal symbol. More precisely, $\mathcal{L}(\alpha)$ is the free R-module generated by formal elements $a_s\delta_s$, satisfying the relations

$$a_s\delta_s+b_s\delta_s=(a_s+b_s)\delta_s$$

and

$$\lambda(a_s\delta_s)=(\lambda a_s)\delta_s,$$

for all $s\in S$, $a_s,b_s\in D_s$ and $\lambda\in R$.

We define a product as the linear extension of the rule

$$(a_s\delta_s)(a_t\delta_t) = \alpha_s(\alpha_{s^*}(a_s)a_t)\delta_{st}. \tag{1.16}$$

Since $\alpha_{s^*}(a_s)a_t \in D_{s^*} \cap D_t$ then

$$\alpha_s(\alpha_{s^*}(a_s)a_t) \in \alpha_s(D_{s^*} \cap D_t) = D_s \cap D_{st},$$

and the product is well-defined.

Remark 1.6.2. In the case of a group partial action, the R-algebra $\mathcal{L}(\alpha)$ is exactly the partial skew group algebra (see [24]).

In general, this product might make $\mathcal{L}(\alpha)$ a non-associative algebra (see [24, Example 3.5]).

Lemma 1.6.3. *Let $\alpha = (\{D_s\}_{s\in S}, \{\alpha_s\}_{s\in S})$ be an algebraic partial action of S on A. A necessary and suficient condition for $\mathcal{L}(\alpha)$ to be associative is that,*

$$a\alpha_s(\alpha_{s^*}(b)c) = \alpha_s(\alpha_{s^*}(ab)c),$$

for all $s \in S$, $b \in D_s$, and $a, c \in A$.

Proof. Given $d\delta_r, b\delta_s, a\delta_t \in \mathcal{L}(\alpha)$ we need to prove that

$$d\delta_r(b\delta_s c\delta_t) = (d\delta_r b\delta_s)c\delta_t \tag{1.17}$$

if, and only if,

$$d\alpha_s(\alpha_{s^*}(b)c) = \alpha_s(\alpha_{s^*}(db)c).$$

Let us start by rewriting the right-hand of (1.17):

$$d\delta_r(b\delta_s c\delta_t) = d\delta_r(\alpha_s[\alpha_{s^*}(b)c])\delta_{st} = \alpha_r(\alpha_{r^*}(d)\alpha_s[\alpha_{s^*}(b)c])\delta_{rst} \tag{1.18}$$

On the other hand, the left side of (1.17) may be rewritten as

$$(d\delta_r b\delta_s)c\delta_t = \alpha_r[\alpha_{r^*}(d)b]\delta_{rs}c\delta_t = \alpha_{rs}[\alpha_{(rs)^*}(\alpha_r[\alpha_{r^*}(d)b])c]\delta_{rst}.$$

Notice that

$$\alpha_r[\alpha_{r^*}(d)b] \in \alpha_r(D_{r^*} \cap D_s) = D_r \cap D_{rs}.$$

On $D_r \cap D_{rs}$,

$$\alpha_{(rs)^*} = \alpha_{s^*} \circ \alpha_{r^*}.$$

Hence

$$(d\delta_r b\delta_s)c\delta_t = \alpha_{rs}(\alpha_{s^*}[\alpha_{r^*}(d)b]c)\delta_{rst}.$$

Observe also that

$$\alpha_{s^*}[\alpha_{r^*}(d)b] \in \theta_{s^*}(D_{r^*} \cap D_s) = D_{s^*} \cap D_{(rs)^*}.$$

We thus finally obtain that

$$(d\delta_r b\delta_s)c\delta_t = \alpha_r[\alpha_s(\alpha_{s^*}[\alpha_{r^*}(d)b]c)]\delta_{rst}. \tag{1.19}$$

Comparing (1.18) and (1.19) we have that the associativity of $\mathcal{L}(\alpha)$ holds if, and only if,

$$\alpha_r(\alpha_{r^*}(d)\alpha_s[\alpha_{s^*}(b)c])\delta_{rst} = \alpha_r[\alpha_s(\alpha_{s^*}[\alpha_{r^*}(d)b]c)]\delta_{rst},$$

which is clearly the same as

$$\alpha_{r^*}(d)\alpha_s(\alpha_{s^*}(b)c) = \alpha_s(\alpha_{s^*}(\alpha_{r^*}(d)b)c).$$

Writing $a = \alpha_{r^*}(d)$, we get the desired equality. $\square$

A ring R is said to have *local units* if for every finite subset F of R, there exists an idempotent $e \in R$ such that $F \subseteq eRe$. In this case, $r = er = re$ holds for each $r \in F$ and the element e will be referred to as a local unit for the set F.

A ring B is *left (right) s-unital* if $b \in Bb$ ($b \in bB$), for all $b \in B$. From [76, Theorem 1] it follows that B is left (right) s-unital if, and only if, for all $n \in \mathbb{N}$ and all $b_1, \cdots, b_n \in B$, there is $u \in B$ such that, for each $i \in \{1, \cdots, n\}$, the equality $ub_i = b_i$ ($b_i = ub_i$) holds. Obviously every ring that has local units is s-unital.

Lemma 1.6.4. *Suppose that $\alpha = (\{D_s\}_{s\in S}, \{\alpha_s\}_{s\in S})$ is an algebraic partial action of an inverse semigroup S on a R-algebra A, and that every ideal D_s is left (right) s-unital. Then $\mathcal{L}(\alpha)$ is associative.*

Proof. Notice that, by Lemma 1.6.3, is it enough to show the equality

$$a\alpha_s(\alpha_{s^*}(b)c) = \alpha_s(\alpha_{s^*}(ab)c),$$

for all $s \in S$, $b \in D_s$, and $a, c \in A$. Since D_s is left s-unital, there is $u \in D_s$ such that $u\alpha_s(\alpha_{s^*}(b)c) = \alpha_s(\alpha_{s^*}(b)c)$ and $ub = b$. Then

$$\begin{aligned} a\alpha_s(\alpha_{s^*}(b)c) &= au\alpha_s(\alpha_{s^*}(b)c) = \alpha_s(\alpha_{s^*}(au))\alpha_s(\alpha_{s^*}(b)c) \\ &= \alpha_s(\alpha_{s^*}(aub)c) = \alpha_s(\alpha_{s^*}(ab)c), \end{aligned}$$

as required.

Since D_s is right s-unital, there is $v \in D_{s^*}$ such that $\alpha_{s^*}(b)v = \alpha_{s^*}(b)$ and $\alpha_{s^*}(ab)v = \alpha_{s^*}(ab)$. Then

$$\begin{aligned} a\alpha_s(\alpha_{s^*}(b)c) &= a\alpha_s(\alpha_{s^*}(b)vc) = a\alpha_{ss^*}(b)\alpha_s(vc) = ab\alpha_s(vc) \\ &= \alpha_{ss^*}(ab)\alpha_s(vc) = \alpha_s(\alpha_{s^*}(ab)vc) = \alpha_s(\alpha_{s^*}(ab)c). \quad \square \end{aligned}$$

Definition 1.6.5. Let $\alpha = (\{D_s\}_{s\in S}, \{\alpha_s\}_{s\in S})$ be an algebraic partial action of S on A. We denote by $\mathcal{N}(\alpha)$ the two-sided ideal

$$\langle\ a\delta_r - a\delta_s \mid r, s \in S,\ r \leq s \text{ and } a \in D_s\ \rangle,$$

that is, $\mathcal{N}(\alpha)$ is the two-sided ideal of $\mathcal{L}(\alpha)$ generated by all elements of the form $a\delta_r - a\delta_s$, where $r \leq s$ (notice that $a \in D_r$ also).

Lemma 1.6.6. *[3, Lemma 2.3] Let $\alpha = (\{D_s\}_{s\in S}, \{\alpha_s\}_{s\in S})$ be an algebraic partial action of S on A. The ideal $\mathcal{N}(\alpha)$ is equal to the additive group generated by the set $\{a\delta_r - a\delta_s \mid r, s \in S,\ r \leq s \text{ and } a \in D_r\}$.*

Proof. It is enough to show that for $r, s, t, u \in S$ with $r \leq s$, and $a \in D_r$, $b \in D_t$, $c \in D_u$, it holds that the elements $b\delta_t(a\delta_r - a\delta_s)$, $(a\delta_r - a\delta_s)c\delta_u$ and $b\delta_t(a\delta_r - a\delta_s)c\delta_u$ are all of the form $x\delta_v - x\delta_w$ for some $v, w \in S$ and $x \in D_v$, such that $v \leq w$. Notice that

$$b\delta_t(a\delta_r - a\delta_s) = b\delta_t a\delta_r - b\delta_t a\delta_s = \alpha_t(\alpha_{t^*}(b)a)\delta_{tr} - \alpha_t(\alpha_{t^*}(b)a)\delta_{ts}$$

and, since $tr \leq ts$, we are done in this case.

In the next case we get

$$(a\delta_r - a\delta_s)c\delta_u = \alpha_r(\alpha_{r^*}(a)c)\delta_{ru} - \alpha_s(\alpha_{s^*}(a)c)\delta_{su}.$$

Using that $r \leq s$ we get that $r^* \leq s^*$ and $ru \leq su$. Then $a \in X_s$, $\alpha_{r^*}(a) = \alpha_{s^*}(a)$ and $\alpha_r(\alpha_{r^*}(a)c) = \alpha_r(\alpha_{s^*}(a)c) = \alpha_s(\alpha_{s^*}(a)c)$ and hence the desired conclusion follows.

To conclude, notice that $b\delta_t(a\delta_r - a\delta_s)c\delta_u$ is of the form $(x\delta_{tr} - x\delta_{ts})c\delta_u$ by the first case, and now, by the second case, $(x\delta_{tr} - x\delta_{ts})c\delta_u$ has the desired form. □

Finally, we define the corresponding skew inverse semigroup algebra associated to a partial action of an inverse semigroup.

Definition 1.6.7. Let $\alpha = (\{D_s\}_{s\in S}, \{\alpha_s\}_{s\in S})$ be a partial action of an inverse semigroup S on an R-algebra A. The *skew inverse semigroup algebra associated to* α, which we denote by $A \rtimes_\alpha S$, is the quotient algebra

$$\frac{\mathcal{L}(\alpha)}{\mathcal{N}(\alpha)}.$$

Elements of $A \rtimes_\alpha S$ will be written as $\overline{x}$, where $x \in \mathcal{L}(\alpha)$.

It is interesting to point out that the quotient involved in the definition of a skew inverse semigroup algebra is motivated by the C*-algebraic definition of crossed products by inverse semigroups (see, for example, [68], [60], [33]).

Remark 1.6.8. Let $s, t \in S$. Notice that if $s \leq t$ and $a \in D_s$, then $\overline{a\delta_s} = \overline{a\delta_t}$.

Remark 1.6.9. In the case that S is a group, the natural order of S coincides with equality, and hence the ideal $\mathcal{N}(\alpha) = \{0\}$. Therefore, the partial skew inverse semigroup is simply

$$A \rtimes_\alpha S = \mathcal{L}(\alpha),$$

which is exactly the partial skew group algebras defined in [24]. Notice that the more general structure of partial skew inverse semigroup algebras does not carry the graded structure presented in the group case.

Instead, we only have that every partial skew inverse semigroup ring admits a pre-grading (defined below) over the semigroup.

Definition 1.6.10. Let B be any R-algebra and let S be an inverse semigroup. A pre-grading of B over S is a family of linear subspaces $\{B_s\}_{s\in S}$ of B, such that for every $s,t \in S$ one has that

(i) $B_sB_t \subseteq B_{st}$,

(ii) if $s \leq t$ then $B_s \subseteq B_t$,

(iii) $B = \operatorname{Span}_R \left(\bigcup_{s\in S} B_s\right)$.

Remark 1.6.11. Notice that if I is an ideal of A then we can give the partial skew inverse semigroup algebra $A \rtimes_\alpha S$ an I-module structure by defining

$$b(\overline{a_s\delta_s}) = \overline{(ba_s)\delta_s}, \quad (\overline{a_s\delta_s})b = \overline{(a_sb)\delta_s},$$

for all $b \in I$ and $\overline{a_s\delta_s} \in A \rtimes_\alpha S$.

1.6.1 Covariant representation

Definition 1.6.12. Let $\alpha = (\{D_s\}_{s\in S}, \{\alpha_s\}_{s\in S})$ be a partial action of S on A. A *covariant representation of* α consists of an algebra B and a pair (π, u), where $\pi : A \to B$ is an R-homomorphism, and $u : S \to B$ is a partial homomorphism of S to the multiplicative semigroup of B, satisfying the *covariance condition*

$$\pi(\alpha_s(a_{s^*})) = u(s)\pi(a_{s^*})u(s^*), \tag{1.20}$$

for all $s \in S$ and $a_s \in D_{s^*}$. We say that (π, u) is *non-degenerate* if $u(s)u(s^*) \in \pi(D_s)$, for all $s \in S$.

Proposition 1.6.13. *[19, Lemma 4.3.14. and Theorem 4.3.15.] Let $\alpha = (\{D_s\}_{s\in S}, \{\alpha_s\}_{s\in S})$ be an algebraic partial action of S on A. Also let (π, u) be a non-degenerate covariant representation of $A \rtimes_\alpha S$ in an R-algebra B. Then, for every $s,t \in S$, we have have that:*

(a) If $a_s \in D_s$, then

$$\pi(a_s) = u(s)u(s^*)\pi(a_s) = \pi(a_s)u(s)u(s^*),$$

(b) If $s \leq t$ and $a_s \in D_s$, then

$$\pi(a_s)u(s) = \pi(a_s)u(t),$$

(c) If D_s has unit 1_s, then

$$\pi(1_s)u(s) = u(s), \quad u(s^*)\pi(1_s) = u(s^*) \quad \text{and} \quad \pi(1_s) = u(s)u(s^*),$$

(d) The linear mapping $\pi \times u : A \rtimes_\alpha S \to B$ determined by

$$(\pi \times u)(\overline{a_s\delta_s}) = \pi(a_s)u(s), \tag{1.21}$$

for all $\overline{a_s\delta_s} \in A \rtimes_\alpha S$, is a R-homomorphism.

Proof. (a) By the covariance relation, we have that

$$\begin{aligned}\pi(a_s) &= \pi(\alpha_s(\alpha_{s^*}(a_s))) = u(s)\pi(\alpha_{s^*}(a_s))u(s^*)\\ &= u(s)u(s^*)\pi(a_s)u(s)u(s^*).\end{aligned}$$

Since u is partial homomorphism of inverse semigroups, we get that

$$\begin{aligned}u(s)u(s^*)\pi(a_s) &= u(s)u(s^*)\left[u(s)u(s^*)\pi(a_s)u(s)u(s^*)\right]\\ &= u(s)u(s^*)\pi(a_s)u(s)u(s^*) = \pi(a_s).\end{aligned}$$

Similarly, $\pi(a_s)u(s)u(s^*) = \pi(a_s)$.

(b) Since $s \leq t$ and u is partial homomorphism, it follows that $u(s^*s) = u(s^*t)$, and by (a),

$$\begin{aligned}\pi(a_s)u(t) &= \pi(a_s)u(s)u(s^*)u(t) = \pi(a_s)u(s)u(s^*t)\\ &= \pi(a_s)u(s)u(s^*)u(s) = \pi(a_s)u(s).\end{aligned}$$

(c) Notice that $\pi(1_s)$ is the unit of $\pi(D_s)$. By non-degeneracy of (π, u), we have that $u(s)u(s^*) \in \pi(D_s)$, so

$$u(s) = u(s)u(s^*)u(s) = \pi(1_s)u(s)u(s^*)u(s) = \pi(1_s)u(s).$$

Analogously, $u(s^*) = u(s^*)\pi(1_s)$. Moreover, multiplying by $u(s^*)$ both sides of the equality $u(s) = \pi(1_s)u(s)$, we obtain

$$u(s)u(s^*) = \pi(1_s)u(s)u(s^*) \overset{(a)}{=} \pi(1_s).$$

(d) Notice that, by item (b), the map $\pi \times u$ given by (1.21) is well-defined. To prove that $\pi \times u$ is multiplicative, we take $\overline{a_s\delta_s}, \overline{b_t\delta_t} \in A \rtimes_\alpha S$, then

$$\begin{aligned}
(\pi \times u)\left(\overline{a_s\delta_s}\ \overline{a_t\delta_t}\right) &= \pi(\alpha_s(\alpha_{s^*}(a_s)b_t))u(st) \\
&\overset{(1.20)}{=} u(s)\pi(\alpha_{s^*}(a_s)b_t)u(s^*)u(st) \\
&= u(s)\pi(\alpha_{s^*}(a_s))\pi(b_t)u(s^*)u(s)u(t) \\
&\overset{(1.20)}{=} u(s)u(s^*)\pi(a_s)u(s)\pi(b_t)u(s^*)u(s)u(t) \\
&\overset{(a)}{=} \pi(a_s)u(s)\pi(b_t)u(s^*)u(s)u(t) \\
&\overset{(c)}{=} \pi(a_s)u(s)\pi(1_{s^*})\pi(b_t)u(s^*)u(s)u(t) \\
&= \pi(a_s)u(s)\pi(1_{s^*}b_t)u(s^*)u(s)u(t) \\
&\overset{(a)}{=} \pi(a_s)u(s)\pi(1_{s^*}b_t)u(t) \\
&\overset{(c)}{=} \pi(a_s)u(s)\pi(b_t)u(t) \\
&= (\pi \times u)(\overline{a_s\delta_s})(\pi \times u)(\overline{b_t\delta_t}).
\end{aligned}$$ □

Let $\alpha = (\{D_s\}_{s\in S}, \{\alpha_s\}_{s\in S})$ be an algebraic partial action of S on A. Just as for the skew group algebras, we may define an additive map $\tau : \mathcal{L}(\alpha) \to A$ by

$$\tau\left(\sum_{s\in S} a_s\delta_s\right) = \sum_{s\in S} a_s. \tag{1.22}$$

Remark 1.6.14. By Lemma 1.6.6, we have that $\tau(\mathcal{N}(\alpha)) = \{0\}$ and hence we get a well-defined additive map $\widetilde{\tau} : \mathcal{A} \rtimes_\pi S \to A$ given by $\widetilde{\tau}(\overline{x}) = \tau(x)$, for $x \in \mathcal{L}(\alpha)$.

Definition 1.6.15. [3, Proposition 3.1.] Let $\alpha = (\{D_s\}_{s\in S}, \{\alpha_s\}_{s\in S})$ be an algebraic partial action of S on A. The *diagonal* subalgebra $\mathcal{D}$ of a partial skew inverse semigroup algebra $A \rtimes_\alpha S$ is the subalgebra generated by elements of the form $\overline{a\delta_e}$, where $e \in E(S)$ and $a \in D_e$, that is,

$$\mathcal{D} = \left\{ \sum_{i=1}^{n} \overline{a_i\delta_{e_i}} \;\middle|\; n \in \mathbb{N},\ e_i \in E(S),\ a_i \in D_{e_i} \right\}.$$

Proposition 1.6.16. *Let $\alpha = (\{D_s\}_{s\in S}, \{\alpha_s\}_{s\in S})$ be an algebraic partial action of S on A that has the property that A and each ideal D_s, for $s \in S$, are left (right) s-unital. Then A is embedded in $A \rtimes_\alpha S$ and is isomorphic to $\mathcal{D}$, which is a subalgebra of $A \rtimes_\alpha S$.*

Proof. It is easy to see that $\mathcal{D}$ is a subring of $A \rtimes_\alpha S$ since $\alpha_e = \mathrm{id}_{X_e}$, for all $e \in E(S)$ and so,

$$\overline{a\delta_{e_1}} \cdot \overline{b\delta_{e_2}} = \overline{ab\delta_{e_1e_2}} \ \in \mathcal{D},$$

for all $\overline{a\delta_{e_1}}, \overline{a\delta_{e_1}} \in \mathcal{D}$.

Next we show that $\mathcal{D}$ is isomorphic to A. Notice that, by Definition 1.5.27 (ii), given $a \in A$ we can write

$$a = \sum_{i=1}^{n} a_{e_i},$$

where $n \in \mathbb{N}$, $e_i \in E(S)$, and $a_{e_i} \in D_{e_i}$ for each $i \in \{1, \ldots, n\}$. Let $\phi : A \to \mathcal{D}$ be the map defined by

$$\phi(a) = \sum_{i=1}^{n} \overline{a_{e_i}\delta_{e_i}},$$

for $a = \sum_{i=1}^{n} a_{e_i} \in \mathcal{A}$. Clearly, ϕ is additive.

We prove by induction that ϕ is well-defined. More precisely, we will show that if $\sum_{i=1}^{n} a_{e_i} = 0$ for $n \in \mathbb{N}$, $e_i \in E(S)$, and $a_{e_i} \in D_{e_i}$, for $i \in \{1, \ldots, n\}$, then $\sum_{i=1}^{n} \overline{a_{e_i}\delta_{e_i}} = 0$.

If $a_{e_1} = 0$, then clearly $\overline{a_{e_1}\delta_{e_1}} = 0$. Let $n \in \mathbb{N}$ be arbitrary. As induction hypothesis, suppose that if $\sum_{i=1}^{n} a_{e_i} = 0$, then $\sum_{i=1}^{n} \overline{a_{e_i}\delta_{e_i}} = 0$.

Now, let $f \in E(S)$, $a_f \in D_f$, and suppose that $a_f + \sum_{i=1}^{n} a_{e_i} = 0$. Take $u \in A$ such that $ua_f = a_f$ and $ua_{e_i} = a_{e_i}$, for $i \in \{1, \cdots\ n\}$ and take $u_f \in D_f$ such that $u_f a_f = a_f$. Then

$$\begin{aligned} 0 = (u - u_f)\left(a_f + \sum_{i=1}^{n} a_{e_i}\right) &= ua_f - u_f a_f + (u - u_f)\left(\sum_{i=1}^{n} a_{e_i}\right) \\ &= a_f - a_f + (u - u_f)\left(\sum_{i=1}^{n} a_{e_i}\right) = \sum_{i=1}^{n} (u - u_f) a_{e_i}. \end{aligned}$$

By the induction hypothesis, we conclude that $\sum_{i=1}^{n} \overline{(u - u_f) a_{e_i} \delta_{e_i}} = 0$. Using this, together with Remark 1.6.8 and the fact that $fe_i \leq e_i$ and $fe_i \leq f$, and hence $D_{fe_i} = D_f \cap D_{e_i}$, for each $i \in \{1, \ldots, n\}$, we get that

$$\begin{aligned} \sum_{i=1}^{n} \overline{a_{e_i}\delta_{e_i}} = \sum_{i=1}^{n} \overline{u_f a_{e_i}\delta_{e_i}} &= \sum_{i=1}^{n} \overline{u_f a_{e_i}\delta_{fe_i}} = \sum_{i=1}^{n} \overline{u_f a_{e_i}\delta_f} \\ &= \overline{\left(\sum_{i=1}^{n} u_f a_{e_i}\right)\delta_f} = \overline{\left(u_f \sum_{i=1}^{n} a_{e_i}\right)\delta_f} = \overline{(u_f(-a_f))\delta_f} \\ &= \overline{-a_f\delta_f}. \end{aligned}$$

Therefore,

$$\overline{a\delta_f} + \sum_{i=1}^{n} \overline{a_{e_i}\delta_{e_i}} = 0,$$

proving that ϕ is well-defined.

Clearly, ϕ is onto and multiplicative (using that $\alpha_e = \mathrm{id}_{X_e}$, for all $e \in E(S)$) and thus a surjective ring morphism. Now, consider the map $\widetilde{\tau}$ which was defined in Remark 1.6.14. Notice that

$$\widetilde{\tau} \circ \phi\left(\sum_{i=1}^{n} a_{e_i}\right) = \widetilde{\tau}\left(\sum_{i=1}^{n} \overline{a_{e_i}\delta_{e_i}}\right) = \sum_{i=1}^{n} a_{e_i},$$

that is, $\widetilde{\tau} \circ \phi = \mathrm{id}_A$, and hence ϕ is injective. □

Remark 1.6.17. Suppose that S is unital, with identity element $1 \in S$. In this case, if $e \in E(S)$, then $e \leq 1$, and therefore for each $a \in D_e$ we have $\overline{a\delta_e} = \overline{a\delta_1}$. Hence, $\overline{A\delta_1} = \mathcal{D}$.

Proposition 1.6.18. *[19, Theorem 4.3.15.] Let $(\{D_s\}_{s\in S}, \{\alpha_s\}_{s\in S})$ be an algebraic partial action of S on A, such that D_s has a unit 1_s, for all $s \in S$. Then, the map*

$$\begin{aligned} \sigma : S &\to A \rtimes_\alpha S \\ s &\mapsto \overline{1_s\delta_s} \end{aligned}$$

is a partial homomorphism of S to the multiplicative semigroup of $A \rtimes_\alpha S$.

Proof. We need to show that σ satisfies the three items of Definition 1.1.8. First, notice that $\alpha_{s^*}(1_s) = 1_{s^*}$, since $\alpha_{s^*} : D_s \to D_{s^*}$ is an R-isomorphism, and so,

$$\sigma(s)\sigma(s^*) = \overline{1_s\delta_s 1_{s^*}\delta_{s^*}} = \overline{1_s\delta_{ss^*}}. \tag{1.23}$$

(i) We have

$$\sigma(s)\sigma(r)\sigma(r^*) \overset{(1.23)}{=} \overline{1_s\delta_s 1_r\delta_{rr^*}} = \overline{\alpha_s(1_{s^*}1_r)\delta_{srr^*}},$$

and

$$\sigma(sr)\sigma(r^*) = \overline{\alpha_{sr}(1_{(sr)^*}1_{r^*})\delta_{srr^*}}.$$

Notice that $1_{s^*}1_r$ is the unit of $D_{s^*} \cap D_r$, and so, $\alpha_s(1_{s^*}1_r)$ is the unit of $\alpha_s(D_{s^*} \cap D_r) = D_s \cap D_{sr}$. On the other hand, $1_{(sr)^*}1_{r^*}$ is the unit of $D_{(sr)^*} \cap D_{r^*}$, and then, $\alpha_{sr}(1_{(sr)^*}1_{r^*})$ is the unit of $\alpha_{sr}(D_{(sr)^*} \cap D_{r^*})$. However

$$\begin{aligned} \alpha_{sr}(D_{(sr)^*} \cap D_{r^*}) = D_{sr} \cap D_{srr^*} &= D_{sr} \cap D_{(sr)(sr)^*s} \\ &= \alpha_{sr}(D_{(sr)^*} \cap D_{(sr)^*s}) \\ &= \alpha_{sr}(\alpha_{(sr)^*}(D_{sr} \cap D_s)) = D_{sr} \cap D_s, \end{aligned}$$

and then, $\alpha_s(1_{s^*}) = \alpha_{sr}(1_{(sr)^*}1_{r^*})$. We can conclude that

$$\sigma(s)\sigma(r)\sigma(r^*) = \sigma(sr)\sigma(r^*).$$

(ii) Similarly as in the previous item, notice that $1_{s^*}1_r$ is the unit of $D_{s^*} \cap D_r$, and that $\alpha_s(1_{s^*}1_{sr})$ is the unit of $\alpha_s(D_{s^*} \cap D_{sr})$. Moreover,

$$\alpha_s(D_{s^*} \cap D_{sr}) = \alpha_s(\alpha_{s^*}(D_s \cap D_r)) = D_{s^*} \cap D_r.$$

Hence, $1_{s^*}1_r = \alpha_s(1_{s^*}1_{sr})$, and then,

$$\begin{aligned}\sigma(s^*)\sigma(s)\sigma(r) &= \overline{1_{s^*}\delta_{s^*s}1_r\delta_r} = \overline{1_{s^*}1_r\delta_{s^*sr}} \\ &= \overline{\alpha_s(1_s 1_{sr})\delta_{s^*sr}} = \sigma(s^*)\sigma(sr).\end{aligned}$$

(iii) This follows easily since

$$\sigma(s)\sigma(s^*)\sigma(s) \overset{(1.23)}{=} \overline{1_s\delta_{ss^*}}\ \overline{1_s\delta_s} = \overline{1_s\delta_{ss^*s}} = \sigma(s).$$

□

Theorem 1.6.19. *[19, Theorem 4.3.15] Let $\alpha = (\{D_s\}_{s\in S}, \{\alpha_s\}_{s\in S})$ be an algebraic partial action of S on A such that A is left (right) s-unital and D_s has a unit 1_s, for all $s \in S$. Then (ι, σ) is a universal non-degenerate covariant representations of $A \rtimes_\alpha S$ in the following sense:*

(a) (ι, σ) is a non-degenerate covariant representation.

(b) If (π, u) is any other non-degenerate covariant representation of $A \rtimes_\alpha S$ in an algebra B, then there exists a unique algebra homomorphism $\Phi : A \rtimes_\alpha S \to B$ such that

$$\pi = \Phi \circ \iota \qquad u = \Phi \circ \sigma.$$

Proof. (a) Let $s \in S$ and $a \in D_{s^*}$. Then $a \in D_{s^*s}$, $\iota(a) = \overline{a\delta_{s^*s}}$, and

$$\begin{aligned}\sigma(s)\iota(a)\sigma(s^*) &= \overline{1_s\delta_s} \cdot \overline{a\delta_{s^*s}} \cdot \overline{1_{s^*}\delta_{s^*}} \\ &= \overline{1_s\delta_s} \cdot \overline{a\delta_{s^*}} = \overline{\alpha_s(a)\delta_{ss^*}} = \iota(\alpha_s(a)),\end{aligned}$$

satisfying the covariance condition. By (1.23) we get the non-degenerance of the pair (ι, σ).

(b) Take $\Phi = \pi \times u$ defined by

$$(\pi \times u)(\overline{a_s \delta_s}) = \pi(a_s)u(s),$$

for all $s \in S$ and $a_s \in D_s$, as in Proposition 1.6.13 (d). We have already seen that $\pi \times u$ is an algebra homomorphism.

Let $e \in E(S)$ and $a_e \in D_e$. Then $u(e) \in E(S)$, and

$$(\pi \times u) \circ \iota(a_e) = \pi(a_e)u(e) = \pi(a_e)u(e)u(e^*) \overset{1.6.13\ (a)}{=} \pi(a_e).$$

For any $s \in S$, we get that

$$(\pi \times u) \circ \sigma(s) = \pi(1_s)u(s) \overset{1.6.13\ (c)}{=} u(s),$$

as required. $\square$

2 THE INTERPLAY BETWEEN STEINBERG ALGEBRAS AND SKEW ALGEBRAS

This chapter is based entirely on the paper [5], produced during the doctorate.

It is our goal in this chapter to link the theory of partial skew algebras (rings) with the theory of Steinberg algebras, in the same way as the theory of partial crossed products is linked to Renault's theory of groupoid C*-algebras. In particular, we provide an "algebraisation" of a result by Abadie (see [1]), that shows that any partial crossed product, associated to a partial action on a topological space, can be seen as a groupoid C*-algebra. The algebraic version of this theorem permits us to join results of Li (see [52]), about continuous orbit equivalence of partial actions on topological spaces, and results of Carlsen and Rout (see [12]), about diagonal-preserving isomorphism between Steinberg algebras, to present results regarding diagonal-preserving isomorphisms of partial skew group ring over commutative algebras.

To complete the interplay between Steinberg algebras and skew algebras (rings), we show an "algebraisation" of [60, Theorem 3.3.1] and [63, Theorem 8.1]: Any Steinberg algebra associated to an ample Hausdorff groupoid can be seen as a skew inverse semigroup algebra. It is interesting to point out that the definition of a skew inverse semigroup algebra involves a quotient by a certain ideal (see Definition 1.6.7). This quotient might seem artificial, and maybe even unnecessary. Our characterization of Steinberg algebras as skew inverse semigroup rings is further evidence that the quotient is necessary in the definition of skew inverse semigroup algebras.

2.1 The Steinberg algebra of a transformation groupoid

Throughout this section, we assume that R is a unital commutative ring, X is a locally compact, Hausdorff, and zero-dimensional topological space, and G is a discrete group.

Given a topological partial action $\theta = (\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ of G on X such that X_g is clopen (closed-open), for every g in G, we will prove that the Steinberg algebra of the transformation groupoid associated to θ can be realized as partial skew group algebras of the form $\mathcal{L}_c(X) \rtimes_\alpha G$.

In Example 1.5.28, we defined $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$ as the commutative R-algebra consisting of all locally constant, compactly supported, R-valued functions on X, with point-wise operations.

Let $\theta = (\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a topological partial action of a discrete group G on X. Similarly to Example 1.5.28, such partial action induces an algebraic partial action $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ of G on $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$ as follows: For each g in G, consider the ideal

$$D_g = \{f \in \mathcal{L}_{\mathrm{c}}(\mathrm{X}) \,:\, \operatorname{supp}(f) \subseteq X_g\} \quad \text{in} \quad \mathcal{L}_{\mathrm{c}}(\mathrm{X}),$$

and define $\alpha_g : D_{g^{-1}} \to D_g$ by setting,

$$\alpha_g(f)(x) = \begin{cases} f \circ \theta_{g^{-1}}(x), & \text{if } x \in X_g \\ 0 & \text{if } x \notin X_g, \end{cases}$$

for all $f \in D_{g^{-1}}$. We can associate to this algebraic partial action α the partial skew group algebra

$$\mathcal{L}_c(X) \rtimes_\alpha G.$$

Recall that a general element $b \in \mathcal{L}_c(X) \rtimes_\alpha G$ is denoted by

$$b = \sum_{g\in G} f_g \delta_g,$$

where each f_g lies in D_g, and $f_g \equiv 0$, for all but finitely many group elements g.

Furthermore, we can also associate to the partial action θ an étale groupoid, denoted by $G \ltimes_\theta X$, and known as the *transformation groupoid*: Let

$$G \ltimes_\theta X = \{(t, x) \mid t \in G \ \text{ and } \ x \in X_{t^{-1}}\}.$$

If $(s, y), (t, x) \in G \ltimes_\theta X$, then $(s, y), (t, x)$ is a composable pair if, and only if, $\theta_t(x) = y$. In this case, we define the product map by

$$(s, y)(t, x) = (st, x).$$

This product makes $G \ltimes_\theta X$ a groupoid, and the inverse of $(t, x) \in G \ltimes_\theta X$ is

$$(t, x)^{-1} = (t^{-1}, \theta_t(x)).$$

We equip $G \ltimes_\theta X$ with the topology inherited from the product topology on $G \times X$. By the continuity of each θ_g $(g \in G)$, the inversion and product maps are continuous. Since $\mathcal{G}$ is discrete and X is Hausdorff, we have that the groupoid $G \ltimes_\theta X$ is Hausdorff. Notice that we can identify X with $(G \ltimes_\theta X)^{(0)}$ via the homeomorphism $i : X \to G \ltimes_\theta X$ given by $x \mapsto (1, x)$, and with this, the unit space $(G \ltimes_\theta X)^{(0)}$ is locally compact, Hausdorff, and zero-dimensional. Moreover, the range and source maps can be simplified by

$$\mathfrak{r}(t, x) = \theta_t(x), \quad \text{and} \quad \mathfrak{s}(t, x) = x,$$

respectively. We have that the range map is a local homeomorphism of $\{t\} \times X_{t^{-1}}$ onto X_t, and the source map is local homeomorphism of $\{t\} \times X_{t^{-1}}$ onto $X_{t^{-1}}$. Therefore, $G \ltimes_\theta X$ is an étale, Hausdorff groupoid. Since the unit space of $G \ltimes_\theta X$ is zero-dimensional, by Proposition 1.2.8, $G \ltimes_\theta X$ is ample. Therefore, we can consider the Steinberg algebra $A_R(G \ltimes_\theta X)$.

We can now prove the following.

Theorem 2.1.1. *[5, Theorem 3.2.] Let $\theta = (\{X_g\}_{g \in G}, \{\theta_g\}_{g \in G})$ be a topological partial action of a discrete group G on a locally compact, Hausdorff and zero-dimensional topological space X, such that each X_g is clopen. Let $\alpha = (\{D_g\}_{g \in G}, \{\alpha_g\}_{g \in G})$ be the corresponding algebraic partial action, and $G \ltimes_\theta X$ be the transformation groupoid associate to θ. Then $\mathcal{L}_c(X) \rtimes_\alpha G$ and $A_R(G \ltimes_\theta X)$ are isomorphic as R-algebras.*

Proof. To define a homomorphism ρ from $\mathcal{L}_c(X) \rtimes_\alpha G$ to $A_R(G \ltimes_\theta X)$, we begin defining it on elements of the form $f_g \delta_g$, and then we extend it linearly to $\mathcal{L}_c(X) \rtimes_\alpha G$. More precisely, for $f_g\delta_g \in \mathcal{L}_c(X) \rtimes_\alpha G$ and $(t,x) \in G \ltimes_\theta X$, let

$$\widetilde{\rho}(f_g\delta_g)(t,x) := \begin{cases} f_g(\theta_g(x)), & \text{if } (t,x) \in \{g\} \times X_{g^{-1}} \\ 0, & \text{otherwise,} \end{cases}$$

and denote the linear extension of $\widetilde{\rho}$ to $\mathcal{L}_c(X) \rtimes_\alpha G$ by ρ.

We claim that ρ is well defined. For this it is enough to prove that, for each $g \in G$, $\widetilde{\rho}(f_g\delta_g)$ is a locally constant function with compact support. Let $(t,x) \in G \ltimes_\theta X$. Suppose that $(t,x) \in \{g\} \times X_g$. Since f_g is locally constant, there is a neighborhood $U \subseteq X_g$ of $\theta_g(x)$ such that $f_g|_U$ is constant. Notice that $V = \{g\} \times \theta_{g^{-1}}(U)$ is a neighborhood of (t,x) and $\widetilde{\rho}(f_g\delta_g)|_V = f_g|_U$, which is constant. Now, suppose that $(t,x) \notin \{g\} \times X_g$. Then $(t,x) \in G \ltimes_\theta X \setminus \operatorname{supp}(\widetilde{\rho}(f_g\delta_g))$, and clearly, $\widetilde{\rho}(f_g\delta_g)$ is constant equal to 0 on this subset. We easily see that

$$\operatorname{supp}(\widetilde{\rho}(f_g\delta_g)) = \{g\} \times \theta_{g^{-1}}(\operatorname{supp}(f_g)),$$

which is compact (and clopen). We conclude that $\tilde{\rho}(f_g\delta_g) \in A_R(G \ltimes X)$.

Next we check that ρ is multiplicative. By linearity, it is enough to check this on elements of the form $f_g\delta_g \in \mathcal{L}_c(X) \rtimes_\alpha G$. So, take $f_g\delta_g$ and $f_h\delta_h$ in $\mathcal{L}_c(X) \rtimes_\alpha G$ and $(t,x) \in G \ltimes_\theta X$.

Form the convolution product definition in $A_R(\mathcal{G})$ (see Equation 1.6), we have that

$$\begin{aligned} \operatorname{supp}[\widetilde{\rho}(f_g\delta_g) * \widetilde{\rho}(f_h\delta_h) &\subseteq \operatorname{supp}(\rho(f_g\delta_g)) \operatorname{supp}(\rho(f_h\delta_h)) \\ &\subseteq \left(\{g\} \times X_{g^{-1}}\right) \left(\{h\} \times X_{h^{-1}}\right). \end{aligned} \tag{2.1}$$

On the other hand, notice that

$$\operatorname{supp}(\alpha_g(\alpha_{g^{-1}}(f_g)f_h)) = \operatorname{supp}(f_g) \cap \theta_g(\operatorname{supp}(f_h)) \subseteq \theta_g(X_{g^{-1}} \cap X_h).$$

Hence,

$$\begin{aligned}
\operatorname{supp}(\rho[(f_g\delta_g)(f_h\delta_h)]) &= \operatorname{supp}(\rho[\alpha_g(\alpha_{g^{-1}}(f_g)f_h)\delta_{gh}]) \\
&= \{gh\} \times X_{(gh)^{-1}} \cap \operatorname{supp}(\alpha_g(\alpha_{g^{-1}}(f_g)f_h)) \\
&\subseteq \{gh\} \times X_{(gh)^{-1}} \cap \theta_g(X_{g^{-1}} \cap X_h) \\
&= \{gh\} \times \theta_{h^{-1}}(X_h \cap X_{g^{-1}}) \\
&= \left(\{g\} \times X_{g^{-1}}\right)(\{h\} \times X_{h^{-1}}). \qquad (2.2)
\end{aligned}$$

By Equations 2.1 and 2.2, we can conclude that

$$[\widetilde{\rho}(f_g\delta_g) * \widetilde{\rho}(f_h\delta_h)](t,x) = 0 = \rho[(f_g\delta_g)(f_h\delta_h)](t,x),$$

for all $(t,x) \in G \ltimes_\theta X \setminus \left(\{g\} \times X_{g^{-1}}\right)(\{h\} \times X_{h^{-1}})$.

If $(t,x) \in \left(\{g\} \times X_{g^{-1}}\right)(\{h\} \times X_{h^{-1}})$, we get that

$$(t,x) = (g, \theta_h(x))(h,x) = (gh,x),$$

and $((g,\theta_h(x)),(h,x))$ is the only pair in $\left(\{g\} \times X_{g^{-1}}\right) \times (\{h\} \times X_{h^{-1}})$ such that the product is (t,x). Then, in this case,

$$\begin{aligned}
[\rho(f_g\delta_g) * \rho(f_h\delta_h)](t,x) &= \rho(f_g\delta_g)(g,\theta_h(x))\rho(f_h\delta_h)(h,x) \\
&= f_g(\theta_g(\theta_h(x)))f_h(\theta_h(x)) \\
&= f_g(\theta_{gh}(x))f_h(\theta_{g^{-1}}(\theta_{gh}(x)) \\
&= \alpha_g(\alpha_{g^{-1}}(f_g)f_h)(\theta_{gh}(x)) \\
&= \rho(\alpha_g(\alpha_{g^{-1}}(f_g)f_h)\delta_{gh})(gh,x) \\
&= \rho[(f_g\delta_g)(f_h\delta_h)](t,x).
\end{aligned}$$

Therefore, ρ is an homomorphism.

To finish, we prove that ρ is a bijection, by showing that it has an inverse $\rho^{-1}: A_R(G \ltimes_\theta X) \to \mathcal{L}_c(X) \rtimes_\alpha G$ given by $\rho^{-1}(f) = \sum f_g\delta_g$, where

$$f_g(x) := \begin{cases} f(g, \theta_{g^{-1}}(x)), & \text{if } x \in X_g \\ 0, & \text{otherwise.} \end{cases}$$

We claim that ρ is well defined. To this end, we need first to prove that $f_g \in D_g$.

Since the topology of $G \ltimes_\theta X$ is the relative product topology from product topology of $G \times X$ and $\mathrm{supp}(f)$ is compact, we have $\mathrm{supp}(f) = (\{g_1, \cdots, g_n\} \times K) \cap (G \ltimes_\theta X)$, where $\{g_1, \cdots, g_n\} \subseteq G$ and $K \subseteq X$ is compact. Thus $\mathrm{supp}(f_g) = K \cap X_g$ is compact.

Let $x \in X_g$. Then, there is an open neighborhood W of (g, x) such that $f|_W$ is constant. Notice that W is of the form

$$(H \times U) \cap (G \ltimes_\theta X),$$

where H is open subset of G and U is open subset of X. We have that $V := U \cap X_g$ is an open neighborhood of x and $f_g|_V$ is constant. If $x \notin X_g$ then f is identically null in the open subset $X \setminus X_g$. Therefore $f_g \in D_g$.

Notice that the set $\{g \in G \mid f(g,x) \neq 0,\ x \in X_g\}$ is finite because G is discrete and $\mathrm{supp}(f)$ is compact. Then the set $\{g \in G \mid f_g \neq 0\}$ is also finite, that is, the sum $\sum f_g \delta_g = \rho^{-1}(f)$ is finite.

It is straightforward to check that ρ^{-1} is the inverse of ρ. $\square$

Remark 2.1.2. Under the assumptions of the theorem above we remark that partial actions such that each X_g is clopen ($g \in G$) are exactly the one's for which the envelope space is Hausdorff (see [30]).

Remark 2.1.3. In Chapter 4 we will prove a similar, and more general, result than Theorem 2.1.1. Furthermore, we will see that the hypothesis of each X_g is closed is not necessary.

From the identification of the unit space $(G \ltimes_\theta X)^{(0)}$ with the space X we have that the diagonal of the Steinberg algebra $A_R(G \ltimes_\theta X)$ is as set

$$D_R(G \ltimes_\theta X) = \mathrm{Span}_R\{1_U \mid U \subseteq X \text{ is compact-open}\},$$

which is a commutative subalgebra of $A_R(G \ltimes_\theta X)$, with point-wise operations.

Corollary 2.1.4. *The isomorphism above maps the diagonal subalgebra $\mathcal{L}_c(\mathrm{X})\,\delta_1$ onto the diagonal subalgebra $D_R(G \ltimes_\theta X)$.*

2.2 Application to diagonal-preserving isomorphisms

In this section we study diagonal-preserving isomorphisms of partial skew group algebras of the form $\mathcal{L}_c(X) \rtimes_\alpha G$. For this purpose we apply the isomorphism of the previous section and we make use of results proved in [12] and [52]. We continue to use the same assumptions on X, G and θ of the previous section.

By Theorem 2.1.1 and Corollary 2.1.4, we obtain an isomorphism between the algebras $\mathcal{L}_c(X) \rtimes_\alpha G$ and $A_R(G \ltimes_\theta X)$ that "preserves diagonal". In [12, Theorem 3.1], Carlsen and Rout characterize diagonal-preserving (graded) isomorphism between two (graded) Steinberg algebras. For our particular case we will use [12, Corollary 3.2] as follows:

Corollary 2.2.1. *[12, Corollary 3.2] Let R be an integral domain. For $i = 1, 2$, let $\mathcal{G}_i$ be an ample Hausdorff groupoid such that there is a dense subset $X_i \subseteq \mathcal{G}_i^0$, such that the group-ring $R((\mathcal{G}_i)_x^x)$ has no zero-divisors and only trivial units for all $x \in X_i$. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic if, and only if, there is a diagonal-preserving isomorphism between $A_R(\mathcal{G}_1)$ and $A_R(\mathcal{G}_2)$.*

Recall that if R is an integral domain and G is a group, the group-ring RG of G is defined by

$$RG = \left\{ \sum_{i=1}^{n} r_i g_i \mid n \in \mathbb{N}, r_i \in R \text{ and } g_i \in G \right\}.$$

An element $x \in RG$ is a *unit* if there exist $y, z \in RG$ such that $xy = 1 = zx$. A unit is *trivial* if it has the form rg for some $r \in R$ and $g \in G$.

By the previous corollary, we need to find sufficient (and necessary, if possible) conditions such that the group-ring $R((G \ltimes_\theta X)_x^x)$ has no zero-divisors and only trivial units. Let $x \in (G \ltimes_\theta X)^{(0)}$ be fixed. We have that

$$\begin{aligned}(G \ltimes_\theta X)_x^x &= \{(g, y) \in G \ltimes_\theta X \mid r(g, y) = x = s(g, y)\} \\ &= \{(g, y) \in G \ltimes_\theta X \mid \theta_g(y) = x = y\} \\ &= \{(g, x) \in G \ltimes_\theta X \mid x = \theta_g(x)\}. \end{aligned} \tag{2.3}$$

Notice that the group $(G \ltimes_\theta X)_x^x$ is isomorphic to a subgroup of G. To study this subgroup we present some results of [45] below.

An infinite group G is said to be *indexed* if we are given a homomorphism γ of G in the additive group of integers, such that $\gamma(G)$ does not consist of zero alone. In general, a group can be indexed in more than one way. An infinite group G is said to be *indicable throughout* if every subgroup of G, not consisting of the unit alone, can be indexed. Notice that any non-trivial subgroup of an indicable throughout group is also indicable throughout.

Example 2.2.2. Since any free group is indexed, and any subgroup of a free group is either a free group or the unit alone, we have that any free group is indicable throughout. Similarly any free Abelian group is indicable throughout.

Theorem 2.2.3. *[45, Theorem 12] If G is indicable throughout and R has no zero-divisors then RG has no zero-divisors.*

Theorem 2.2.4. *[45, Theorem 13] If G is indicable throughout and R has no zero-divisors then all the units of RG are trivial.*

With these two theorems we obtain the following lemma:

Lemma 2.2.5. *Let G be an indicable throughout group and let $\theta = (\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial action of G on X. Then the group-ring $R((G \ltimes_\theta X)_x^x)$ has no zero-divisors and only trivial units, for all $x \in (G \ltimes_\theta X)^{(0)}$.*

Proof. We have that $(G \ltimes_\theta X)_x^x$ is isomorphic to a subgroup of G. Then this subgroup is indicable throughout or the unit alone. In the first case, Theorems 2.2.3 and 2.2.4 ensure that $R((G \ltimes_\theta X)_x^x)$ has no zero-divisors and only trivial units. In the second case, $R((G \ltimes_\theta X)_x^x) = R(\{x\}) \cong R$, that also has no zero-divisors and only trivial units. □

Theorem 2.2.6. *[5, Theorem 4.5] Let R be an integral domain, let G, H be indicable throughout, discrete groups, let X, Y be locally compact, Hausdorff, and zero-dimensional topological spaces, and let $\theta =$*

$(\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ and $\gamma = (\{Y_h\}_{h\in H}, \{\theta_h\}_{h\in H})$ be partial actions. Then the following are equivalent:

(i) the transformation groupoids $G \ltimes_\theta X$ and $H \ltimes_\gamma Y$ are isomorphic as topological groupoids,

(ii) there exists a diagonal-preserving isomorphism

$$\Gamma : A_R(G \ltimes_\theta X) \longrightarrow A_R(H \ltimes_\gamma Y),$$

(iii) there exists a diagonal-preserving isomorphism

$$\Phi : \mathcal{L}_c(X) \rtimes G \longrightarrow \mathcal{L}_c(Y) \rtimes H.$$

Proof. "(i) $\Leftrightarrow$ (ii)" By Lemma 2.2.5, the hypotheses of Corollary 2.2.1 are satisfied and thus we get the two implications.

"(ii) $\Rightarrow$ (iii)" Let $\rho_G : \mathcal{L}_c(X) \rtimes G \longrightarrow A_R(G \ltimes_\theta X)$ and $\rho_H : \mathcal{L}_c(Y) \rtimes H \longrightarrow A_R(H \ltimes_\gamma Y)$ be the isomorphisms given by Theorem 2.1.1. Define $\Phi :\ \mathcal{L}_c(X) \rtimes G \longrightarrow \mathcal{L}_c(Y) \rtimes H$ by $\Phi = \rho_H^{-1} \circ \Gamma \circ \rho_G$. Clearly Φ is an isomorphisms and

$$\begin{aligned}\Phi(\mathcal{L}_\mathrm{c}(\mathrm{X})\,\delta_1) = \rho_H^{-1} \circ \Gamma \circ \rho_G(\mathcal{L}_\mathrm{c}(\mathrm{X})\,\delta_1) &\overset{2.1.4}{=} \rho_H^{-1} \circ \Gamma(D_R(G \ltimes_\theta X)) \\ &\overset{(iii)}{=} \rho_H^{-1}(D_R(H \ltimes_\gamma Y)) \overset{2.1.4}{=} \mathcal{L}_c(Y)\delta_1.\end{aligned}$$

"(iii) $\Rightarrow$ (ii)" Similar to the previous one, just take $\Gamma = \rho_H \circ \Phi \circ \rho_G^{-1}$. □

The above theorem can be applied, for example, to Leavitt path algebras, since they can be seen as partial skew group rings associated to a partial action of the free group, generated by edges of the graph (see [40][Theorem 3.3]).

In [52, Theorem 2.7] X. Li characterizes diagonal-preserving isomorphisms of partial C*-crossed products, over commutative algebras, in terms of continuous orbit equivalence of the associated partial actions:

Theorem 2.2.7. *[52, Theorem 2.7] Let G, H be discrete and countable groups, let X, Y be locally compact, Hausdorff topological spaces, and let*

$\theta = (\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ and $\gamma = (\{Y_h\}_{h\in H}, \{\theta_h\}_{h\in H})$ be topologically free partial actions. Then the following are equivalent:

(i) θ and γ are continuously orbit equivalent,

(ii) the transformation groupoids $G \ltimes_\theta X$ and $H \ltimes_\gamma Y$ are isomorphic as topological groupoids,

(iii) there exists an isomorphism $\Phi : C_0(X) \rtimes_r G \longrightarrow C_0(Y) \rtimes_r H$ with $\Phi(C_0(X)) = C_0(Y)$

Moreover, "(ii) $\Rightarrow$ (i)" holds in general (i.e., without the assumption of topological freeness).

We are now able to add two additional equivalent conditions to continuous orbit equivalence of partial actions, in terms of Steinberg algebras and partial skew group rings. Before we do this we recall the notion of topologically free partial action below.

Definition 2.2.8. A partial action $\theta = (\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ of group G on locally compact, Hausdorff topological space X is *topologically free* (or *effective*) if, for every $g \in G \setminus \{1\}$, the set

$$\{x \in X_{g^{-1}} \mid \theta_g(x) \neq x\}$$

is dense in $X_{g^{-1}}$. This is equivalent to saying that, for every $1 \neq g \in G$, the set

$$\{x \in X_{g^{-1}} \mid \theta_g(x) = x\}$$

has empty interior.

Remark 2.2.9. Let $\theta = (\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial action of a group G on locally compact, Hausdorff topological space X. Then, θ is topologically free if, and only if, the transformation groupoip $G \ltimes_\theta X$ is effective. In Chapter 4, we will prove this result for the more general case of partial actions of inverse semigroups, (see Proposition 4.2.1).

Definition 2.2.10. Let $\theta = (\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial action of a group G on a topological space X. We define,

(a) for $x \in X$ fixed, the subset $G_x = \left\{g \in G \mid x \in X_{g^{-1}}\right\}$ of G.

(b) the subset $\Lambda = \{x \in X \mid \theta_g(x) \neq x, \text{ for all } g \in G_x \setminus \{1\}\}$ of X,

We say that θ is *topological principal* if Λ is dense in X.

Lemma 2.2.11. *Let $\theta = (\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial action of a group G on a locally compact, Hausdorff topological space X. Then θ is topologically principal if, and only if, the transformation groupoid $G \ltimes_\theta X$ is topologically principal.*

Proof. It suffices to prove that

$$\Lambda = \{x \in X \mid (G \ltimes_\theta X)_x^x = \{x\}\}. \tag{2.4}$$

By Equation 2.3, we have already seen that, for any $x \in X$,

$$(G \ltimes_\theta X)_x^x = \{(g, x) \in G \ltimes_\theta X \mid \theta_g(x) = x\}.$$

Thus $(G \ltimes_\theta X)_x^x = \{x\}$ if, and only if, $\theta_g(x) \neq x$, for all $g \in G_x \setminus \{1\}$, which shows that Equality 2.4 holds. □

Lemma 2.2.12. *[52, Lemma 2.4] Let $\theta = (\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial action of a discrete group G on a locally compact, Hausdorff topological space X. If θ is topologically principal, then θ is topologically free. The converse is true if G is countable.*

Proof. For any $g \in G$ with $g \neq 1$, notice that

$$\Lambda \cap X_{g^{-1}} \subseteq \left\{x \in X_{g^{-1}} \mid \theta_g(x) \neq x\right\}.$$

Then, by the density of Λ in X, $\left\{x \in X_{g^{-1}} \mid \theta_g(x) \neq x\right\}$ is dense in $X_{g^{-1}}$.

To prove the converse, notice that the subset $\{x \in X_{g^{-1}} \mid \theta_g(x) = x\}$ is an open subset of $X_{g^{-1}}$, and that $\{x \in X_{g^{-1}} \mid \theta_g(x) = x\} \cup \left(X \setminus \overline{X_{g^{-1}}}\right)$ is dense in X. By the Baire category theorem, we conclude that

$$\Lambda = \bigcap_{g \in G\setminus\{1\}} \{x \in X_{g^{-1}} \mid \theta_g(x) \neq x\} \cup \left(X \setminus \overline{X_{g^{-1}}}\right)$$

is dense in X. □

Remark 2.2.13. Let $\theta = (\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial action of a discrete group G on a locally compact, Hausdorff topological space X. Notice that by the combination of Remark 2.2.9, Lemma 2.2.11 and Lemma 2.2.12, we obtain as result a particular case, for the transformation groupoids, of Renault's Proposition about effective and topological principal groupoids (see Proposition 1.2.18). More precisely, if the transformation groupoid $G \ltimes_\theta X$ is topologically principal, then $G \ltimes_\theta X$ is effective. The converse is true if we add that $G \ltimes_\theta X$ is second countable.

Let θ be a partial action of a group G on a topological space X. We denoted by $G*X$ the subset $\{(g,x) \in G \times X \mid x \in X_{g^{-1}}\}$ of $G\times X$.

Definition 2.2.14. [52, Definition 2.6.] Two topological partial actions $\theta = (\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ and $\gamma = (\{Y_h\}_{h\in H}, \{\gamma\}_{h\in H})$ are called *continuously orbit equivalent* if, there are a homeomorphism $\varphi : X \to Y$, and continuous maps $a : G * X \to H$, $b : H * Y \to G$ such that

(i) $\varphi(\theta_g(x)) = \gamma_{a(g,x)}(\varphi(x))$,

(ii) $\varphi^{-1}(\gamma_h(y)) = \theta_{b(h,y)}(\varphi^{-1}(y))$.

Implicitly, we require that $a(g,x) \in H_{\varphi(x)}$ and $b(h,y) \in G_{\varphi^{-1}(y)}$.

Before we state our next Theorem we need the following lemma.

Lemma 2.2.15. *Let R be an integral domain and θ be a partial action of G on X. If θ is topologically principal, then there is a dense subset $Z \subseteq (G \ltimes_\theta X)^{(0)}$ such that the group ring $R((G \ltimes_\theta X)_z^z))$ has no zero-divisors and only trivial units for all $z \in Z$.*

Proof. By Lemma (2.2.11) the groupoid $G \ltimes_\theta X$ is topologically principal, that is, there exist a dense subset Z of $(G \ltimes_\theta X)^{(0)}$ such that $(G \ltimes_\theta X)_z^z = \{z\}$, for every $z \in Z$. Thus for $z \in Z$ we have that $R((G \ltimes_\theta X)_z^z) = R(\{z\}) \cong R$, which has no zero-divisors and only trivial units. $\square$

Theorem 2.2.16. *[5, Theorem 4.9] Let R be an integral domain, let G, H be discrete groups, let X, Y be locally compact, Hausdorff, zero-dimensional spaces, and let $\theta = (\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ and $\gamma = (\{Y_h\}_{h\in H}, \{\theta_h\}_{h\in H})$ be topologically principal partial actions. Then the following are equivalent:*

(i) θ and γ are continuously orbit equivalent,

(ii) the transformation groupoids $G \ltimes_\theta X$ and $H \ltimes_\gamma Y$ are isomorphic as topological groupoids,

(iii) there is a diagonal-preserving isomorphism $\Gamma : A_R(G \ltimes_\theta X) \to A_R(H \ltimes_\gamma Y)$,

(iv) there is a diagonal-preserving isomorphism $\Phi : \mathcal{L}_c(X) \rtimes G \to \mathcal{L}_c(Y) \rtimes H$.

Moreover, "(ii) $\Rightarrow$ (i)" holds in general (i.e., without the assumption of topological freeness).

Proof. "(i) $\Leftrightarrow$ (ii)" See Theorem 2.2.7.

"(ii) $\Leftrightarrow$ (iii)" By Lemma (2.2.15), the hypotheses of Corollary 2.2.1 are satisfied and hence we get the desired implications.

"(iii) $\Leftrightarrow$ (iv)" Analogous to the proof of "(ii) $\Leftrightarrow$ (iii)" in Theorem 2.2.6. $\square$

Remark 2.2.17. If we change the assumptions about the groups G, H and the partial actions θ, γ in the previous theorem to: let G, H be discrete and **countable** groups and let $\theta = (\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ and $\gamma = (\{Y_h\}_{h\in H}, \{\theta_h\}_{h\in H})$ be **topologically free**, by Lemma 2.2.12, we obtain the same equivalent conditions (i) - (iv).

2.3 Steinberg algebras realized as partial skew inverse semigroup algebras

In this section we will show that every Steinberg algebra can be realized as a skew inverse semigroup algebra. This is an "algebraisation" of [60, Theorem 3.3.1] and [63, Theorem 8.1].

From now on we fix an ample Hausdorff groupoid $\mathcal{G}$. By Propositions 1.2.6 and 1.2.8, the set of all compact-open bisections in $\mathcal{G}$, denoted by $\mathcal{G}^a$, is an inverse semigroup under the operations defined by

$$BC = \{bc \in \mathcal{G} \mid b \in B,\ c \in C \text{ and } \mathfrak{s}(b) = \mathfrak{r}(c)\},$$

and

$$B^{-1} = \{b^{-1} \mid b \in B\},$$

for all $B, C \in \mathcal{G}^a$. The idempotent semilattice of $\mathcal{G}^a$ consists precisely of the compact-open subsets of $\mathcal{G}^{(0)}$, and the inverse semigroup partial order in $\mathcal{G}^a$ is defined by

$$A \leq B \quad \Longleftrightarrow \quad A \subseteq B,$$

where $A, B \in \mathcal{G}^a$.

Similarly to Example 1.5.9, we get a topological action of inverse semigroup which is intrinsic to every ample Hausdorff groupoid $\mathcal{G}$. More precisely, the inverse semigroup $\mathcal{G}^a$ acts on the $\mathcal{G}^{(0)}$, which is a locally compact, Hausdorff, and zero-dimensional topological space as follows: Given a compact-open bisection B of $\mathcal{G}$, we have that the map θ_B : $\mathfrak{s}(B) \to \mathfrak{r}(B)$ defined by

$$\theta_B(u) = \mathfrak{r}(\mathfrak{s}_B^{-1}(u)),$$

for all $u \in \mathfrak{s}(B)$, is a homeomorphisms, and the collection

$$\theta = \left(\{\mathfrak{s}(B)\}_{B\in\mathcal{G}^a}, \{\theta_B\}_{B\in\mathcal{G}^a}\right)$$

is a topological action of $\mathcal{G}^a$ on the $\mathcal{G}^{(0)}$. We say that θ is the *canonical action* of $\mathcal{G}^a$ on $\mathcal{G}^{(0)}$.

As mentioned in Example 1.5.28, from the canonical action θ of $\mathcal{G}^a$ on $\mathcal{G}^{(0)}$, we get a corresponding action

$$\alpha = \left(\{D_B\}_{B\in\mathcal{G}^a}, \{\alpha_B\}_{B\in\mathcal{G}^a}\right)$$

of $\mathcal{G}^a$ on the R-algebra $\mathcal{L}_c(\mathcal{G}^{(0)})$ of all locally constant, compactly supported, R-valued functions on $\mathcal{G}^{(0)}$, where R is a unital commutative ring. More precisely, for every $B \in \mathcal{G}^a$, we have that α_B is an isomorphism from

$$D_{B^*} = \{f \in \mathcal{L}_c(\mathcal{G}^{(0)}) \mid \operatorname{supp}(f) \subseteq \mathfrak{s}(B)\}$$

onto

$$D_B = \{f \in \mathcal{L}_c(\mathcal{G}^{(0)}) \mid \operatorname{supp}(f) \subseteq \mathfrak{r}(B)\},$$

which is defined by

$$\alpha_B(f)(x) = \begin{cases} f \circ \theta_{B^*}(x), & \text{if } x \in \mathfrak{r}(B) \\ 0 & \text{if } x \notin \mathfrak{r}(B). \end{cases}$$

We can now prove last theorem of the this chapter.

Theorem 2.3.1. *[5, Theorem 5.2] Let $\mathcal{G}$ be an ample and Hausdorff groupoid, let θ be the canonical action of the inverse semigroup $\mathcal{G}^a$ over the unit space $\mathcal{G}^{(0)}$, and let α be the corresponding action of $\mathcal{G}^a$ on $\mathcal{L}_c(\mathcal{G}^{(0)})$. Then the Steinberg algebra $A_R(\mathcal{G})$ is isomorphic to the skew inverse semigroup algebra $\mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes_\alpha \mathcal{G}^a$.*

Proof. We will first show the existence of an epimorphism

$$\psi : \mathcal{L}(\alpha) \to A_R(\mathcal{G}),$$

that vanishes in the ideal $\mathcal{N}(\alpha)$ (thus, we can extend ψ to an epimorphism $\widetilde{\psi}$ of the quotient $\mathcal{L}(\alpha)/\mathcal{N}(\alpha) = \mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes_\alpha \mathcal{G}^a$). To conclude, we will show that $\widetilde{\psi}$ admits a right inverse map φ. Define the homomorphism $\psi : \mathcal{L}(\alpha) \to A_R(\mathcal{G})$, on the elements of the form $f_B\delta_B$, by

$$\psi(f_B\delta_B)(x) = \begin{cases} f_B(\mathfrak{r}(x)) & \text{if } x \in B \\ 0 & \text{if } x \notin B, \end{cases}$$

and extend it linearly to $\mathcal{L}(\alpha)$.

We need to show that ψ is well defined, that is, the function $\psi(f_B\delta_B)$ is locally constant and has compact support.

If $x \in \mathcal{G} \setminus B$ then $\mathcal{G} \setminus B$ is an open neighborhood of x and $\psi(f_B\delta_B)|_{\mathcal{G}\setminus B} \equiv 0$. Now, if $x \in B$ then $\psi(f_B\delta_B)(x) = f_B(\mathfrak{r}(x))$. Since

f_B is constant locally, there is an open neighborhood V of $\mathfrak{r}(x)$ such that $f_B|_V$ is constant. We can take $V \subseteq \mathfrak{r}(B)$ because $\mathfrak{r}(B)$ is open. Thus $\mathfrak{r}_B^{-1}(V)$ is an open neighborhood of x and $\psi(f_B\delta_B)|_{\mathfrak{r}_B^{-1}(V)} = f_B|_V$ is constant. Moreover,

$$\operatorname{supp}(\psi(f_B\delta_B)) = B \cap \operatorname{supp}(f_B \circ \mathfrak{r}),$$

is a compact subset of $\mathcal{G}^{(0)}$, since $\mathfrak{r}$ is a homeomorphism and $\operatorname{supp}(f_B)$ is compact subset of $\mathcal{G}^{(0)}$. Hence $\psi(f_B\delta_B) \in A_R(\mathcal{G})$.

Next, we will verify that ψ is multiplicative. By linearity, it is enough to verify that this application is multiplicative in the homogeneous terms. So, let $f_B\delta_B, f_C\delta_C \in \mathcal{L}(\alpha)$ and $x \in \mathcal{G}$. Then,

$$\begin{aligned}
&\psi(f_B\delta_B f_C\delta_C)(x)\\
&\quad= \psi(\alpha_B(\alpha_{B^*}(f_B)f_C)\delta_{BC})(x)\\
&\quad= \begin{cases} \alpha_B(\alpha_{B^*}(f_B)f_C)(\mathfrak{r}(x)), & \text{if}\, x \in BC\\ 0, & \text{if}\, x \notin BC\end{cases}\\
&\quad\overset{\mathfrak{r}(x)\in\mathfrak{r}(B)}{=} \begin{cases} \alpha_{B^*}(f_B)f_C(\theta_{B^*}(\mathfrak{r}(x)), & \text{if}\, x \in BC\\ 0, & \text{if}\, x \notin BC\end{cases}\\
&\quad= \begin{cases} \alpha_{B^*}(f_B(\theta_{B^*}(\mathfrak{r}(x))f_C(\theta_{B^*}(\mathfrak{r}(x)), & \text{if}\, x \in BC\\ 0, & \text{if}\, x \notin BC\end{cases}\\
&\quad\overset{\theta_{B^*}(\mathfrak{r}(x))\in\mathfrak{s}(B)}{=} \begin{cases} f_B(\mathfrak{r}(x))f_C(\theta_{B^*}(\mathfrak{r}(x)), & \text{if}\, x \in BC\\ 0, & \text{if}\, x \notin BC\end{cases}\\
&\quad= \begin{cases} f_B(\mathfrak{r}(x))f_C(\mathfrak{s}_B(\mathfrak{r}_B^{-1}(\mathfrak{r}(x)))), & \text{if}\, x \in BC\\ 0, & \text{if}\, x \notin BC\end{cases}
\end{aligned}$$

If $x \in BC$ then there are $b \in B$ and $c \in C$ such that $\mathfrak{s}(b) = \mathfrak{r}(c)$ and $x = bc$. Notice that b is the only element of B such that $\mathfrak{r}(b) = \mathfrak{r}(x) \in r(B)$. Thus

$$\mathfrak{r}_B^{-1}(\mathfrak{r}(x)) = b, \text{ and } \mathfrak{s}_B(\mathfrak{r}_B^{-1}(\mathfrak{r}(x))) = \mathfrak{s}_B(b) = \mathfrak{s}(b).$$

Hence

$$
\begin{aligned}
&= \begin{cases} f_B(\mathfrak{r}(x))f_C(\mathfrak{s}(b)), & \text{if } b \in B, c \in C \text{ and } x = bc \\ 0, & \text{otherwise.} \end{cases} \\
&\overset{\mathfrak{s}(b)=\mathfrak{r}(c)}{=} \begin{cases} f_B(\mathfrak{r}(x))f_C(\mathfrak{r}(c)), & \text{if } b \in B, c \in C \text{ and } x = bc \\ 0, & \text{otherwise.} \end{cases} \\
&\overset{\mathfrak{r}(x)=\mathfrak{r}(b)}{=} \begin{cases} f_B(\mathfrak{r}(b))f_C(\mathfrak{r}(c)), & \text{if } b \in B, c \in C \text{ and } x = bc \\ 0, & \text{otherwise.} \end{cases} \\
&= \sum_{\substack{\mathfrak{s}(b)=\mathfrak{r}(c) \\ x=bc}} \psi(f_B\delta_B)(b)\psi(f_C\delta_C)(c) \\
&= \psi(f_B\delta_B)\psi(f_C\delta_C)(x).
\end{aligned}
$$

In order to prove the surjectivity of ψ, let $f \in A_R(\mathcal{G})$. We have seen that f can be written as

$$\sum_{i=1}^{n} r_i 1_{B_i},$$

where $n \in \mathbb{N}$, $r_i \in R$, and B_i are pairwise disjoint compact-open bisections, for all $i = 1, \cdots, n$. For each i, we define f_{B_i} by

$$f_{B_i}(x) := \begin{cases} r_i, & \text{if } x \in \mathfrak{r}(B_i) \\ 0, & \text{if } x \in \mathcal{G}^{(0)} \setminus \mathfrak{r}(B_i). \end{cases}$$

Clearly f_{B_i} belongs to D_{B_i}. For $y \in \mathcal{G}$ we have that

$$
\begin{aligned}
\psi(f_{B_i}\delta_{B_i})(y) &= \begin{cases} f_{B_i}(\mathfrak{r}(y)), & \text{if } y \in B_i \\ 0, & \text{if } y \notin B_i \end{cases} \\
&= \begin{cases} r_i, & \text{if } y \in B_i \\ 0, & \text{if } y \notin B_i, \end{cases}
\end{aligned}
$$

that is, $\psi(f_{B_i}\delta_{B_i}) = r_i 1_{B_i}$. Taking $F = \sum_{i=1}^{n} f_{B_i}\delta_{B_i} \in \mathcal{L}(\alpha)$ we obtain that

$$\psi(F) = \sum_{i=1}^{n} r_i 1_{B_i} = f,$$

and thus ψ is surjective.

Next we will show that $\psi(\mathcal{N}(\alpha)) = \{0\}$, where $\mathcal{N}(\alpha)$ is the ideal of $\mathcal{L}(\alpha)$ generated by the set

$$\{f\delta_B - f\delta_A \mid B \subseteq A \text{ and } f \in D_B\}.$$

Since ψ is a homomorphism it is enough to show that ψ is zero on the generators of $\mathcal{N}(\alpha)$. Let $x \in \mathcal{G}$.

- If $x \notin A$, then $x \notin B$ and

$$\psi(f\delta_B)(x) - \psi(f\delta_A)(x) = 0.$$

- If $x \in A \setminus B$, then $\mathfrak{r}(x) \in \mathfrak{r}(A) \setminus \mathfrak{r}(B)$ ($\mathfrak{r}$ is injective in A). Thus

$$\psi(f\delta_B)(x) - \psi(f\delta_A)(x) = \psi(f\delta_A)(x) = f(\mathfrak{r}(x)) = 0,$$

  because $f \in D_B$ and $\mathfrak{r}(x) \notin \mathfrak{r}(B)$.

- If $x \in B$, then $x \in A$ and

$$\psi(f\delta_B)(x) - \psi(f\delta_A)(x) = f(\mathfrak{r}(x)) - f(\mathfrak{r}(x)) = 0.$$

This proves that ψ vanishes on $\mathcal{N}(\alpha)$.

We can now define a map $\widetilde{\psi}$ from the quotient $\mathcal{L}/\mathcal{I} = \mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes_\alpha \mathcal{G}^a$ to $A_R(\mathcal{G})$. Given $\overline{f_{B_i}\delta_{B_i}} \in \mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes_\alpha \mathcal{G}^a$ let

$$\widetilde{\psi}\left(\sum_{i=1}^{n} \overline{f_{B_i}\delta_{B_i}}\right) := \psi\left(\sum_{i=1}^{n} f_{B_i}\delta_{B_i}\right).$$

Notice that t ψ is well defined since $\psi(\mathcal{N}(\alpha)) = 0$. Clearly $\widetilde{\psi}$ is a surjective homomorphism.

In order to prove that $\widetilde{\psi}$ is an isomorphism, it suffices to verify that $\widetilde{\psi}$ admits a left inverse. To this end, consider the map $\varphi : A_R(\mathcal{G}) \to \mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes_\alpha \mathcal{G}^a$ defined as follows: Given $f \in A_R(\mathcal{G})$, we can writte f as

$$f = \sum_{j=1}^{n} b_j 1_{B_j},$$

where $n \in \mathbb{N}$, B_j are pairwise disjoint compact-open bisections of $\mathcal{G}$ and $b_j \neq 0$, for all $j = 1, \cdots, n$. Define

$$\varphi(f) = \varphi\left(\sum_{i=1}^{n} b_j 1_{B_j}\right) = \sum_{j=1}^{n} \overline{b_j 1_{\mathfrak{r}(B_j)} \delta_{B_j}}.$$

We claim that φ is well defined. Suppose that f can also be written as $f = \sum_{k=1}^{m} c_k 1_{C_k}$, where $C_1, \cdots, C_m$ are pairwise disjoint compact-open bisections of $\mathcal{G}$ and $c_1, \cdots, c_n$ are nonzero elements of R. Notice that,

$$\bigcup_{j=1}^{n} B_j = \operatorname{supp}(f) = \bigcup_{k=1}^{m} C_k,$$

where these unions are disjoint. Moreover, for any j, B_j is equal to the disjoint union $\bigcup_{k=1}^{m}(B_j \cap C_k)$, as well as, for any k, C_K is equal to the disjoint union $\bigcup_{j=1}^{n}(B_j \cap C_k)$. We can then conclude that

$$\sum_{j=1}^{n}\sum_{k=1}^{m} b_j 1_{B_j \cap C_k} = f = \sum_{k=1}^{m}\sum_{j=1}^{n} c_k 1_{C_k \cap B_j}.$$

Since the collection $\{C_j \cap B_k\}_{j,k}$ is pairwise disjoint, this implies that

$$b_j 1_{B_j \cap C_k} = c_k 1_{C_k \cap B_j},$$

for every pair i, j. Composing both maps, on the right side, with $\mathfrak{r}^{-1}_{B_j \cap C_k}$, we obtain

$$b_j 1_{\mathfrak{r}(B_j \cap C_k)} = c_k 1_{\mathfrak{r}(C_k \cap B_j)},$$

for every pair i, j. Since, for any j, B_j is a bisection, we get

$$\mathfrak{r}(B_j) = \bigcup_{k=1}^{m} \mathfrak{r}(B_j \cap C_k).$$

Analogously, for any k,

$$\mathfrak{r}(C_k) = \bigcup_{j=1}^{n} \mathfrak{r}(B_j \cap C_k).$$

With this we obtain

$$\begin{aligned}\sum_{j=1}^{n}\overline{b_j 1_{\mathfrak{r}(B_j)\delta_{B_j}}} &= \sum_{j=1}^{n}\overline{b_j 1_{\cup_{k=1}^{m}\mathfrak{r}(B_j\cap C_k)}\delta_{B_j}} = \sum_{j=1}^{n}\sum_{k=1}^{m}\overline{b_j 1_{\mathfrak{r}(B_j\cap C_k)}\delta_{B_j}} \\ &= \sum_{j=1}^{n}\sum_{k=1}^{m}\overline{b_j 1_{\mathfrak{r}(B_j\cap C_k)}\delta_{B_j\cap C_k}} = \sum_{k=1}^{m}\sum_{j=1}^{n}\overline{c_k 1_{\mathfrak{r}(B_j\cap C_k)}\delta_{C_K}} \\ &= \sum_{k=1}^{m}\overline{c_k 1_{\cup_{j=1}^{n}\mathfrak{r}(B_j\cap C_k)}\delta_{B_j}} = \sum_{k=1}^{m}\overline{c_k 1_{\mathfrak{r}(C_k)}\delta_{C_k}},\end{aligned}$$

proving that φ is well defined.

We need to show that $\varphi\circ\widetilde{\psi}$ is the identity map of $\mathcal{L}_c(\mathcal{G}^{(0)})\rtimes_\alpha \mathcal{G}^a$. Notice that, since each $f_B\in D_B$ can be written as

$$f_B=\sum_{k=1}^{m}c_k 1_{B_k},$$

where $m\in N, c_k\in R$ and B_k are pairwise disjoint compact-open subsets of $\mathfrak{r}(B)$ such that $\cup_{k=1}^{m}B_k\subseteq \mathfrak{r}(B)$, we have that

$$\begin{aligned}\psi(f_B\delta_B)(x) &= \begin{cases}\sum_{k=1}^{m}c_k 1_{B_k}(\mathfrak{r}(x)), & \text{if } x\in B\\ 0, & \text{if } x\notin B\end{cases}\\ &= \begin{cases} c_k, & \text{if } x\in B\cap \mathfrak{r}^{-1}(B_k)\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Since the subsets $B\cap\mathfrak{r}^{-1}(B_k)$ are pairwise disjoint compact-open, we have that

$$\psi(f_B\delta_B)=\sum_{k=1}^{m}c_k 1_{B\cap\mathfrak{r}^{-1}(B_k)}.$$

Thus,

$$\begin{aligned}\varphi\circ\widetilde{\psi}(\overline{f_B\delta_B}) &= \varphi\circ\psi(f_B\delta_B)=\varphi\left(\sum_{k=1}^{m}c_k 1_{B\cap\mathfrak{r}^{-1}(B_k)}\right)\\ &= \sum_{k=1}^{m}\overline{c_k 1_{\mathfrak{r}(B\cap\mathfrak{r}^{-1}(B_k))}\delta_{B\cap\mathfrak{r}^{-1}(B_k)}} \overset{\mathfrak{r}(B\cap r^{-1}(B_k))=B_k}{=} \sum_{k=1}^{m}\overline{c_k 1_{B_k}\delta_{B\cap\mathfrak{r}^{-1}(B_k)}}\\ &\overset{B\cap\mathfrak{r}^{-1}(B_k)\subseteq B}{=} \sum_{k=1}^{m}\overline{c_k 1_{B_k}\delta_B} = \overline{\sum_{k=1}^{m}c_k 1_{B_k}\delta_B} = \overline{\left(\sum_{k=1}^{m}c_k 1_{B_k}\right)\delta_B} = \overline{f_B\delta_B}.\end{aligned}$$

Notice that $\widetilde{\psi}$ is additive. If we prove that φ is also additive, then we have that

$$\varphi \circ \widetilde{\psi}(\sum f_B \delta_B) = \sum (\varphi \circ \widetilde{\psi}(f_B \delta_B)) = \sum f_B \delta_B.$$

and so we conclude that $\widetilde{\psi}$ is an isomorphism as desired.

So it remains to prove that φ is additive. Suppose that $f, g \in A_R(\mathcal{G})$ have representations

$$f = \sum_{i=1}^{n} r_i 1_{A_i} \quad \text{and} \quad \sum_{j=1}^{m} s_j 1_{B_j},$$

where A_i's and B_j's are pairwise disjoint compact-open bisections. We can, if necessary, add terms of the form $0 \cdot 1_{B_j \setminus \text{supp}(f)}$ to the representation of f, and similarly for g, add terms of the form $0 \cdot 1_{C_i \setminus \text{supp}(g)}$, and assume that $\bigcup_{i=1}^{n} C_i = \bigcup_{j=1}^{m} D_j$. Therefore we may rewrite

$$f = \sum_{i,j} r_i 1_{C_i \cap D_j} \quad \text{and} \quad g = \sum_{i,j} s_j 1_{C_i \cap D_j},$$

and hence

$$f + g = \sum_{i,j} (r_i + s_j) 1_{C_i \cap D_j},$$

and the definition of φ readily implies $\varphi(f + \lambda g) = \varphi(f) + \varphi(g)$. □

Remark 2.3.2. Let $\theta = (\{X_g\}_{g \in G}, \{\theta_g\}_{g \in G})$ be a partial action of a discrete group G on a locally compact, Hausdorff, and zero-dimensional topological space X. By Theorems 2.1.1, we get

$$A_R(G \ltimes_\theta X) \cong \mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\beta (G \ltimes_\theta X)^a,$$

where β is the action of the inverse semigroup $(G \ltimes_\theta X)^a$ on the R-algebra $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$ associated to canonical action of $(G \ltimes_\theta X)^a$ on the unit space $(G \ltimes_\theta X)^{(0)} \cong X$. On the other hand, by Theorem 2.1.1, we have that

$$A_R(G \ltimes_\theta X) \cong \mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\alpha G,$$

where α is the partial action of the group G on the R-algebra $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$ associated to the partial action θ.

3 SIMPLICITY OF SKEW INVERSE SEMIGROUP RINGS

This chapter is based entirely on the paper [3], produced during the doctorate. We shall be concerned with the simplicity of skew inverse semigroup rings $A \rtimes_\alpha S$ when A is a commutative ring. Our interest to study this class of rings comes from its connections with topological dynamics (see Section 3.2), and the fact that any Steinberg algebra, associated with an ample Hausdorff groupoid, can be realized as a skew inverse semigroup ring (see Theorem 2.3.1).

The interplay between topological dynamics and crossed products algebras is a driving force in the field of C*-algebras and has motivated the study of relations between topological dynamics and purely algebraic objects (as Steinberg algebras). By applying our main results we can describe connections between simplicity of the skew inverse semigroup ring associated with a topological partial action and topological properties of the action. The techniques we employ here are quite different from the ones used in [37].

This chapter is organized as follows. In the first section, we prove the main result of this chapter, which yields a complete characterization of simplicity of skew inverse semigroup rings in the case when A is commutative (see Theorem 3.1.5). In Section 3.2 we apply our result in the context of topological dynamics: Given topological partial action of an inverse semigroup on a locally compact, Hausdorff and zero-dimensional space, we show that the associated skew inverse semigroup ring is simple if, and only if, the action is minimal, topologically principal and a certain condition on the existence of functions with non-empty support on ideals of the skew inverse semigroup ring holds. (The aforementioned condition has the same flavour as the one presented in [15] for groupoids. We were not aware of the work in [15] while developing Section 3.2). Finally, in Section 3.3, based on the previous chapters, we apply our main result to get a new proof of the simplicity criterion for a Steinberg algebra $A_R(\mathcal{G})$ associated with a Hausdorff and ample groupoid $\mathcal{G}$ (see Theorem 3.3.1).

3.1 Simplicity of skew inverse semigroup rings

Throughout this section we shall make the following assumptions: Any given partial action $\alpha = (\{\alpha_s\}_{s\in S}, \{D_s\}_{s\in S})$ of S on a ring A has the property that A and each ideal D_s for $s \in S$, are s-unital.

Our goal is to give a characterization of simplicity for skew inverse semigroup rings $A \rtimes_\alpha S$ in the case when A is commutative (see Theorem 3.1.5).

Before we proceed, let us first recall some of the notations and results established in Section 1.6.1. By Lemma 1.6.14, the map $\widetilde{\tau} : A \rtimes_\alpha S \to A$ defined by

$$\widetilde{\tau}\left(\sum_{s\in S} \overline{a_s\delta_s}\right) = \sum_{s\in S} a_s, \tag{3.1}$$

is a well-defined additive map. Recall that the diagonal $\mathcal{D}$ of the partial skew inverse semigroup ring $A \rtimes_\alpha S$ is the subring generated by the elements of the form $\overline{a\delta_e}$, where $e \in E(S)$ and $a \in D_e$. By Proposition 1.6.16, the ring A is isomorphic to $\mathcal{D}$ via the isomorphism $\phi : A \to \mathcal{D}$ defined by

$$\phi\left(\sum_{i=1}^{n} a_{e_i}\right) = \sum_{i=1}^{n} \overline{a_{e_i}\delta_{e_i}}, \tag{3.2}$$

where $n \in \mathbb{N}$, $e_i \in E(S)$ and $a_{e_i} \in D_{e_1}$ for $i \in \{1, \cdots, n\}$. Clearly, if A is a commutative ring, then the diagonal $\mathcal{D}$ is also commutative. Therefore, we will identify A and $\mathcal{D}$ and use A and $\mathcal{D}$ interchangeably.

It does not make sense to speak of the support-length of an element in the quotient ring $A \rtimes_\alpha S$. However, given any element $a \in A \rtimes_\alpha S$ we may speak of the *minimal support-length of a representative of* a, i.e. an element $x \in \mathcal{L}(\alpha)$ such that $a = \overline{x}$. We make the following definition.

Definition 3.1.1. For each non-zero $a \in A \rtimes_\alpha S$ we define the number

$$n(a) = \min\left\{ |F| \;\middle|\; a = \sum_{s\in F} \overline{a_s\delta_s} \;\text{ and }\; a_s \neq 0 \text{ for all } s \in F \right\},$$

where $|F|$ denotes the cardinality of the finite set F.

Recall that the *centralizer* of a non-empty subset M of a ring R is the set of all the elements of R that commute with each element of M. If the centralizer of M is M itself, we say that this set is *maximal commutative in* R. Notice that a maximal commutative subring is necessarily commutative.

Theorem 3.1.2. *[3, Theorem 3.4] Let A be a commutative ring. Then $A \cong \mathcal{D}$ is a maximal commutative subring of $A \rtimes_\alpha S$ if, and only if, $\mathcal{J} \cap A \neq \{0\}$ for each non-zero ideal $\mathcal{J}$ of $A \rtimes_\alpha S$.*

Proof. We first show the "if" statement. To this end, suppose that $A \cong \mathcal{D}$ is not a maximal commutative subring of $A \rtimes_\alpha S$. We now wish to conclude that there is some non-zero ideal $\mathcal{J}$ of $A \rtimes_\alpha S$ such that $\mathcal{J} \cap \mathcal{D} = \{0\}$.

Let $c = \sum_{s\in F} \overline{c_s\delta_s} \in (A \rtimes_\alpha S) \setminus \mathcal{D}$ be an element that commutes with all the elements of $\mathcal{D}$. Since c commutes with $\overline{a\delta_e}$ for each $e \in E(S)$ and $a \in D_e$, we get that

$$\sum_{s\in F} \overline{ac_s\delta_{es}} = \sum_{s\in F} \overline{\alpha_s(\alpha_{s^*}(c_s)a)\delta_{se}},$$

and hence

$$\sum_{s\in F} ac_s\delta_{es} - \sum_{s\in F} \alpha_s(\alpha_{s^*}(c_s)a)\delta_{se} \in \mathcal{N}(\alpha).$$

Using that $\tau(\mathcal{N}(\alpha)) = \{0\}$ we get that

$$\sum_{s\in F} (ac_s - \alpha_s(\alpha_{s^*}(c_s)a)) = 0. \tag{3.3}$$

Notice that $x := \sum_{s\in F} \overline{c_s\delta_{ss^*}} - \sum_{s\in F} \overline{c_s\delta_s} \neq \overline{0}$. Otherwise we would have $c = \sum_{s\in F} \overline{c_s\delta_s} = \sum_{s\in F} \overline{c_s\delta_{ss^*}} \in \mathcal{D}$.

Now, let $\mathcal{J}$ be the non-zero ideal of $A \rtimes_\alpha S$ generated by the element x. Each element of $\mathcal{J}$ is a finite sum of elements of the form $\overline{a_u\delta_u}x\overline{a_v\delta_v}$, $\overline{a_u\delta_u}x$ and $x\overline{a_v\delta_v}$ for $u, v \in S$ and $a_u \in D_u$, $b_v \in D_v$. By the fact that $ss^* \in E(S)$ and $\alpha_{ss^*} = \mathrm{id}_{D_{ss^*}}$, we notice that

$$\begin{aligned}\overline{a_u\delta_u}x\overline{a_v\delta_v} &= \overline{a_u\delta_u}\left(\sum_{s\in F}\overline{c_s\delta_{ss^*}} - \sum_{s\in F}\overline{c_s\delta_s}\right)\overline{a_v\delta_v}\\ &= \sum_{s\in F}\overline{\alpha_u(\alpha_{u^*}(a_u)c_sa_v)\delta_{uss^*v}} - \sum_{s\in F}\overline{\alpha_u(\alpha_{u^*}(a_u)\alpha_s(\alpha_{s^*}(c_s)a_v))\delta_{usv}},\end{aligned}$$

and hence, by Equation (3.3), we get that

$$\begin{aligned}\widetilde{\tau}\left(\overline{a_u\delta_u}x\overline{a_v\delta_v}\right) &= \sum_{s\in F}\alpha_u(\alpha_{u^*}(a_u)c_sa_v) - \sum_{s\in F}\alpha_u(\alpha_{u^*}(a_u)\alpha_s(\alpha_{s^*}(c_s)a_v))\\ &= \alpha_u\left(\alpha_{u^*}(a_u)\sum_{s\in F}(c_sa_v - \alpha_s(\alpha_{s^*}(c_s)a_v))\right) = 0.\end{aligned}$$

Analogously, one may show that $\widetilde{\tau}(\overline{a_u\delta_u}x) = 0$ and $\widetilde{\tau}(x\overline{a_v\delta_v}) = 0$. This shows that $\widetilde{\tau}(\mathcal{J}) = \{0\}$.

Take any $y \in \mathcal{J} \cap \mathcal{D}$. Then $y = \sum_{i=1}^n \overline{a_i\delta_{e_i}}$, for some $n \in \mathbb{Z}_+$, $e_i \in E(S)$ and $a_i \in D_{e_i}$ for $i \in \{1, \ldots, n\}$. Notice that

$$\sum_{i=1}^n a_i = \widetilde{\tau}\left(\sum_{i=1}^n \overline{a_i\delta_{e_i}}\right) = \widetilde{\tau}(y) = 0.$$

Hence $y = 0$ (by the same reason that ϕ is well-defined in Proposition 1.6.16). We now conclude that $\mathcal{J} \cap \mathcal{D} = \{0\}$.

Now we show the "only if" statement. Suppose that $\mathcal{D} \cong A$ is a maximal commutative subring of $A \rtimes_\alpha S$. Let $\mathcal{J}$ be a non-zero ideal of $A \rtimes_\alpha S$. Take $x \in \mathcal{J} \setminus \{0\}$ such that $n(x) = \min\{n(y) \mid y \in \mathcal{J} \setminus \{0\}\}$ and write $x = \sum\limits_{s\in F}\overline{x_s\delta_s}$, where $|F| = n(x)$. Choose some $h \in F$, and let $1_h \in D_h$ be such that $1_hx_h = x_h$. Since $D_h \subseteq D_{hh^*}$ then $1_h \in D_{hh^*}$, and

$$\overline{1_h\delta_{hh^*}}x = \overline{x_h\delta_h} + \sum_{s\in F\setminus\{h\}}\overline{1_hx_s\delta_{hh^*s}}.$$

Using that $hh^*s \leq s$, for each $s \in S$, we get that $\overline{1_h x_s \delta_{hh^*s}} = \overline{1_h x_s \delta_s}$ and hence

$$\overline{1_h \delta_{hh^*}} x = \overline{x_h \delta_h} + \sum_{s \in F \setminus \{h\}} \overline{1_h x_s \delta_s}.$$

Let

$$y = x - \overline{1_h \delta_{hh^*}} x \quad = \sum_{s \in F \setminus \{h\}} \overline{(1_h x_s - x_s)\delta_s}$$

and notice that $y \in \mathcal{J}$. Using that $n(x)$ is minimal and $y \in \mathcal{J}$ we conclude that $y = 0$. Thus, we have that $\sum\limits_{s \in F \setminus \{h\}} \overline{1_h x_s \delta_s} = \sum\limits_{s \in F \setminus \{h\}} \overline{x_s \delta_s}$ and hence

$$x = \overline{x_h \delta_h} + \sum_{s \in F \setminus \{h\}} \overline{1_h x_s \delta_s}.$$

In particular, $\overline{1_h \delta_{hh^*}}\ x = x \neq 0$ and, since

$$\overline{1_h \delta_{hh^*}}\ x = \overline{1_h \delta_h}\ \overline{\alpha_{h^*}(1_h)\delta_{h^*}}\ x,$$

we have that $\overline{\alpha_{h^*}(1_h)\delta_{h^*}}\ x \neq 0$. Let $z = \overline{\alpha_{h^*}(1_h)\delta_{h^*}}\ x \in \mathcal{J}$ and notice that z is non-zero and

$$\begin{aligned} z = \overline{\alpha_{h^*}(1_h)\delta_{h^*}} x = \overline{\alpha_{h^*}(x_h)\delta_{h^*h}} \quad &+ \sum_{s \in F \setminus \{h\}} \overline{\alpha_{h^*}(1_h)\delta_{h^*} x_s \delta_s} \\ = \overline{\alpha_{h^*}(x_h)\delta_{h^*h}} \quad &+ \sum_{s \in F \setminus \{h\}} \overline{\alpha_{h^*}(1_h x_s)\delta_{h^*s}}. \end{aligned}$$

Now, let $\overline{a\delta_e} \in \mathcal{D}$ be arbitrary and consider the element $p = \overline{a\delta_e} \cdot z - z \cdot \overline{a\delta_e} \in \mathcal{J}$. We have that

$$\begin{aligned} p \quad = \quad & \overline{a\alpha_{h^*}(x_h)\delta_{ehh^*}} \quad + \sum_{s \in F \setminus \{h\}} \overline{a\alpha_{h^*}(1_h x_s)\delta_{eh^*s}} \\ & - \overline{\alpha_{h^*}(x_h) a \delta_{hh^*e}} - \sum_{s \in F \setminus \{h\}} \overline{\alpha_{h^*s}(\alpha_{s^*h}(\alpha_{h^*}(1_h x_s))a)\delta_{h^*se}}. \end{aligned}$$

Since A and $E(S)$ are commutative, we have that

$$p \quad = \sum_{s \in F \setminus \{h\}} \overline{a\alpha_{h^*}(1_h x_s)\delta_{eh^*s}} - \sum_{s \in F \setminus \{h\}} \overline{\alpha_{h^*s}(\alpha_{s^*h}(\alpha_{h^*}(1_h x_s))a)\delta_{h^*se}}.$$

Using that $eh^*s \leq h^*s$ and $h^*se \leq h^*s$, we have that

$$p = \sum_{s\in F\setminus\{h\}} \overline{a\alpha_{h^*}(1_h x_s)\delta_{h^*s}} - \sum_{s\in F\setminus\{h\}} \overline{\alpha_{h^*s}(\alpha_{s^*h}(\alpha_{h^*}(1_h x_s))a)\delta_{h^*s}}.$$

Hence, $n(p) < n(x)$ and by the minimality of $n(x)$ we conclude that $p = 0$.

But this implies that $\overline{a\delta_e} \cdot z = z \cdot \overline{a\delta_e}$. Therefore

$$\sum_{i=1}^{n} \overline{a_i\delta_{e_i}} \cdot z = z \cdot \sum_{i=1}^{n} \overline{a_i\delta_{e_i}},$$

for all $\sum_{i=1}^{n} \overline{a_i\delta_{e_i}} \in \mathcal{D}$. Since $\mathcal{D} \cong A$ is maximal commutative, we get that $z \in \mathcal{D}$. We conclude that $\mathcal{J} \cap \mathcal{D} \neq \{0\}$. □

Corollary 3.1.3. *[3, Corollary 3.5] Let A be a commutative ring. If $A \rtimes_\alpha S$ is simple, then $A \cong \mathcal{D}$ is a maximal commutative subring of $A \rtimes_\alpha S$.*

Recall that an ideal I of A is S-*invariant* if $\alpha_s(I \cap D_{s^*}) \subseteq I$ holds for each $s \in S$. The ring A is said to be S-*simple* if A has no non-zero S-invariant proper ideal.

Proposition 3.1.4. *[3, Propostion 3.6] Let A be a ring. If $A \rtimes_\alpha S$ is simple, then A is S-simple.*

Proof. Let I be a non-zero S-invariant ideal of A. Define the set

$$\mathcal{H} = \left\{ \sum_{s\in S} \overline{a_s\delta_s} \in A \rtimes_\alpha S \;\middle|\; a_s \in I \cap D_s,\ s \in S \right\}.$$

Notice that $\mathcal{H} \neq \{0\}$. Indeed, let $a \in I$ be non-zero and let $u \in A$ be such that $ua = a$. By the non-degeneracy of α there are idempotents $e_1, \ldots, e_n \in E(S)$ such that $u = \sum_{i=1}^{n} u_i$, with $u_i \in D_{e_i}$ for $i \in \{1, \ldots, n\}$. Clearly,

$$0 \neq a = ua = \sum_{i=1}^{n} u_i a.$$

Using that I is an ideal of A, we get that $u_i a \in I \cap D_{e_i}$ for $i \in \{1, \ldots, n\}$, and hence $\sum_{i=1}^{n} \overline{u_i a \delta_{e_i}} \in \mathcal{H}$. Let ϕ denote the ring isomorphism from the proof of Proposition 1.6.16. Using that $a \neq 0$, we get that

$$\sum_{i=1}^{n} \overline{u_i a \delta_{e_i}} = \phi\left(\sum_{i=1}^{n} u_i\right) = \phi(a) \neq 0.$$

Moreover, $\mathcal{H}$ is a left ideal of $A \rtimes_\alpha S$. Indeed, if $\overline{a_r \delta_r} \in A \rtimes_\alpha S$ and $a_s \in I \cap D_s$ then $(\overline{a_r \delta_r})(\overline{a_s \delta_s}) = \overline{\alpha_r(\alpha_{r^*}(a_r)a_s)\delta_{rs}}$. Since I is S-invariant, $\alpha_r(\alpha_{r^*}(a_r)a_s) \in I$, and from the definition of a partial action we get that $\alpha_r(\alpha_{r^*}(a_r)a_s) \in D_{rs}$. Hence, $\overline{a_r \delta_r a_s \delta_s} \in \mathcal{H}$.

Similarly, $\mathcal{H}$ is a right ideal of $A \rtimes_\alpha S$ and hence, by the simplicity of $A \rtimes_\alpha S$, we obtain that $\mathcal{H} = A \rtimes_\alpha S$. From the definition of $\mathcal{H}$ we immediately see that $\widetilde{\tau}(\mathcal{H}) \subseteq I$, and from what was done above, $\widetilde{\tau}(\mathcal{H}) = \widetilde{\tau}(A \rtimes_\alpha S) = A$. Thus, $I = A$ and therefore A is S-simple. ☐

We are now ready to state and prove the main result of this section.

Theorem 3.1.5. *[3, Theorem 3.7] If A is a commutative ring, then the following two assertions are equivalent:*

(i) The skew inverse semigroup ring $A \rtimes_\alpha S$ is simple;

(ii) A is S-simple, and $A \cong \mathcal{D}$ is a maximal commutative subring of $A \rtimes_\alpha S$.

Proof. (i)⇒(ii): This follows from Corollary 3.1.3 and Proposition 3.1.4.

(ii)⇒(i): Let $\mathcal{J}$ be a non-zero ideal of $A \rtimes_\alpha S$. By Theorem 3.1.2, $\mathcal{J} \cap \mathcal{D} \neq \{0\}$.

Put $\mathcal{K} = \mathcal{J} \cap \mathcal{D}$ and $\mathcal{K}' = \phi^{-1}(\mathcal{K})$, where $\phi : A \to \mathcal{D}$ is the ring isomorphism from Proposition 1.6.16. Clearly, $\mathcal{K}'$ is a non-zero ideal of A. Now we show that $\mathcal{K}'$ is S-invariant.

Take an arbitrary $s \in S$ and an arbitrary $a_s \in \mathcal{K}' \cap D_s$. Pick $1_s \in D_s$ such that $1_s a_s = a_s$. By the definition of A there are idempotents $e_1, \ldots, e_n \in S$, and elements $a_{e_i} \in D_{e_i}$, for $i \in \{1, \ldots, n\}$, such that $a_s = \sum_{i=1}^{n} a_{e_i}$ and $\phi(a_s) = \sum_{i=1}^{n} \overline{a_{e_i} \delta_{e_i}} \in \mathcal{K}$. We notice that

$$\begin{aligned}
\overline{\alpha_{s^*}(1_s)\delta_{s^*}} \cdot \sum_{i=1}^{n} \overline{a_{e_i}\delta_{e_i}} \cdot \overline{1_s\delta_s} &= \overline{\alpha_{s^*}(1_s)\delta_{s^*} \cdot \sum_{i=1}^{n} a_{e_i}\delta_{e_i} \cdot 1_s\delta_s} \\
&= \overline{\sum_{i=1}^{n} \alpha_{s^*}(1_s a_{e_i})\delta_{s^* e_i} \cdot 1_s\delta_s} \\
&\overset{s^*e_i \leq s^*}{=} \overline{\sum_{i=1}^{n} \alpha_{s^*}(1_s a_{e_i})\delta_{s^*} \cdot 1_s\delta_s} \\
&= \overline{\sum_{i=1}^{n} \alpha_{s^*}(1_s a_{e_i} 1_s)\delta_{s^* s}} \\
&= \overline{\alpha_{s^*}\left(1_s \left(\sum_{i=1}^{n} a_{e_i}\right) 1_s\right)\delta_{s^* s}} \\
&= \overline{\alpha_{s^*}(a_s)\delta_{s^* s}}
\end{aligned}$$

is in $\mathcal{J} \cap A = \mathcal{K}$ and hence $\alpha_{s^*}(a_s) = \phi^{-1}(\overline{\alpha_{s^*}(a_s)\delta_{s^*s}}) \in \mathcal{K}'$. Therefore $\mathcal{K}'$ is S-invariant. Using that A is S-simple we conclude that $\mathcal{K}' = A$.

Now, consider the arbitrary element $\overline{a_s\delta_s} \in A \rtimes_\alpha S$. By letting $1_s \in D_s$ be such that $1_s a_s = a_s$. We have that $1_s \in A = \mathcal{K}'$. Hence there are idempotents $f_1, \ldots, f_m \in E(S)$ and $u_j \in D_{f_j}$, for $j \in \{1, \ldots, m\}$, such that $1_s = \sum_{j=1}^{m} u_j \in \mathcal{K}'$ and $\phi(1_s) = \sum_{j=1}^{m} \overline{u_j\delta_{f_j}} \in \mathcal{K} \subseteq \mathcal{J}$. Thus,

$$\begin{aligned}
\overline{a_s\delta_s} = \overline{1_s a_s\delta_s} = \overline{\left(\sum_{j=1}^{m} u_j\right) a_s\delta_s} = \overline{\sum_{j=1}^{m} u_j a_s\delta_s} \overset{f_j s \leq s}{=} \overline{\sum_{j=1}^{m} u_j a_s \delta_{f_j s}} \\
= \overline{\left(\sum_{j=1}^{m} u_j\delta_{f_j}\right)(a_s\delta_s)} = \left(\sum_{j=1}^{m} \overline{u_j\delta_{f_j}}\right) \overline{a_s\delta_s} \in \mathcal{J}.
\end{aligned}$$

This shows that $A \rtimes_\alpha S = \mathcal{J}$ as desired. □

3.2 An application to topological dynamics

In this section we will apply our main results and connect topological properties of a partial action of an inverse semigroup S on a topological space X with algebraic properties of the associated skew inverse semigroup ring $\mathcal{L}_c(\mathrm{X}) \rtimes_\alpha S$.

With this purpose, throughout this section, we assume that $\theta = (\{\theta_s\}_{s\in S}, \{X_s\}_{s\in S})$ is a topological partial action of an inverse semigroup S on a locally compact, Hausdorff, zero-dimensional topological space X. From the partial action θ we get a corresponding partial action $\alpha = (\{\alpha_s\}_{s\in S}, \{D_s\}_{s\in S})$ of S on the R-algebra $\mathcal{L}_c(\mathrm{X})$, of all locally constant, compactly supported, R-valued functions on X, where R is a unital and commutative ring (see Example 1.5.28).

For a subset $T \subseteq X$, we define

$$\mathbf{J}(T) = \{f \in \mathcal{L}_c(\mathrm{X}) \mid f(x) = 0, \text{ for all } x \in T\}.$$

Clearly, the set $I(T)$ is an ideal of $\mathcal{L}_c(\mathrm{X})$. Moreover, since every function in $\mathcal{L}_c(\mathrm{X})$ is continuous, we conclude that $\mathbf{J}(T) = \mathbf{J}\left(\overline{T}\right)$, where $\overline{T}$ denotes the closure of T.

Lemma 3.2.1. *Let R be a field. Then every ideal J of $\mathcal{L}_c(\mathrm{X})$ is of the form*

$$\mathbf{J}(F) = \{f \in \mathcal{L}_c(\mathrm{X}) \mid f(x) = 0, \text{ for all } x \in F\},$$

where F is a closed subset of X given by

$$F = \{x \in X \mid f(x) = 0, \text{ for all } f \in J\}.$$

Proof. Let J be an ideal of $\mathcal{L}_c(\mathrm{X})$. Using that every function $f \in \mathcal{L}_c(\mathrm{X})$ is continuous, we have that the subset $F = \{x \in X \mid f(x) = 0, \text{ for all } f \in J\}$ is closed in X. Clearly, $J \subseteq \mathbf{J}(F)$.

Now, take any $f \in \mathbf{J}(F)$. Consider the set $U = \mathrm{supp}(f)$. Notice that $U \cap F = \varnothing$. If $x \in U$, then $x \notin F$ and there exists some $f_x \in J$ such that $f_x(x) \neq 0$. We have that

$$U \subseteq \bigcup_{x\in U} \{y \in X \mid f_x(y) \neq 0\} = \bigcup_{x\in U} \mathrm{supp}(f_x).$$

By compactness of U we may find finitely many points $x_1, \ldots, x_n$ such that

$$U \subseteq \bigcup_{i=1}^{n} \{y \in X \mid f_{x_i}(y) \neq 0\} = \bigcup_{i=1}^{n} \mathrm{supp}(f_{x_i}).$$

Consider $U_1 = \mathrm{supp}(f_{x_1})$ and $U_i := \mathrm{supp}(f_{x_i}) \setminus \bigcup_{k=1}^{i-1} \mathrm{supp}(f_{x_k})$ for all $i \in \{2, \ldots, n\}$. We have that

$$\bigcup_{i=1}^{n} \mathrm{supp}(f_{x_i}) = \bigcup_{i=1}^{n} U_i,$$

where the last union is a disjoint union of compact-open subsets.

Let $g = \sum_{i=1}^{n} f_{x_i} \cdot 1_{U_i}$. Using that $f_{x_i} \in J$, for each $i \in \{1, \ldots, n\}$, we have that $g \in J$. Notice that $g(x) \neq 0$ for all $x \in U$. We define

$$h(x) = \begin{cases} \frac{1}{g(x)} & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

and notice that $h \in \mathcal{L}_{\mathrm{c}}(\mathrm{X})$. Clearly, $f = f \cdot g \cdot h \in J$. In fact, $g \cdot h$ is a local unit for f. $\square$

Remark 3.2.2. Let R be a field. Notice that, by Lemma 3.2.1, every ideal J of $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$ is of the form

$$\mathbf{I}(U) := \mathbf{J}(X \setminus U) = \{f \in \mathcal{L}_{\mathrm{c}}(\mathrm{X}) \mid \mathrm{supp}(f) \subseteq U\},$$

where U is an open subset of X defined as

$$U = \{x \in X \mid \text{ there is } f \in J \text{ such that } f(x) \neq 0\} = \bigcup_{f \in J} \mathrm{supp}(f).$$

Let $\theta = (\{\theta_s\}_{s\in S}, \{X_s\}_{s\in S})$ be a topological partial action of an inverse semigroup S on a locally compact Hausdorff space X. Recall that a subset U of X is *invariant* if $\theta_s(U \cap X_{s^*}) \subseteq U$ for all $s \in S$. The topological partial action θ is *minimal* if there is no non-empty, proper and open invariant subset of X.

Let R be a field. It is easy to see that if U is an open invariant subset of X then the associated ideal $\mathbf{I}(U)$ is invariant. Conversely, every invariant ideal corresponds to an open invariant subset of X. Indeed, suppose that I is an invariant ideal of $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$. By Remark 3.2.2 there is an open subset U of X such that $I = \mathbf{I}(U)$. Take $s \in S$, $x \in U \cap X_{s^*}$, and suppose that $\theta_s(x) \notin U$. Let $K \subseteq U$ be a compact-open neighbourhood of x (it exists since X is zero-dimensional). Notice that the function 1_K is contained in $\mathbf{I}(U)$. Since I is invariant, $\alpha_s(1_K) \in \mathbf{I}(U)$, that is, $1_K \circ \theta_{s^*} \in \mathbf{I}(U)$. But then we get that

$$1 = 1_K(x) = 1_K(\theta_{s^*} \circ \theta_s(x)) = 1_K \circ \theta_{s^*}(\theta_s(x)) = 0.$$

Therefore U is invariant.

From the previous paragraph we obtain the following result.

Proposition 3.2.3. *Let R be a field and let $\theta = (\{\theta_s\}_{s\in S}, \{X_s\}_{s\in S})$ be a topological partial action of an inverse semigroup S on a locally compact, Hausdorff and zero-dimensional space X. Then θ is minimal if, and only if, $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$ is S-simple (with respect to the action α associated with θ).*

The notion of a topologically free (or effective) topological partial action is already well-known for partial group actions, see Definition 2.2.8. In this case, the freeness of a topological partial action is directly linked with the diagonal maximality of the skew group ring associated with this partial action.

Proposition 3.2.4. *Suppose that $\theta = (\{X_t\}_{t\in G}, \{\theta_t\}_{t\in G})$ is a topologically free partial action of a group G on X. Then $\mathcal{L}_{\mathrm{c}}(\mathrm{X})\,\delta_1$ is maximal commutative in $\mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\alpha G$.*

Proof. The proof is analogous to the proof of [37, Proposition 4.7]. □

With intention of generalizing the above result, we will now present the notion of topologically principal partial action of an inverse semigroup, which was introduced in [2], and the definition of effective (or topologically free) partial action of inverse semigroup, which was introduced in [32].

Let $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ be a partial action of an inverse semigroup S on a set X. The subset $\{s \in S \mid x \in X_{s^*}\}$ of S will be denoted by S_x.

Definition 3.2.5. [32, Definition 4.1] Let $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ be a partial action of an inverse semigroup S on a topological space X. Given $x \in S$ and $s \in S_x$, we say that

(a) x is *fixed* by s if $\theta_s(x) = x$,

(b) x is *trivially fixed* by s if there is $e \in E(S)$ such that $e \leq s$ and $x \in X_e$,

(c) θ is *effective* (or *topologically free*) if, for every s in S, the interior of the set of fixed points for s coincides with the set of points trivially fixed by s. Symbolically, θ is effective if

$$\operatorname{int}\{x \in X_{s^*} \mid \theta_s(x) = x\} = \{x \in X_{s^*} \mid \exists\, e \in E(S),\ e \leq s \text{ and } x \in X_e\}. \tag{3.4}$$

Remark 3.2.6. The inclusion "$\supseteq$" in (3.4) is always satisfied. Indeed, notice that if x is a trivial fixed point for s, then there is $e \in E(S)$ such that $e \leq s$ and $x \in X_e$, and

$$\theta_s(x) = \theta_e(x) = x,$$

that is, x is a fixed point for s. Moreover, since $X_e \subseteq X_s$, we have that every $y \in X_e$ is trivially fixed for s. This show that the set of trivial fixed points for s is open, and hence it is necessarily contained in the interior of the set of fixed points for s. Symbolically, we get that

$$\{x \in X_{s^*} \mid \exists\, e \in E(S), e \leq s \text{ and } x \in X_e\} \subseteq \operatorname{int}\{x \in X_{s^*} \mid \theta_s(x) = x\},$$

for all $s \in S$, as required.

We should mention, however, that partial actions which correspond to effective groupoids of germs were defined under the name "topologically free" in [32], so, in order to avoid confusion throughout this thesis, we will call the class of partial actions defined in [32] by *effective*.

It follows from Definition 3.2.5 that in the case when θ is a partial action of a group G on a topological space X, θ is effective if, for all $g \in G \setminus \{1\}$,

$$\operatorname{int}\{x \in X_{g^*} \mid \theta_g(x) = x\} = \varnothing.$$

This is exactly the definition of a group topologically free partial action (see Definition 2.2.8).

Definition 3.2.7. [2, Definition 7.1] Let $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ be a topological partial action of an inverse semigroup S on a topological space X. The subset

$$\{x \in X \mid \text{if } s \in S_x \text{ and } x \text{ is fixed by } s \text{ then } x \text{ is trivially fixed by } s\} \tag{3.5}$$

of X will be denoted by $\Lambda(\theta)$.

We say that θ is *topologically principal* if, and only if, $\Lambda(\theta)$ is dense in X.

The notion of a topologically principal partial action stems from the fact that the groupoid of germs $S \ltimes X$ associated with a topological partial action $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ is topologically principal if, and only if, the partial action θ is topologically principal. We will also prove this correspondence in Proposition 4.2.2.

The following proposition is a useful rewording of principality of partial actions.

Lemma 3.2.8. *Let $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ be a topological partial action of S on X. Then $\Lambda(\theta)$ coincides with the subset*

$$\{x \in X \mid \forall s,t \in S_x, \textit{if } \theta_s(x) = \theta_t(x) \textit{ then } \exists u \in S \textit{ with } u \leq s,t \textit{ and } x \in X_{u^*}\}\,. \tag{3.6}$$

In particular, given $x \in X$ and $s,t \in S_x$, if $\theta_s(x) = \theta_t(x)$ then there is $u \leq s,t$ with $x \in X_{u^*}$, and θ_s and θ_t coincide in the neighbourhood X_{u^*} of x.

Proof. Suppose that $x \in \Lambda(\theta)$. For any $s,t \in S_x$, if $\theta_s(x) = \theta_t(x)$ then $x = \theta_{s^*t}(x)$, and so, there is $e \in E(S)$ such that $e \leq s^*t$ and $x \in X_e$. Taking $u = se$, we get that $\Lambda(\theta)$ is contained in the subset (3.6).

For the reverse inclusion, suppose that x belongs to subset (3.6). For any $s \in S_x$, if $\theta_s(x) = x$ then $\theta_s(x) = \theta_{s^*s}(x)$. By hypothesis, there is $u \leq s, s^*s$ and $x \in X_{u^*}$. Using $e = uu^*$, we get the desired result. □

Proposition 3.2.9. *[3, Proposition 4.10] Let $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ be a topological partial action of a countable inverse semigroup S on a* *locally compact, Hausdorff topological space X. Then θ is topologically principal if, and only if, for any $s \in S$, the set*

$$\Lambda_s(\theta) = \{x \in X_{s^*} \mid \text{if } x \text{ is fixed by } s \text{ then } x \text{ is trivially fixed by } s\} \tag{3.7}$$

is dense in X_{s^*}.

Proof. For any $s \in S$, we have that

$$\Lambda(\theta) \cap X_{s^*} \subseteq \Lambda_s(\theta),$$

and if θ is topologically principal, we can conclude that $\Lambda_s(\theta)$ is dense in X_{s^*}. Notice that in this direction we do not need to use the fact that S is countable.

Conversely, suppose that $\Lambda_s(\theta)$ is dense in X_{s^*} for $s \in S$. Notice that $\Lambda_s(\theta)$ is an open subset of X and that

$$X = \overline{X_{s^*}} \cup (X \setminus \overline{X_{s^*}}) = \overline{\Lambda_s(\theta)} \cup \operatorname{int}(X \setminus X_{s^*}) \subseteq \overline{\Lambda_s(\theta) \cup \operatorname{int}(X \setminus X_{s^*})},$$

this means that, $\Lambda_s(\theta) \cup \operatorname{int}(X \setminus X_{s^*})$ is dense in X. Thus

$$\Lambda(\theta) = \bigcap_{s \in S} \left(\Lambda_s(\theta) \bigcup \operatorname{int}(X \setminus X_{s^*})\right)$$

is dense in X by the Baire category theorem. $\square$

Lemma 3.2.10. *[3, Lemma 4.13] Let S be a countable inverse semigroup and let X be a locally compact, Hausdorff space X. If $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ is an effective partial action of S on X, then θ is topologically principal.*

Proof. Suppose that θ is not topologically principal. We will show that θ is not topologically free. By Proposition 3.2.9, there is some $s \in S$ such that $\Lambda_s(\theta)$ is not dense in X_{s^*}. Now, pick some $y \in X_{s^*}$ such that $y \notin \Lambda_s(\theta)$ and y is not a limit point of $\Lambda_s(\theta)$. Notice that

$$\{x \in X_{s^*} \mid \theta_s(x) \neq x\} \subseteq \Lambda_s(\theta).$$

Clearly, $y \in \{x \in X_{s^*} \mid \theta_s(x) = x\}$. Moreover, there is an open neighbourhood U of y such that $U \cap \Lambda_s(\theta) = \varnothing$. Thus,

$$U \subseteq \{x \in X_{s^*} \mid \theta_s(x) = x\}.$$

This shows that $y \in \operatorname{int}\{x \in X_{s^*} \mid \theta_s(x) = x\}$ and therefore θ is not effective. $\square$

The next example shows that the conclusion of Lemma 3.2.10 does not hold for an arbitrary inverse semigroup S.

Example 3.2.11. [7, Example 6.4] or [3, Example 4.14] Let K denote the Cantor set and equip $\mathbb{T} = \{e^{i\omega} \mid \omega \in \mathbb{R}\}$ with the discrete topology. Consider the topological product space $X = (K \cap (0,1)) \times \mathbb{T}$. Define an action θ of the additive group $\mathbb{R}$ on X by

$$\theta_t(s, e^{i\omega}) = (s, e^{i(\omega + 2st\alpha)})$$

for $t \in \mathbb{R}$ and $(s, e^{i\omega}) \in X$. For any $t \in \mathbb{R} \setminus \{0\}$, we have that

$$\operatorname{int}\{(s, e^{i\omega}) \in X \mid \theta_t(s, e^{i\omega}) = (s, e^{i\omega})\} = \operatorname{int}\{(s, e^{i\omega}) \in X \mid st \in \mathbb{Z}\} = \varnothing,$$

and therefore θ is effective. However, θ is not topologically principal. Indeed, let $x = (s, e^{i\omega}) \in X$ be arbitrary. Put $t = \frac{1}{s}$ and notice that $\theta_t(x) = \theta_{\frac{1}{s}}(s, e^{i\omega}) = (s, e^{i(\omega+2\alpha)}) = (s, e^{i\omega}) = x$. But 0 is the only idempotent element of the additive group $\mathbb{R}$, and using that $t = \frac{1}{s} \neq 0$, we conclude that $0 \not\leq \frac{1}{s}$. In other words, $(s, e^{i\omega}) \notin \Lambda(\theta)$. This shows that $\Lambda(\theta) = \varnothing$, and in particular θ is not topologically principal.

The next example shows that the converse of Lemma 3.2.10 does not hold. That is, there is a topologically principal partial action θ of a countable inverse semigroup S on a locally compact, Hausdorff topological space X such that θ is not effective. This example also shows the loss of some properties that one has when generalizing partial action of groups to partial action of inverse semigroups (for example, in the case of group the converse of Lemma 3.2.10 holds, see Lemma 2.2.12).

Example 3.2.12. [3, Example 4.15] We shall consider a particular case of the *Munn representation* (see Example 1.4.8). As in Example 1.1.5, let $S = \mathbb{N} \cup \{\infty, z\}$ be the inverse semigroup whose product, for any $m, n \in \mathbb{N}$, is given by

$$nm = \min(n, m), \quad n\infty = \infty n = nz = zn = n, \qquad z\infty = \infty z = z$$

$$\text{and} \quad zz = \infty\infty = \infty.$$

Then $X = E(S) = \mathbb{N} \cup \{\infty\}$ can be seen as the one-point compactification of the natural numbers. Notice that the compact-open sets of X are either cofinite or contained in $\mathbb{N}$. Now, let θ be the Munn representation of S on X. More precisely,

- for $n \in \mathbb{N}$, $X_n = \{1, 2, \dots, n\}$ and $\theta_n = \mathrm{id}_{X_n}$,
- $X_\infty = \mathbb{N} \cup \{\infty\}$ and $\theta_\infty = \mathrm{id}_{X_\infty}$,
- $X_z = \mathbb{N} \cup \{\infty\}$ and $\theta_z = \mathrm{id}_{X_z}$.

Notice that S is countable and that X is a locally compact and Hausdorff space.

Since $n, \infty \in E(S)$, clearly $\Lambda_n(\theta) = X_n$ and $\Lambda_\infty(\theta) = X_\infty$. Moreover, we notice that $\Lambda_z(\theta) = \mathbb{N}$, which is a dense subset of $\mathbb{N} \cup \{\infty\} = X_z$. Hence, by Proposition 3.2.9, θ is topologically principal.

We claim that θ is not an effective partial action. Indeed,

$$\mathrm{int}\{x \in X_{z^*} \mid \theta_z(x) = x\} = \mathrm{int}\,(\mathbb{N} \cup \{\infty\}) = \mathbb{N} \cup \{\infty\},$$

but $\{x \in X_{z^*} \mid \text{ there is } e \in E(S) \text{ such that } e \leq z \text{ and } x \in X_e\} = \mathbb{N}$.

Let α be the partial action of S on $\mathcal{L}_\mathrm{c}(\mathrm{X})$ associated with θ, then

- for $n \in \mathbb{N}$, $D_n \cong \mathcal{L}_c(\{1, 2, \dots, n\})$ and $\alpha_n = \mathrm{id}_{D_n}$.
- $D_\infty \cong \mathcal{L}_c(\mathbb{N} \cup \{\infty\})$ and $\alpha_\infty = \mathrm{id}_{D_\infty}$.
- $D_z = \mathcal{L}_c(\mathbb{N} \cup \{\infty\})$ and $\alpha_z = \mathrm{id}_{D_z}$.

Now we will see that the diagonal of $\mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\alpha S$ is not a maximal commutative subring. Fix $n \in \mathbb{N}$. We denote by $1_{[n,\infty]}$ the characteristic function of the set $\{n, n+1, n+2, \ldots\} \cup \{\infty\}$. We have that $1_{[n,\infty]} \in D_z \cong \mathcal{L}_c(\mathbb{N} \cup \{\infty\})$ and that $\overline{1_{[n,\infty]}\delta_z}$ does not belong to the diagonal $\mathcal{D}$ of $\mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\alpha S$. It is not difficult to see that $\overline{1_{[n,\infty]}\delta_z}$ commutes with all elements of the diagonal.

It is also worth noticing that in this example the ring $\mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\alpha S$ has an infinite number of non-zero ideals whose intersection with the diagonal is zero. Indeed, for each $n \in \mathbb{N}$, consider the ideal $\mathcal{J}$ generated by

$$\overline{1_{[n,\infty]}\delta_z} - \overline{1_{[n,\infty]}\delta_\infty}$$

and observe that

$$\mathcal{J} = \left\{ \sum_{i=1}^{k} \overline{r_i 1_{[n_i,\infty]}\delta_z} - \overline{r_i 1_{[n_i,\infty]}\delta_\infty} \;\middle|\; n_i \geq n \text{ and } r_i \in R \right\}.$$

It is important to note that in this case, $\operatorname{supp}\left(\widetilde{\tau}(\overline{f})\right) = \varnothing$, for every $\overline{f} \in \mathcal{J}$.

Remark 3.2.13. Notice that given $\overline{f} = \sum_{i=1}^{n} \overline{f_i\delta_{s_i}} \in \mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\alpha S$, we have that $\operatorname{supp}\left(\widetilde{\tau}(\overline{f})\right) = \varnothing$ if, and only if, $\sum_{i=1}^{n} f_i = 0$.

Remark 3.2.14. The set

$$\mathcal{J} = \left\{ \sum_{i=1}^{n} \overline{f_i\delta_{s_i}} \in \mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\alpha S \;\middle|\; \sum_{i=1}^{n} f_i = 0 \right\}$$

is a left ideal of $\mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\alpha S$.

Indeed, for every $\overline{f} = \sum_{i=1}^{n} \overline{f_i\delta_{s_i}} \in \mathcal{J}$, and for every $\overline{1_K\delta_t} \in \mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\alpha S$ ($t \in S$ and K compact-open subset of X_t), we have that

$$\overline{1_K\delta_t} \cdot \overline{f} = \sum_{i=1}^{n} \overline{\alpha_s(\alpha_{s^*}(1_U)f_i)\delta_{ts_i}}.$$

Since α_s is a ring homomorphism, we obtain

$$\sum_{i=1}^{n} \alpha_s(\alpha_{s^*}(1_U)f_i) = \alpha_s\left(\alpha_{s^*}(1_U)\sum_{i=1}^{n} f_i\right) = \alpha_s(\alpha_{s^*}(1_U)0) = 0.$$

Since every function of D_t $(t \in S)$ can be written as a linear combination of characteristic function of compact-open subset of X_t, we conclude that $\mathcal{J}$ is a left ideal of $\mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\alpha S$.

However, $\mathcal{J}$ is not necessarily a right ideal. For example, let $X = \{a, b, c\}$ be a set with there points and $S = \mathcal{I}(X)$ (the set of every partial bijections of X) acting naturally on X, that is, $f \circ x = f(x)$, for all $x \in \mathrm{dom}(f)$. Let

$$s = \text{“}a \to b\text{”} \qquad t = \text{“}c \to b\text{”} \qquad s = \text{“}a \to a\text{”}.$$

We have $X_s = X_c = \{b\}$, so $1_{\{b\}} \in D_s \cap D_t$ and $\overline{1_{\{b\}}\delta_s} - \overline{1_{\{b\}}\delta_t} \in \mathcal{J}$, but $1_{\{a\}}\delta_u \in D_u$ and

$$(\overline{1_{\{b\}}\delta_s} - \overline{1_{\{b\}}\delta_t})(\overline{1_{\{a\}}\delta_u}) = \overline{1_{\{b\}}\delta_{su}} \notin \mathcal{J}.$$

Next we present a sufficient condition to obtain the ideal intersection property for the skew inverse semigroup ring arising from a topologically principal partial action.

Proposition 3.2.15. *[3, Proposition 4.16] Let* $\theta = (\{\theta_s\}_{s \in S}, \{X_s\}_{s \in S})$ *be a topologically principal partial action of* S *on a zero-dimensional locally compact Hausdorff space* X*. If* $\mathcal{I}$ *is a non-zero ideal of* $\mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\alpha S$*, and there is some* $\overline{f} \in \mathcal{I}$ *such that* $\mathrm{supp}\left(\widetilde{\tau}\left(\overline{f}\right)\right) \neq \varnothing$*, then* $\mathcal{I} \cap \mathcal{D} \neq \{0\}$*.*

Proof. Let $\mathcal{I}$ be a non-zero ideal of $\mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\alpha S$ and let

$$\overline{f} = \sum_{s \in F} \overline{f_s\delta_s} \in \mathcal{I}$$

be such that $\mathrm{supp}(\widetilde{\tau}(\overline{f})) \neq \varnothing$. Since $\mathrm{supp}(\widetilde{\tau}(\overline{f}))$ is an open subset of X, and θ is topologically principal, there is some $x \in \mathrm{supp}\left(\widetilde{\tau}\left(\overline{f}\right)\right) \cap \Lambda(\theta)$. We fix this x throughout the rest of the proof.

Notice that the subset

$$\{\theta_{s^*}(x) \mid s \in F \text{ and } f_s(x) \neq 0\}$$

is non-empty and finite.

We choose $s_1 \in F$ such that $f_{s_1}(x) \neq 0$ and

$$r = \sum_{s \in T} f_s(x) \neq 0,$$

where $T = \{s \in F \mid f_s(x) \neq 0 \text{ and } \theta_{s^*}(x) = \theta_{s_1^*}(x)\}$.

Let $y = \theta_{s_1^*}(x)$. Furthermore, let B be a compact-open neighbourhood of x contained in $X_{s_1} \cap X_e$, for some $e \in E(S)$ such that

$$\{\theta_{s^*}(x) \mid s \in F \text{ and } f_s(x) \neq 0\} \cap B = \{y\}.$$

We have that $\overline{g} = \overline{f} \cdot \overline{1_B \delta_e} \in \mathcal{I}$ and that

$$\overline{g} = \sum_{s \in F} \overline{\alpha_s(\alpha_{s^*}(f_s)1_B)\delta_{es}} = \sum_{s \in F} \overline{\alpha_s(\alpha_{s^*}(f_s)1_B)\delta_s}.$$

Put $g_s = \alpha_s(\alpha_{s^*}(f_s)1_B)$, and notice that $\operatorname{supp}(g_s) \subseteq \operatorname{supp}(f_s) \cap \theta_s(B)$. Then

$$\{\theta_{s^*}(x) \mid s \in F \text{ and } g_s(x) \neq 0\} = \{y\}.$$

Furthermore,

$$\{s \in F \mid g_s(x) \neq 0\} = \{s \in F \mid f_s(x) \neq 0 \text{ and } \theta_{s^*}(x) = \theta_{s_1^*}(x)\} = T.$$

Using that $x \in \Lambda(\theta)$, by Lemma 3.2.8, there is some $u \in S$ such that $x \in X_{u^*}$ and $u \leq s^*$, for all $s \in T$. Let $C \subseteq X_{u^*}$ be a compact-open neighbourhood of x. We may now rewrite $\overline{g}$ as

$$\begin{aligned}
\overline{g} &= \sum_{s \in F: g_s(x) = 0} \overline{g_s \delta_s} + \sum_{s \in F: g_s(x) \neq 0} \overline{(g_s 1_C - g_s 1_{X \setminus C})\delta_s} \\
&= \sum_{s \in F: g_s(x) = 0} \overline{g_s \delta_s} + \sum_{s \in F: g_s(x) \neq 0} \overline{g_s 1_C \delta_s} - \sum_{s \in F: g_s(x) \neq 0} \overline{g_s 1_{X \setminus C} \delta_s} \\
&= \sum_{s \in F: g_s(x) = 0} \overline{g_s \delta_s} + \sum_{s \in F: g_s(x) \neq 0} \overline{g_s 1_C \delta_{u^*}} - \sum_{s \in F: g_s(x) \neq 0} \overline{g_s 1_{X \setminus C} \delta_s}.
\end{aligned}$$

Since each g_s is locally constant, we can find another compact-open neighbourhood K of x contained in C such that $g_s|_K$ is constant, for all $s \in F$, and $K \subseteq X_v$, for some $v \in E(S)$.

Thus, $\overline{1_K \delta_v} \cdot \overline{g} \cdot \overline{\alpha_u(1_K)\delta_u} \in \mathcal{I}$ and

$$\begin{aligned}
\overline{1_K\delta_v}\cdot\overline{g}\cdot\overline{\alpha_u(1_K)\delta_u} &= \sum_{s\in F:g_s(x)\neq 0}\overline{1_K g_s 1_C\delta_{vu^*}}\cdot\overline{\alpha_u(1_K)\delta_u}\\
&= \overline{\left(\sum_{s\in F:g_s(x)\neq 0} g_s(x)\right)1_K\delta_{vu^*}}\cdot\overline{\alpha_u(1_K)\delta_u}\\
&= \overline{r1_K\delta_{u^*}}\cdot\overline{\alpha_u(1_K)\delta_u}\\
&= \overline{\alpha_{u^*}(\alpha_u(r1_K)\alpha_u(1_K))\delta_{u^*u}}\\
&= \overline{r1_K\delta_{u^*u}}\in\mathcal{I}\cap\mathcal{D},
\end{aligned}$$

where $r\neq 0$. □

Corollary 3.2.16. *[3, Corollary 4.17] Let $\theta=(\{\theta_s\}_{s\in S},\{X_s\}_{s\in S})$ be a topologically principal partial action of S on a zero-dimensional locally compact Hausdorff space X. If for each non-zero ideal $\mathcal{I}$ of $\mathcal{L}_{\mathrm{c}}(\mathrm{X})\rtimes_\alpha S$ there is some $\overline{f}\in\mathcal{I}$ such that $\operatorname{supp}(\widetilde{\tau}(\overline{f}))\neq\varnothing$, then the diagonal $\mathcal{D}$ is a maximal commutative subring of $\mathcal{L}_{\mathrm{c}}(\mathrm{X})\rtimes_\alpha S$.*

Proof. This follows from Proposition 3.2.15 and Theorem 3.1.2. □

Finally, we show that maximal commutativity of $\mathcal{D}$ in $\mathcal{L}_{\mathrm{c}}(\mathrm{X})\rtimes_\alpha S$ implies that the underlying action is topologically principal (we also show the condition involving ideals and support of elements in $\mathcal{L}_{\mathrm{c}}(\mathrm{X})\rtimes_\alpha S$).

Proposition 3.2.17. *[3, Proposition 4.18] Suppose that S is countable, X is a locally compact, Hausdorff, and zero-dimensional space, and $\theta=(\{\theta_s\}_{s\in S},\{X_s\}_{s\in S})$ is a topological partial action of S on X. If the diagonal $\mathcal{D}$ is a maximal commutative subring of $\mathcal{L}_{\mathrm{c}}(\mathrm{X})\rtimes_\alpha S$, then θ is topologically principal, and for each non-zero ideal $\mathcal{I}$ of $\mathcal{L}_{\mathrm{c}}(\mathrm{X})\rtimes_\alpha S$ there is some $\overline{f}\in\mathcal{I}$ such that $\operatorname{supp}(\widetilde{\tau}(\overline{f}))\neq\varnothing$.*

Proof. We show the contrapositive statement. Suppose that θ is not topologically principal. Then there is $s\in S$ and there is a compact-open subset B of X_{s^*} such that $B\cap\Lambda_s(\theta)=\varnothing$. This means that, for each $x\in B$, $\theta_s(x)=x$ and there is no $e\in E(S)$ such that $e\leq s$

and $x \in X_e$ (equivalently, there is no $e \in E(S)$ such that $e \leq s^*$ and $x \in X_e$). Hence, $\overline{1_B\delta_{s^*}} \notin \mathcal{D}$.

Let $e \in E(S)$ be arbitrary and take an arbitrary compact-open subset D of X_e. If $x \in B$, then

$$\alpha_{s^*}(\alpha_s(1_B)1_D)(x) = 1_B(x)1_D(\theta_s(x)) = 1_B(x)1_D(x) = (1_B 1_D)(x).$$

And if $x \in X \setminus B$, then

$$\alpha_{s^*}(\alpha_s(1_B)1_D)(x) = 1_B(x)1_D(\theta_s(x)) = 0 = 1_B(x)1_D(x) = (1_B 1_D)(x).$$

Hence, $\alpha_{s^*}(\alpha_s(1_B)1_D) = 1_B 1_D$. Thus,

$$\overline{1_D\delta_e}\cdot\overline{1_B\delta_{s^*}} = \overline{1_D 1_B\delta_{s^*}} = \overline{1_B 1_D\delta_{s^*}} = \overline{\alpha_{s^*}(\alpha_s(1_B)1_D)\delta_{s^*}} = \overline{1_B\delta_{s^*}}\cdot\overline{1_D\delta_e}.$$

This implies that $\overline{1_B\delta_{s^*}}$ commutes with all elements of the diagonal $\mathcal{D}$, and hence $\mathcal{D}$ is not maximal commutative.

By Theorem 3.1.2, for every non-zero ideal $\mathcal{I}$ of $\mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\alpha S$ we have that $\mathcal{I} \cap \mathcal{D} \neq \{0\}$. Let $\overline{f} = \sum_{i=1}^n f_i\delta_{e_i} \in \mathcal{I} \cap \mathcal{D}$ be non-zero. By the isomorphism of the diagonal $\mathcal{D}$ with $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$ we have that $\overline{f} = \overline{0}$ if, and only if, $\sum_{i=1}^n f_{e_i} = 0$, and thus $\operatorname{supp}\left(\widetilde{\tau}\left(\overline{f}\right)\right) = \operatorname{supp}\left(\sum_{i=1}^n f_{e_i}\right) \neq \varnothing$. □

Theorem 3.2.18. *[3, Corollary 4.19] Let S be a countable inverse semigroup, let X be a locally compact, Hausdorff, and zero-dimensional space, and let R be a field. Then the skew inverse semigroup ring $\mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\alpha S$ is simple if, and only if, the following three conditions are satisfied*

- *θ is minimal,*
- *θ is topologically principal, and*
- *for every non-zero ideal $\mathcal{I}$ of $\mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\alpha S$ there is some $\overline{f} \in \mathcal{I}$ such that $\operatorname{supp}(\overline{f}) \neq \varnothing$.*

Proof. This follows from Proposition 3.2.3, Corollary 3.2.16, Lemma 3.2.17 and Theorem 3.1.5. Notice that for the "if" statement we do not need to use the fact that S is countable. □

Corollary 3.2.19. *[3, Corollary 4.20] Let S, X and R be as in Theorem 3.2.18. Suppose that $\theta = (\{\theta_s\}_{s\in S}, \{X_s\}_{s\in S})$ is a partial action such that the following three assertions hold:*

- *θ is effective*
- *θ is minimal, and*
- *for every non-zero ideal $\mathcal{I}$ of $\mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\alpha S$ there is $\overline{f} \in \mathcal{I}$ such that* $\mathrm{supp}(\overline{f}) \neq \varnothing$.

Then the skew inverse semigroup ring $\mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\alpha S$ is simple.

Proof. This follows from Lemma 3.2.10 and Corollary 3.2.18. □

3.3 An application to Steinberg algebras

Using that there is a description of Steinberg algebras via skew inverse semigroup rings (see [5, Theorem 2.3.1]), which satisfy the assumptions of the Section 3.1, we can apply the main result in this chapter to characterize simplicity of Steinberg algebras. We then obtain a new proof of the following result, which was first proved in [7] for functions over the complex numbers.

Theorem 3.3.1. *[13, Corollary 4.6.] Let $\mathcal{G}$ be an ample Hausdorff groupoid, and let R be a unital and commutative ring. Then the Steinberg algebra $A_R(\mathcal{G})$ is simple if, and only if, $\mathcal{G}$ is effective, minimal, and R is a field.*

Remark 3.3.2. The isomorphism of Theorem 2.3.1, between the skew inverse semigroup algebra $\mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes_\alpha \mathcal{G}^a$ and Steinberg algebra $A_R(\mathcal{G})$, is given by the map $\tilde{\psi} : \mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes_\alpha \mathcal{G}^a \to A_R(\mathcal{G})$, which is defined on the elements of the form $\overline{f_B \delta_B}$, by

$$\psi(f_B\delta_B)(x) = \begin{cases} f_B(\mathfrak{r}(x)) & \text{if } x \in B \\ 0 & \text{if } x \notin B, \end{cases}$$

and extended linearly to $\mathcal{L}(\alpha)$.

In the proof of Theorem 2.3.1 it was shown that $\tilde{\psi}$ admits a left inverse, namely the map $\varphi : A_R(\mathcal{G}) \to \mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes_\alpha \mathcal{G}^a$ defined as follows: Given $f = \sum_{j=1}^n b_j 1_{B_j} \in A_R(\mathcal{G})$, where the B_j's are pairwise disjoint compact bisections of $\mathcal{G}$, let

$$\varphi(f) = \varphi\left(\sum_{i=1}^n b_j 1_{B_j}\right) := \sum_{j=1}^n \overline{b_j 1_{\mathfrak{r}(B_j)} \delta_{B_j}}.$$

Actually φ is the inverse of $\tilde{\psi}$, and, in particular, it is bijective. By the surjectivity of φ, given any $f \in \mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes_\alpha \mathcal{G}^a$ we can write

$$f = \sum_{j=1}^n \overline{b_j 1_{\mathfrak{r}(B_j)} \delta_{B_j}},$$

where the B_j's are pairwise disjoint compact bisections of $\mathcal{G}$. Furthermore, by the injectivity of φ, if

$$\sum_{j=1}^n \overline{b_j 1_{\mathfrak{r}(B_j)} \delta_{B_j}} = \sum_{j=1}^n \overline{c_j 1_{\mathfrak{r}(C_j)} \delta_{C_j}},$$

where the B_j's and C_j's are pairwise disjoint compact bisections, then

$$\sum_{i=1}^n b_i 1_{B_i} = \sum_{i=1}^n c_i 1_{C_i}.$$

Our first step towards a proof of Theorem 3.3.1 is to characterize minimality of $\mathcal{G}$ in terms of $\mathcal{G}^a$-simplicity of $\mathcal{L}_c(\mathcal{G}^{(0)})$.

Proposition 3.3.3. *[3, Propostion 5.4] Let $\mathcal{G}$ be an ample Hausdorff groupoid, and let R be a field. Then $\mathcal{G}$ is minimal if, and only if, $\mathcal{L}_c(\mathcal{G}^{(0)})$ is $\mathcal{G}^a$-simple.*

Proof. Suppose that $\mathcal{G}$ is minimal. Let J be a $\mathcal{G}^a$-invariant non-zero ideal of $\mathcal{L}_c(\mathcal{G}^{(0)})$. By Remark 3.2.2, we know that

$$J = \{f \in \mathcal{L}_c(\mathcal{G}^{(0)}) \mid \operatorname{supp}(f) \subseteq U\},$$

where U is an open subset of $\mathcal{G}^{(0)}$ given by

$$U = \{u \in \mathcal{G}^{(0)} \mid \exists\ f \in J \text{ such that } f(u) \neq 0\}.$$

Notice that, since $\mathcal{G}$ is minimal, if we prove that U is an invariant subset of $\mathcal{G}^{(0)}$, then $U = \mathcal{G}^{(0)}$ and hence $J = \mathcal{L}_c(\mathcal{G}^{(0)})$. We prove the invariance of U below.

Let $x \in \mathcal{G}$ be such that $\mathfrak{s}(x) \in U$. Then there exists a function $g \in J$ such that $g(\mathfrak{s}(x)) \neq 0$. Furthermore, we can take $x \in B$, where B is a compact bisection of $\mathcal{G}$. Since U and $\mathfrak{s}(B)$ are open, we can consider

$$g \in J \cap I(\mathfrak{s}(B)) = \{f \in \mathcal{L}_c(\mathcal{G}^{(0)}) \mid \mathrm{supp}(f) \subseteq U \cap \mathfrak{s}(B)\}.$$

Using that J is $\mathcal{G}^a$-invariant we get that $\alpha_B(g) \in J$. Notice that

$$\alpha_B(g)(\mathfrak{r}(x)) = g(\theta_{B^*}(\mathfrak{r}(x))) = g(\mathfrak{s}(\mathfrak{r}_B^{-1}(\mathfrak{r}(x)))) = g(\mathfrak{s}(x)) \neq 0.$$

Therefore, $\mathfrak{r}(x) \in U$ and hence U is $\mathcal{G}^a$-invariant, as desired.

Now, suppose that $\mathcal{L}_c(\mathcal{G}^{(0)})$ is $\mathcal{G}^a$-simple. Let $U \subseteq \mathcal{G}^{(0)}$ be a non-empty invariant open subset. Consider the set

$$J = \{f \in \mathcal{L}_c(\mathcal{G}^{(0)}) \mid \mathrm{supp}(f) \subseteq U\}.$$

Clearly, J is an ideal of $\mathcal{L}_c(\mathcal{G}^{(0)})$. To see that J is $\mathcal{G}^a$-invariant, suppose that $B \in \mathcal{G}^a$, $g \in J \cap D_{B^*}$, and $x \in \mathcal{G}^{(0)} \setminus U$. If $x \in \mathfrak{r}(B)$, then there exists some $y \in B$ such that $x = \mathfrak{r}(y)$, and hence

$$\alpha_B(g)(x) = \alpha_B(g)(\mathfrak{r}(y)) = g(\mathfrak{s}(\mathfrak{r}_B^{-1}(\mathfrak{r}(y)))) = g(\mathfrak{s}(y)).$$

Since U is invariant, and $\mathfrak{r}(y) = x \notin U$, we have that $\mathfrak{s}(y) \notin U$. Hence, $g(\mathfrak{s}(y)) = 0$. If $x \notin \mathfrak{r}(B)$, then from the definition of α_B, we also have that $\alpha_B(g)(x) = 0$. Therefore, $\alpha_B(g) \in J$, and hence J is $\mathcal{G}^a$-invariant. Using that $\mathcal{L}_c(\mathcal{G}^{(0)})$ is $\mathcal{G}^a$-simple it follows that $J = \mathcal{L}_c(\mathcal{G}^{(0)})$ and $U = \mathcal{G}^{(0)}$.

Notice that, for the "f" statement, we do not need to use the fact that R is a field. $\square$

Proposition 3.3.4. *[3, Proposition 5.5] Let $\mathcal{G}$ be an ample Hausdorff groupoid, and let R be a commutative ring with unit. Then $\mathcal{G}$ is effective if, and only if, the diagonal*

$$\mathcal{D} = \left\{ \sum_{i=1}^{n} \overline{f_i \delta_{U_i}} \mid n \in \mathbb{N}, U_i \in E(\mathcal{G}^a) \text{ and } f_i \in I(\mathfrak{r}(U_i)) \right\} \cong \mathcal{L}_c(\mathcal{G}^{(0)})$$

is a maximal commutative subring of $\mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes_\alpha \mathcal{G}^a$.

Proof. Suppose that $\mathcal{G}$ is effective. We already know that $\mathcal{D}$ is a commutative subring.

Let $0 \neq f = \sum_{i=1}^n \overline{r_i 1_{\mathfrak{r}(B_i)} \delta_{B_i}} \in \mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes_\alpha \mathcal{G}^a$, where $r_i \in R \setminus \{0\}$ and the B_i's are pairwise disjoint compact bisections of $\mathcal{G}$ for all $i \in \{1, \ldots, n\}$. Suppose that f is an element which commutes with all elements of $\mathcal{D}$. We need to show that $f \in \mathcal{D}$.

By the effectiveness of $\mathcal{G}$ it suffices to show that $B_i \subseteq \mathrm{Iso}(\mathcal{G})$ for every $i \in \{1, \ldots, n\}$ (since B_i is open and $B_i \subseteq \mathrm{int}(\mathrm{Iso}(\mathcal{G})) = \mathcal{G}^{(0)}$). To this end, suppose that there exists some $k \in \{1, \ldots, n\}$, and $b \in B_k$, such that $\mathfrak{r}(b) \neq \mathfrak{s}(b)$. Since $\mathcal{G}$ is Hausdorff, there exists a compact bisection $U \subseteq \mathcal{G}^{(0)}$ such that $\mathfrak{r}(b) \in U$ and $\mathfrak{s}(b) \notin U$. Notice that $U \in E(\mathcal{G}^a)$.

Using the fact that f belongs to the centralizer of $\mathcal{D}$ we have that

$$\overline{1_U \delta_U} \cdot f = f \cdot \overline{1_U \delta_U}.$$

This implies that

$$\sum_{i=1}^n \overline{r_i 1_U 1_{\mathfrak{r}(B_i)} \delta_{UB_i}} = \sum_{i=1}^n \overline{r_i \alpha_{B_i}(\alpha_{B_i^*}(1_{\mathfrak{r}(B_i)}) 1_U) \delta_{B_i U}}.$$

Since $UB_i, B_iU \subseteq B_i$, for all $i \in \{1, \ldots, n\}$, we get that

$$\sum_{i=1}^n \overline{r_i 1_U 1_{\mathfrak{r}(B_i)} \delta_{B_i}} = \sum_{i=1}^n \overline{r_i \alpha_{B_i}(\alpha_{B_i^*}(1_{\mathfrak{r}(B_i)}) 1_U) \delta_{B_i}}. \tag{3.8}$$

Developing the left side of (3.8) we obtain

$$\sum_{i=1}^n \overline{r_i 1_U 1_{\mathfrak{r}(B_i)} \delta_{B_i}} = \sum_{i=1}^n \overline{r_i 1_{U \cap \mathfrak{r}(B_i)} \delta_{B_i}}.$$

For each $i \in \{1, \ldots, n\}$, define $C_i := \mathfrak{r}_{B_i}^{-1}(U \cap \mathfrak{r}(B_i))$. Notice that $C_i \subseteq B_i$ and $\mathfrak{r}(C_i) = U \cap \mathfrak{r}(B_i)$. Thus

$$\sum_{i=1}^n \overline{r_i 1_U 1_{\mathfrak{r}(B_i)} \delta_{B_i}} = \sum_{i=1}^n \overline{r_i 1_{U \cap \mathfrak{r}(B_i)} \delta_{B_i}} = \sum_{i=1}^n \overline{r_i 1_{\mathfrak{r}(C_i)} \delta_{C_i}}. \tag{3.9}$$

Now, developing the right side of (3.8) we get that

$$\begin{aligned}\sum_{i=1}^{n}\overline{r_i\alpha_{B_i}(\alpha_{B_i^*}(1_{\mathfrak{r}(B_i)})1_U)\delta_{B_i}} &= \sum_{i=1}^{n}\overline{r_i\alpha_{B_i}(1_{\mathfrak{s}(B_i)}1_U)\delta_{B_i}}\\ &= \sum_{i=1}^{n}\overline{r_i\alpha_{B_i}(1_{\mathfrak{s}(B_i)\cap U})\delta_{B_i}}\\ &= \sum_{i=1}^{n}\overline{r_i 1_{\mathfrak{r}(B_i)\cap\theta_{B_i}(\mathfrak{s}(B_i)\cap U)}\delta_{B_i}}.\end{aligned}$$

Define $D_i := \mathfrak{r}_{B_i}^{-1}(\mathfrak{r}(B_i)\cap\theta_{B_i}(\mathfrak{s}(B_i)\cap U))$. Notice that $D_i \subseteq B_i$ and $\mathfrak{r}(D_i) = \mathfrak{r}(B_i)\cap\theta_{B_i}(\mathfrak{s}(B_i)\cap U)$. Then

$$\sum_{i=1}^{n}\overline{r_i\alpha_{B_i}(\alpha_{B_i^*}(1_{\mathfrak{r}(B_i)})1_U)\delta_{B_i}} = \sum_{i=1}^{n}\overline{r_i 1_{\mathfrak{r}(D_i)}\delta_{D_i}}. \tag{3.10}$$

By substituting (3.9) and (3.10) into Equation (3.8) we obtain that

$$\sum_{i=1}^{n}\overline{r_i 1_{\mathfrak{r}(C_i)}\delta_{C_i}} = \sum_{i=1}^{n}\overline{r_i 1_{\mathfrak{r}(D_i)}\delta_{D_i}}.$$

Since $C_i \subseteq B_i$, for each $i \in \{1,\ldots,n\}$, the C_i's are pairwise disjoint compact bisections, and similarly the D_i's are also pairwise disjoint compact bisections. By Remark 3.3.2 we have that

$$\sum_{i=1}^{n} b_i 1_{C_i} = \sum_{i=1}^{n} b_i 1_{D_i}.$$

Next we evaluate the above equality on the element b of B_k such that $\mathfrak{r}(b) \neq \mathfrak{s}(b)$. Since the B_i's are pairwise disjoint we have that $b \notin C_i, b \notin D_i$ for $i \neq k$ and hence

$$b_k 1_{C_k}(b) = b_k 1_{D_k}(b). \tag{3.11}$$

Notice that

$$b \in C_k = \mathfrak{r}_{B_k}^{-1}(U\cap\mathfrak{r}(B_k)) \Longleftrightarrow \mathfrak{r}(b) \in U\cap\mathfrak{r}(B_k),$$

and

$$\begin{aligned}
b \in D_k &\iff b \in \mathfrak{r}_{B_k}^{-1}(\mathfrak{r}(B_k) \cap \theta_{B_k}(\mathfrak{s}(B_k) \cap U)) \\
&\iff \mathfrak{r}(b) \in \mathfrak{r}(B_k) \cap \theta_{B_k}(\mathfrak{s}(B_k) \cap U) \\
&\iff \mathfrak{r}(b) \in \mathfrak{r}(B_k) \text{ and } \mathfrak{r}(b) \in \mathfrak{r}(s_{B_k}^{-1}(\mathfrak{s}(B_k) \cap U)) \\
&\overset{b \in B_k}{\iff} \mathfrak{r}(b) \in \mathfrak{r}(B_k) \text{ and } b \in \mathfrak{s}_{B_k}^{-1}(\mathfrak{s}(B_k) \cap U) \\
&\iff \mathfrak{r}(b) \in \mathfrak{r}(B_k) \text{ and } \mathfrak{s}(b) \in \mathfrak{s}(B_k) \cap U \\
&\overset{b \in B_k}{\iff} \mathfrak{r}(b) \in \mathfrak{r}(B_k) \text{ and } \mathfrak{s}(b) \in U.
\end{aligned}$$

Recall that, by construction, $b \in C_k$ and $\mathfrak{s}(b) \notin U$. Thus, Equation (3.11) yields $b_k = 0$, a contradiction. Therefore, $\mathfrak{r}(b) = \mathfrak{s}(b)$, $b \in \mathrm{Iso}(G)$ and $B_i \in \mathrm{Iso}(G)$ as desired.

In order to prove the converse we show the contrapositive statement. Suppose that $\mathcal{G}$ is not effective. Then there exists a bisection $B \subseteq \mathcal{G} \setminus \mathcal{G}^{(0)}$ such that $\mathfrak{s}(b) = \mathfrak{r}(b)$ for all $b \in B$.

Recall that $\theta_B : \mathfrak{s}(B) \to \mathfrak{r}(B)$ is defined by $\mathfrak{r}(u) = \mathfrak{r}(\mathfrak{s}_B^{-1}(u))$. Thus, in this case, $\theta_B(\mathfrak{s}(b)) = \mathfrak{r}(\mathfrak{s}_B^{-1}(\mathfrak{s}(b))) = \mathfrak{r}(b) = \mathfrak{s}(b)$, that is, $\theta_B = \mathrm{id}_{\mathfrak{s}(B)}$. Similarly, $\theta_{B^*} = \mathrm{id}_{\mathfrak{r}(B)}$. This implies that $\alpha_B = \mathrm{id}_{D_{B^*}}$ and $\alpha_{B^*} = \mathrm{id}_{D_B}$.

Notice that $\overline{1_{\mathfrak{r}(B)}\delta_B} \notin \mathcal{D}$. Take any $\overline{f\delta_U} \in \mathcal{D}$. Then

$$\begin{aligned}
\overline{f\delta_U} \cdot \overline{1_{\mathfrak{r}(B)}\delta_B} &= \overline{f1_{\mathfrak{r}(B)}\delta_{UB}} \overset{UB \subseteq B}{=} \overline{f1_{\mathfrak{r}(B)}\delta_B} \\
&= \overline{1_{\mathfrak{r}(B)}f\delta_B} \overset{\mathfrak{r}(B)=\mathfrak{r}(UB)}{=} \overline{1_{\mathfrak{r}(B)}f\delta_{BU}} \\
&= \overline{\alpha_B(\alpha_{B^*}(1_{\mathfrak{r}(B)}f))\delta_{BU}} = \overline{1_{\mathfrak{r}(B)}\delta_B} \cdot \overline{f\delta_U},
\end{aligned}$$

that is, $\overline{1_{\mathfrak{r}(B)}\delta_B}$ commutes with all of $\mathcal{D}$. This shows that $\mathcal{D}$ is not maximal commutative. $\square$

Remark 3.3.5. Since $\mathcal{D}$ is isomorphic to $\mathcal{L}_c(\mathcal{G}^{(0)}) \cong A_R(\mathcal{G}^{(0)})$, it follows from Proposition 3.3.4 and Theorem 3.1.2 that $\mathcal{G}$ is effective if, and only if, $A_R(\mathcal{G}_0)$ is maximal commutative if, and only if, every non-zero ideal I of $A_R(\mathcal{G})$ has non-zero intersection with $A_R(\mathcal{G}^{(0)})$. The characterization of effectiveness in terms of the ideal intersection property was first given in [14], and the equivalence between effectiveness of $\mathcal{G}$ and maximal commutativeness of $A_R(\mathcal{G}_0)$ was first proven in [75].

In order to apply Theorem 3.1.5 we need to verify that the assumption about the local units is satisfied. In fact, for any finite subset $\{f_1, \cdots, f_n\}$ of $\mathcal{L}_c(\mathcal{G}^{(0)})$ consider $U = \bigcup_{i=1}^n \mathrm{supp}(f_i)$. Clearly, $1_U \in \mathcal{L}_c(\mathcal{G}^{(0)})$ and this element is a local unit for $\{f_1, \cdots, f_n\}$. Moreover, $1_{\mathfrak{r}(B)}$ and $1_{\mathfrak{s}(B)}$ are multiplicative identity elements in D_B and D_{B^*}, respectively.

Proof of Theorem 3.3.1. Let $\mathcal{G}$ be a Hausdorff and ample groupoid, and let R be a unital and commutative ring. We will use Theorem 2.3.1 to identify the Steinberg algebra $A_R(\mathcal{G})$ with a certain skew inverse semigroup ring $\mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes \mathcal{G}^a$.

Suppose that R is a field, and that $\mathcal{G}$ is minimal and effective. By Proposition 3.3.3 and Proposition 3.3.4, respectively, we get that $\mathcal{L}_c(\mathcal{G}^{(0)})$ is $\mathcal{G}^a$-simple and a maximal commutative subring of $\mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes \mathcal{G}^a$ (by identifying $\mathcal{L}_c(\mathcal{G}^{(0)})$ with the diagonal $\mathcal{D}$). Therefore, by Theorem 3.1.5, we conclude that $\mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes \mathcal{G}^a$, and hence also $A_R(\mathcal{G})$, is simple.

Conversely, suppose that the Steinberg algebra $A_R(\mathcal{G})$ is simple. We claim that R is a field. Seeking a contradiction, suppose that I is a nontrivial ideal of R. Then $IA_R(\mathcal{G})$ is a nontrivial ideal of $A_R(\mathcal{G})$ which is a contradiction. By Theorem 3.1.5, $\mathcal{L}_c(\mathcal{G}^{(0)})$ is $\mathcal{G}^a$-simple and a maximal commutative subring of $\mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes \mathcal{G}^a$. It follows from Proposition 3.3.3 and Proposition 3.3.4, respectively, that $\mathcal{G}$ is minimal and effective. $\square$

4 THE DYNAMICS OF PARTIAL INVERSE SEMIGROUP ACTIONS

The preparation of this chapter is mostly based on joint work with Luiz Cordeiro (see [2]), which was produced during my 6-month visit to the University of Ottawa under the supervision of Professor Thierry Giordano.

In this chapter we will construct the groupoid of germs associated to a partial inverse semigroup action on a topological space in a similar manner to that of [28]. However we also need to take the necessary care as in the construction of the transformation groupoid of a partial group action. With our construction we obtain a common ground for the study of both partial group actions and inverse semigroup actions.

The Hausdorff assumption we make on the groupoid of germs has been necessary throughout the recent papers in this direction [12, 70, 18], and it is always satisfied by semigroups which are weak semilattices, as long as we restrict ourselves to ample actions. Even more strongly, all transformation groupoids (or simply groupoids of germs) of partial group actions on Hausdorff spaces are always Hausdorff, and the same will also be true for all E-unitary inverse semigroups. These are the inverse subsemigroups of semidirect products of lattices by groups ([48, Theorem 7.1.5]).

The first problem we tackle is to prove, in this general setting, that the Steinberg algebra of the associated groupoid of germs, as long as this groupoid is Hausdorff, is always isomorphic to the partial skew inverse semigroup algebra of a partial inverse semigroup action (Thereom 4.3.4). This result generalize both Theorems 2.1.1 and 2.3.1 presented in Chapter 2, which are also about isomorphisms of Steinberg algebras and skew algebras.

The theory of disjoint continuous function of [18] (which in the topological context works with the reconstruction of a topological space from subset classes, or function classes) can be applied in the context of inverse semigroup partial actions, and with this we will obtain a description of partial skew inverse semigroups of commutative algebras as Steinberg algebras (in general, only the opposite direction has been considered).

Orbit equivalence and full groups for actions of $\mathbb{Z}$ were initially studied by Giordano, Putnam and Skaw [35, 36], and by Li in [52, 53] for partial actions of discrete groups. The notion of continuous orbit equivalence can be immediately extended to partial inverse semigroup actions. We prove that two ample, topologically principal partial inverse semigroup actions are continuously orbit equivalent if, and only if, their corresponding groupoids of germs are isomorphic.

We finish this chapter with an application of our results, by realizing Leavitt path algebras $L_R(E)$ of directed graph E as skew inverse semigroup algebras. This description is similar of that of [40] and [42], where $L_R(E)$ was described as a partial skew group ring and partial skew groupoid ring, respectively. We can then compare our notion of continuous orbit equivalence with the one for graphs given in [9], and reobtain results regarding equivalence of the graphs satisfying the condition (L), isomorphism of the Leavitt path algebras and related notions.

4.1 Groupoids of germs

Groupoids of germs were already considered by Paterson in ([60]) for localizations of inverse semigroups, and for natural actions of pseudogroups by Renault (see [66]). In [28], Exel defined groupoids of germs for arbitrary actions of inverse semigroups on topological spaces in a similar, albeit more general, manner than both previous definitions of groupoids of germs.

The objective in this section is to construct a groupoid of germs associated to any topological partial action of an inverse semigroup in a way that generalizes both groupoids of germs of inverse semigroup actions, and transformation groupoids of partial group actions.

Let $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ be a partial action of an inverse semigroup S on a topological space X. We denote by $S * X$ the subset of $S \times X$ given by

$$S * X = \{(s,x) \in S \times X \mid x \in X_{s^*}\},$$

and we define the following equivalence relation $\sim$ on $S * X$: for every $(s,x), (t,y) \in S * X$,

$$(s,x) \sim (t,y) \tag{4.1}$$

if $x = y$, and there is $u \in S$ such that $u \leq s, t$ and $x \in X_{u^*}$. We say that the equivalence class of (s,x) is the *germ of s at x*, and we denote it by $[s,x]$.

Remark 4.1.1. If $u, s \in S$ such that $u \leq s$ and $x \in X_{u^*}$, then $x \in X_{s^*}$ and $[s,x] = [u,x]$.

Remark 4.1.2. Notice that if $(s,x), (t,y) \in S * X$ then $(s,x) \sim (t,y)$ if, and only if, $x = y$, and there is $e \in E(S)$ such that $x \in X_e$ and $se = te$.

Lemma 4.1.3. *Let $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ be a partial action of an inverse semigroup S on a topological space X. Suppose $(s,x) \in S * X$. Then*

(a) $(s^*, \theta_s(x)) \in S * X$*,*

(b) if $(t,y) \in S * X$ *and* $\theta_t(y) = x$*, then* $(st, y) \in S * X$*.*

Proof. (a) Since $\theta_s(x) \in X_s = X_{(s^*)^*}$ then $(s^*, \theta_s(x)) \in S * X$.

(b) By assumption, $y = \theta_{t^*}(x) \in \theta_{t^*}(X_t \cap X_{s^*}) \subseteq X_{(st)^*}$. $\square$

Lemma 4.1.4. *Let $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ be a partial action of an inverse semigroup S on a topological space X. Suppose (s_1, x), (s_2, x), (t_1, y), $(t_2, y) \in S * X$ with $[s_1, x] = [s_2, x]$, $[t_1, y] = [t_2, y]$, and $\theta_{t_1}(y) = x$. Then*

(a) $\theta_{s_1}(x) = \theta_{s_2}(x)$ *and* $\theta_{t_2}(y) = \theta_{t_1}(y) = x$*;*

(b) $[s_1t_1, y] = [s_2t_2, y]$,

(c) $[s_1^*, \theta_{s_1}(x)] = [s_2^*, \theta_{s_2}(x)]$.

Proof. Take $u, v \in S$, where $u \leq s_1, s_2$, $v \leq t_1, t_2$, $x \in X_{u^*}$ and $y \in X_{v^*}$.

(a) Since $u \leq s_1, s_2$ and $x \in X_{u^*}$ then by Remark 4.1.1,

$$\theta_{s_1}(x) = \theta_u(x) = \theta_{s_2}(x),$$

and similarly $\theta_{t_2}(y) = \theta_v(y) = \theta_{t_1}(y) = x$.

(b) We have $uv \leq s_1t_1, s_2t_2$, and since $\theta_v(y) = \theta_{t_1}(y) = x \in X_{u^*}$, then

$$y = \theta_{v^*}(x) \in \theta_{v^*}(X_v \cap X_{u^*}) \subseteq X_{(uv)^*},$$

which proves that $(s_1t_1, y) \sim (s_2t_2, y)$.

(c) Since $u \leq s_1, s_2$ and $x \in X_{u^*}$, then $u^* \leq s_1^*, s_2^*$ and by Remark 4.1.1,

$$\theta_{s_1}(x) = \theta_u(x) = \theta_{s_2}(x) \in X_u,$$

proving that $(s_1^*, \theta_{s_1}(x)) \sim (s_2^*, \theta_{s_2}(x))$. □

Definition 4.1.5. Let $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ be a partial action of an inverse semigroup S on a topological space X, we let

$$S \ltimes X = \{[s, x] \mid s \in S, x \in X_{s^*}\} = (S * X)/\sim$$

be the set of all the germs. We call $S \ltimes_\theta X$ (or $S \ltimes X$ for short) the *groupoid of germs* associated to θ.

To describe the groupoid structure of $(S \ltimes X)^{(2)}$, we define the set of *composable pairs* as

$$(S \ltimes X)^{(2)} = \{([s, x], [t, y]) \mid x = \theta_t(y)\},$$

(note that $\theta_t(y)$ depends only on the class $[t, y]$, by Lemma 4.1.4 (a)). Given $([s, x], [t, y]) \in (S \ltimes X)^{(2)}$, define their product as

$$[s, x][t, y] = [st, y],$$

which is well defined by Lemma 4.1.4 (b). The inverse of $[s, x] \in S \ltimes X$ is defined by

$$[s, x]^{-1} = [s^*, \theta_s(x)],$$

which is also well defined by Lemma 4.1.4 (c).

It is routine to check that these operations define a groupoid structure on $S \ltimes X$ with unit space

$$(S \ltimes X)^{(0)} = \{[e, x] \mid e \in E(S) \text{ and } x \in X_e\}.$$

The range e source maps are defined by

$$\mathfrak{r}[s, x] = [ss^*, \theta_s(x)] \quad \text{and} \quad \mathfrak{s}[s, x] = [s^*s, x],$$

respectively.

We would now like to endow $S \ltimes X$ with an appropriate topology.

Lemma 4.1.6. *Let $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ be a partial action of an inverse semigroup S on a topological space X. For every $s \in S$ and $U \subseteq X_{s^*}$, let*

$$[s, U] = \{[s, x] \in S \ltimes X \mid x \in U\}.$$

The collection of all sets of the form $[s, U]$, where $s \in S$ and U is a open subset of X_{s^*}, is basis for a topology on $S \ltimes X$.

Proof. Let $s, t \in S$ and let $U \subseteq X_{s^*}$ and $V \subseteq X_{t^*}$ be open subsets. Our task is to prove that if

$$[r, z] \in [s, U] \cap [t, V]$$

then there is an element $u \in S$ and an open set $W \subseteq X_{u^*}$ such that

$$[r, z] \in [u, W] \subseteq [s, U] \cap [t, V].$$

However, by definition of germs in (4.1), we obtain

$$[s, U] \cap [t, V] = \bigcup_{u\leq s,t} [u, U \cap V \cap X_{u^*}],$$

and so it is sufficient to take $u \leq r, s, t$ and $W = U \cap V \cap X_{u^*}$. ∎

Proposition 4.1.7. *Let $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ be a partial action of an inverse semigroup S on a topological space X. The groupoid of germs $S \ltimes X$ is a topological groupoid with the topology induced by the basis of Lemma 4.1.6.*

Proof. Let $m : (S \ltimes X)^{(2)} \to S \ltimes X$ be the product map, and suppose $[r, V]$ a basic open set in $S \ltimes X$. We claim that

$$m^{-1}([r, V]) = \bigcup_{\substack{s,t\in S\\ st\leq r}} ([s, X_{s^*}] \times [t, V \cap X_{t^*} \cap X_{u^*}]) \cap (S \ltimes X)^{(2)} . \tag{4.2}$$

Indeed the inclusion '$\supseteq$' is immediate from the definition of the product. For the converse inclusion, if $([s, y], [t, x]) \in m^{-1}([r, V])$ then $[st, x] \in [r, V]$. This means that there is $u \in S$ such that $u \leq st, r$ and $x \in X_{u^*} \cap V \cap X_{t^*}$. Thus $[st, x] = [u, x] = [r, x]$ and

$$([s, y], [t, x]) = ([s, y], [tu^*u, x])$$

which belongs to the set on the right-hand side of (4.2), since $stu^*u = u \leq r, st$.

Now, notice that if $[s, U]$ is a basic open set in $S \ltimes X$ then

$$[s, U]^{-1} = [s^*, \theta_s(U)],$$

and so continuity of the inversion follows immediately. □

From now on, we always consider $S \ltimes X$ with the topology induced by the basis consisting of all sets of the form $[s, U]$, where $s \in S$ and $U \subseteq X_{s^*}$ open, as in the Lemma 4.1.6.

The unit space $(S \ltimes X)^{(0)}$ of $S \ltimes X$ can be naturally identified with X under the homeomorphism

$$\phi : (S \ltimes X)^{(0)} \to X \qquad [e, x] \mapsto x, \tag{4.3}$$

where $e \in E(S)$. To check that this map is injective, just note that if

$$[e, x],\ [f, y] \in (S \ltimes X)^{(0)} \quad \text{such that} \quad \phi([e, x]) = \phi[f, y],$$

then $x = y \in X_e \cap X_f \subseteq X_{ef}$ and $[e, x] = [ef, x] = [ef, y] = [f, y]$. The surjectivity immediately follows from the fact that we only consider non-degenerate actions.

A basic open set of $(S \ltimes X)^{(0)}$ has the form $[e, U]$ for some $e \in E(S)$ and $U \subseteq X_e$ open, and $\phi([e, U]) = U$. So ϕ takes basic open sets of $(S \ltimes X)^{(0)}$ to basic open sets of X, and is therefore a homeomorphism.

Since the source and range maps of $S \ltimes X$ are given by $\mathfrak{s}[s, x] = [s^*s, x]$ and $\mathfrak{r}[s, x] = [ss^*, \theta_s(x)]$, then enforcing the identification referred to in (4.3), we will write

$$\mathfrak{s}[s, x] = x \qquad \text{and} \qquad \mathfrak{r}[s, x] = \theta_s(x).$$

Proposition 4.1.8. *Let $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ be a partial action of an inverse semigroup S on a locally compact, Hausdorff topological space X. The groupoid $S \ltimes X$ is étale, and each basic open set $[s, U]$, where $s \in S$ and $U \subseteq X_{s^*}$ is open, is a bisection of $S \ltimes X$.*

Proof. By the natural identification between the unit space $(S \ltimes X)^{(0)}$ and X given by the homeomorphism in (4.3), we already obtain that the unit space of $S \ltimes X$ is locally compact and Hausdorff.

Given $s \in S$ and $U \subset X_{s^*}$ an open set, the source map on $[s, U]$ is given by (under the identification $(S \ltimes X)^{(0)}$)

$$\mathfrak{s} : [s, U] \to U \quad [s, x] \mapsto x,$$

and the injectivity of $\mathfrak{s}$ follows immediately from the definition of germs in (4.1). In particular, $\mathfrak{s}([s, U]) = U$ is an open subset of $X = (S \ltimes X)^{(0)}$. Since the basic open subsets of $[s, U])$ and U are (respectively) of the form $[s, V]$ and V, where $V \subseteq U$, we can conclude that the source map is a local homeomorphism from $[s, U]$ to U. Therefore, $S \ltimes X$ is étale. □

Notice that, if $\mathcal{B}$ is a basis for the topology of X, then a basis for $S \ltimes X$ consists of those sets of the form $[s, U]$ with $U \in \mathcal{B}$. Hence, if X is zero-dimensional then the collection of sets of the form $[s, U]$ with U compact-open subset of X, is a basis for $S \ltimes X$.

Corollary 4.1.9. *Let $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ be a partial action of an inverse semigroup S on a locally compact, Hausdorff and zero-dimensional topological space X. Then $S \ltimes X$ is an ample groupoid.*

Proof. This immediately follows from Propositions 4.1.8 and 1.2.9. $\square$

Example 4.1.10. Following Paterson [60], a *localization* consists of an action $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ of an inverse semigroup S on a topological space X such that $\{X_s\}_{s\in S}$ is a basis for the topology of S. The groupoid of germs in the sense of Paterson [60] coincides with the definition above of groupoids of germs.

Example 4.1.11. Let X be a locally compact Hausdorff space. The *canonical action* of $\mathcal{I}(X)$ on X is the action τ given by $\tau_\phi = \phi$, for all $\phi \in \mathcal{I}(X)$. A *pseudogroup* on X is an inverse subsemigroup of $\mathcal{I}(X)$ whose elements are homeomorphisms between open subsets of X.

Let $\mathcal{B}$ be a basis for the topology of X, and for each $B \in \mathcal{B}$ consider its identity function $\mathrm{id}_B : B \to B$.

Given a pseudogroup $\mathcal{G}$ on X, let $\mathcal{GB}$ be the inverse subsemigroup of $\mathcal{I}(X)$ generated by $\mathcal{G} \cup \{\mathrm{id}_B : B \in \mathcal{B}\}$, which is again a pseudogroup on X, and in fact the canonical action of $\mathcal{GB}$ on X is a localization.

The groupoid of germs in the sense of Renault [66] coincides with the groupoid of germs $\mathcal{GB} \ltimes X$ defined above.

The following are natural and well-known examples of constructions which are possible with groupoids of germs (and already appear in some form in [73]).

Example 4.1.12. (*Transformation groupoids*) In the case that S is a discrete group, the equivalence relation on $S * X$ is trivial and the topology is the product topology, that is, $S \ltimes X$ is the transformation groupoid (already seen in the Section 2.1). In particular $S \ltimes X$ is Hausdorff if, and only if, X is Hausdorff.

Example 4.1.13. (*Maximal group image*) An easy example is the case when X is a one-point set on which S acts trivially, that is, θ_s is simply the identity on X, for all $s \in S$. It is then straightforward to see that $S \ltimes X$ is the maximal group image $\mathbf{G}(S)$ of S (see Section 1.5.2 for the definition of $\mathbf{G}(S)$). Indeed, two elements of $S \cong S \ltimes X$ have the same germ if, and only if, they have a common lower bound.

Example 4.1.14. (*Restricted product groupoid*) Another example is the case when S is an arbitrary inverse semigroup, $X = E(S)$ with the discrete topology, and $\theta = (\{X_s\}_{s \in S}, \{\theta_s\}_{s \in S})$ is the *Munn representation* of S: $X_s = \{e \in E(S) : e \leq ss^*\}$ and $\theta_s(e) = ses^*$, for all $e \in X_{s^*}$.

Now from S we can construct the *restricted product groupoid* $(S, \cdot)$, which is the same as S but the product $s \cdot t = st$ is defined only when $s^*s = tt^*$ (see Example 1.2.5).

Then $S \ltimes E(S)$ is a discrete groupoid, and the map

$$S \ltimes E(S) \to (S, \cdot), \qquad [s, e] \mapsto se$$

is an isomorphism of discrete groupoid with inverse $s \mapsto [s, s^*s]$.

Example 4.1.15. Let $S = \mathbb{N} \cup \{\infty, z\}$ as in Example 1.1.5. Let $X = E(S) = \mathbb{N} \cup \{\infty\}$, seen as the one-point compactification of the natural numbers, and let θ be the Munn representation of S, so that $S \ltimes X = (S, \cdot)$, however with the topology whose open sets are either cofinite or contained in $\mathbb{N}$. In particular, $S \ltimes X$ is not Hausdorff.

Example 4.1.16. [28, Proposition 5.4] Every étale groupoid is isomorphic to a groupoid of germs. Indeed, let $\mathcal{G}$ be an étale groupoid, θ be the canonical action of the inverse semigroup of open bisections $\mathcal{G}^{op}$ on the unit space $\mathcal{G}^{(0)}$ (see Example 1.5.9), and S be any inverse subsemigroup of $\mathcal{G}^{op}$ which covers $\mathcal{G}$ (that is, $\mathcal{G} = \bigcup_{A \in S} A$), and which is closed under intersections. Then the map

$$S \ltimes \mathcal{G}^{(0)} \to \mathcal{G}, \qquad [A, x] \mapsto \mathfrak{s}|_A^{-1}(x),$$

is an isomorphism of topological groupoids.

In particular, if $\mathcal{G}$ is an ample Hausdorff groupoid, then the groupoid of germs $\mathcal{G}^a \ltimes \mathcal{G}^{(0)}$, of the restriction of the canonical action on $\mathcal{G}^a$, is isomorphic to the groupoid $\mathcal{G}$.

We will be mostly interested in Hausdorff groupoids, and in particular conditions on inverse semigroups which garantee that groupoids of germs are Hausdorff.

Definition 4.1.17. A poset $(P, \leq)$ is a $\wedge$-*weak semilattice* if for all $s, t \in P$ there exists a finite subset $F \subseteq L$ (possibly empty) such that

$$\{x \in P \mid x \leq s \text{ and } x \leq t\} = \bigcup_{f \in F} \{x \in P \mid x \leq f\}.$$

Example 4.1.18. If $\mathcal{G}$ is a Hausdorff ample groupoid, then $\mathcal{G}^{op}$ and $\mathcal{G}^a$ are meet semilattices, and $U \wedge V = U \cap V$.

Proposition 4.1.19. *[73, Theorem 5.17] Let S be an inverse semigroup which is $\wedge$-weak semilattice and let $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ be a partial action of S on a locally compact Hausdorff X such that X_s is clopen, for all $s \in S$. Then the groupoid of germs $S \ltimes X$ is Hausdorff.*

In particular, if S is a weak semilattice and X is zero-dimensional, then the groupoid of germs $S \ltimes X$ is an ample Hausdorff groupoid.

Proof. Suppose $[s, x] \neq [t, y]$ are elements of $S \ltimes X$. If $x \neq y$, then choose disjoint neighborhoods U, V of x and y in X, respectively. Clearly, $[s, U \cap X_{s^*}]$ and $[t, V \cap X_{t^*}]$ are disjoint neighborhoods of $[s, x]$ and $[t, y]$, respectively.

Next assume $x = y$. If $\{s' \in S \mid s' \leq s\} \cap \{t' \in S \mid t' \leq t\} = \emptyset$, then $[s, X_{s^*}]$ and $[t, X_{t^*}]$ are disjoint neighborhoods of $[s, x]$ and $[t, x]$. So we are left with the case $\{s' \in S \ : \ s' \leq s\} \cap \{t' \in S \ : \ t' \leq t\} \neq \emptyset$. Since S is a weak semilattice, we can find elements $u_1, \cdots, u_n \in S$ so that $u \leq s, t$ if, and only if, $u \leq u_i$, for $i \in \{1, \cdots, n\}$. Let

$$V = X \setminus \bigcup_{i=1}^{n} X_{u_i^*} = \bigcap_{i=1}^{n} (X \setminus X_{u_i^*}),$$

which is an open set by hypothesis. If $x \in X_{u_i^*}$ for some i, then as $u_i \leq s, t$, it follows $[s, x] = [t, x]$, a contradiction. Thus $x \in V$. Define $W = V \cap X_{s^*} \cap X_{t^*}$. We claim $[s, W]$ and $[t, W]$ are disjoint neighborhoods of $[s, x]$ and $[t, x]$, respectively. Indeed, if

$$[r, z] \in [s, W] \cap [t, W],$$

then

$$[s, z] = [r, z] = [t, z],$$

and hence, there exists $u \leq s, t$ with $z \in X_{u^*}$. But then $u \leq u_i$ for some $i \in \{1, \cdots, n\}$ and so $z \in X_{u_i^*}$, contradicting that $z \in W \subseteq V$. □

Remark 4.1.20. Steinberg proved in [73] that an inverse semigroup S is a weak semilattice if, and only if, for any partial action $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ such that X_s is clopen for all $s \in S$, the groupoid of germs $S \ltimes X$ is Hausdorff.

Remark 4.1.21. The hypothesis that the domains of the partial action are clopen is necessary. For example, even if $\mathcal{G}$ is a non-Hausdorff ample groupoid then $\mathcal{G}^{op}$ is still a semilattice, however, as in Example 4.1.16, the groupoid of germs $\mathcal{G}^{op} \ltimes \mathcal{G}^{(0)} \cong \mathcal{G}$ is not Hausdorff.

Next we present two examples of isomorphic groupoid germs.

Example 4.1.22. If S is E^*-unitary then it is a $\wedge$-semilattice: Indeed, given $s, t \in S$, if $\{s, t\}$ does not admit any nonzero lower bound then $s \wedge t = 0$. If $\{s, t\}$ admits a nonzero lower bound, then s and t are compatible, so $s \wedge t = ts^*s$.

As a consequence, every E-unitary inverse semigroup S is a weak semilattice: We can embed S into an E^*-unitary semigroup S_0 by adjoining a 0. Given $s, t \in S$, let $F = \{s \wedge t\} \setminus \{0\}$, which is either empty or equal to $\{s \wedge t\}$, but in any case a finite subset of S, so that $\{x \in S \mid x \leq s, t\} = \bigcup_{f\in F} \{x \in S \mid x \leq f\}$.

A version of the next example has been proven in [57], when considering the canonical action of S on the spectrum of its idempotent set $E(S)$. We prove the result for general partial actions of inverse semigroups on arbitrary topological spaces.

Example 4.1.23. Let $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ be a partial action of an E-unitary semigroup S on a space X and $\widetilde{\theta}$ be the induced action of the maximal group homomorphic image $\mathbf{G}(S)$ on X (see Proposition 1.5.23). Then

$$S \ltimes_\theta X \simeq \mathbf{G}(S) \ltimes_{\widetilde{\theta}} X.$$

Indeed, consider the map $[s,x] \mapsto ([s],x)$, which is well-defined by the definitions of the relations involved (see Equations (4.1) and (1.15)). It is clearly a homomorphism. The surjectivity follows from the fact that given $(\gamma,x) \in \mathbf{G}(S) \ltimes_{\widetilde{\theta}} X$ there is $s \in S$ such that $[s] = \gamma$ and $x \in X_{s^*}$, and so, $[s,x] \mapsto ([\gamma],x)$. As for injectivity, suppose $([s],x) = ([t],y)$, where $[s,x],[t,y] \in S \ltimes_\theta X$. Then $x = y$ and $[s] = [t]$, so $x \in X_{s^*} \cap X_{t^*}$. By Lemma 1.5.22, s and t are compatible, which implies $s(s^*st^*t) = t(s^*st^*t)$ (as both products describe the meet $s \wedge t$). Since $x \in X_{s^*} \cap X_{t^*} \subseteq X_{s^*s} \cap X_{t^*t} \subseteq X_{s^*st^*t}$ we conclude that $[s,x] = [t,y]$.

Example 4.1.24. Suppose $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ be a partial action of an group G on a topological space X, $\overline{\theta}$ is the corresponding global action of $\mathbf{S}(G)$ on X (see Proposition 1.5.25). Then

$$G \ltimes_\theta X \cong \mathbf{S}(G) \ltimes_{\widetilde{\theta}} X.$$

Indeed, let $\gamma = \widetilde{\widetilde{\theta}}$, the partial action of $\mathbf{G}(\mathbf{S}(G))$ induced by $\widetilde{\theta}$ as in Proposition 1.5.23. Let us prove that for all $g \in G$, $\theta_g = \gamma_{[[g]]}$. From this fact and Proposition 1.5.26, it follows easily that

$$G \ltimes_\theta X \to \mathbf{G}(\mathbf{S}(G)) \ltimes_\gamma X, \qquad (g,x) \mapsto ([[g]],x)$$

is a topological groupoid isomorphism. Example 4.1.23 provides the isomorphism $\mathbf{G}(\mathbf{S}(G)) \ltimes_\gamma X \cong \mathbf{S}(G) \ltimes_{\widetilde{\theta}} X$, so we are done.

Let $g \in G$ be fixed. By definition, $\gamma_{[[g]]}$ is the supremum of $\left\{\widetilde{\theta}_s : s \sim [g]\right\}$. From the uniqueness of the standard form of each $s \in \mathbf{S}(G)$, it follows that $s \sim [g]$ if and only if $s \leq [g]$, and thus we conclude that $\gamma_{[[g]]} = \widetilde{\theta}_{[g]}$.

4.2 Topologically principal and effective partial actions

In this section we will work again with the notions of topologically principal and effective (or topologically free) partial actions of inverse semigroups (see Definition 3.2.7 and 3.2.5). These concepts will be used later in our study of continuous orbit equivalence.

Let $\mathcal{G}$ be a topological groupoid. Recall that $\mathcal{G}$ is *effective* if, and only if, the interior of the isotropy subgroupoid $\mathrm{Iso}(\mathcal{G})$ is just the unit space $\mathcal{G}^{(0)}$.

Proposition 4.2.1. *[32, Theorem 4.7] Given a partial action $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ of an inverse semigroup S on a locally compact Hausdorff space X, the corresponding groupoid of germs $S \ltimes X$ is effective if, and only if, θ is effective.*

Proof. Suppose that $S \ltimes X$ is an effective groupoid. By Remark 3.2.6, for every $s \in S$

$$\{x \in X_{s^*} \mid \exists\, e \in E(S),\ e \leq s \text{ and } x \in X_e\} \subseteq \operatorname{int}\{x \in X_{s^*} \mid \theta_s(x) = x\}.$$

Given $s \in S$ take $z \in \operatorname{int}\{x \in X_{s^*} \mid \theta_s(x) = x\}$. Then there is an open subset U of X_{s^*} such that $z \in U$ and $\theta_s(x) = x$, for all $x \in U$. Hence

$$\mathfrak{r}([s,x]) = \theta_s(x) = x = \mathfrak{s}([s,x]),$$

and so $[s,x] \in \mathrm{Iso}(S \ltimes X)$. This implies that the basic open subset $[s,U]$ is contained in the interior of $\mathrm{Iso}(S \ltimes X)$. From the hypothesis, $[s,U] \subseteq (S \ltimes X)^{(0)}$, and so, there is $e \in E(s)$ such that $z \in X_e$ and $[s,z] = [e,z]$. By the definition of germs in (4.1), there is $f \in E(S)$, $f \leq s, e$ such that $x \in X_f$ and

$$[s,z] = [e,z] = [f,z].$$

Applying the range map on the right and left side of this equality, we can conclude that z is a trivial fixed point for s.

Conversely, assume that θ is an effective partial action. Taking $[s,x] \in \operatorname{int}(\mathrm{Iso}(S \ltimes X))$ there is a basic open subset $[s,U]$ in $S \ltimes X$ such that

$$[s,x] \in [s,U] \subseteq \mathrm{Iso}(S \ltimes X).$$

For any $y \in U$, we have that $[s, y] \in \mathrm{Iso}(S \ltimes X)$, so

$$\theta_s(y) = \mathfrak{r}([s, y]) = \mathfrak{s}([s, y]) = y,$$

and we see that y is a fixed point for s. It follows that U is contained in the set of fixed points for s. In particular x is an interior fixed point, and hence, by hypothesis, x is a trivial fixed point. Then there is $e \in E(S)$ such that $e \leq x$ and $x \in X_e$, and consequently $[s, x] = [e, x] \in (S \ltimes X)^{(0)}$. This shows that $S \ltimes X$ is effective. □

Recall that a topological groupoid $\mathcal{G}$ is *topologically principal* if, and only if, the set of points in $\mathcal{G}^{(0)}$ with trivial isotropy group (this means $\mathcal{G}_u^u = \{u\}$) is dense in $\mathcal{G}^{(0)}$.

We will now reword topological principality of a partial action in terms of the groupoid of germs $S \ltimes X$.

Proposition 4.2.2. *Suppose that $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ is a partial action of an inverse semigroup S on a locally compact Hausdorff space X. Then the groupoid of germs $S \ltimes X$ is topologically principal if, and only if, the partial action θ is topologically principal.*

Proof. It is enough to prove that, under the identification of X with $(S \ltimes X)^{(0)}$,

$$\Lambda(\theta) = \{x \in X : (S \ltimes X)_x^x = \{x\}\}.$$

Let $x \in X$ be given. First suppose $x \in \Lambda(\theta)$ and $[s, x] \in (S \ltimes X)_x^x$. This means that $x = \mathfrak{r}([s, x]) = \theta_s(x)$, so there is $e \in E(S) \cap S_x$, $e \leq s$, which implies $[s, x] = [e, x] = x$.

Conversely suppose $(S \ltimes X)_x^x = \{x\}$ and let $s \in S_x$ with $\theta_s(x) = x$. This means that $[s, x] \in (S \ltimes X)_x^x$, and so $[s, x] = [e, x]$ for some idempotent $e \in S_x$. By the definition of the groupoid of germs, we can find another idempotent $f \in S_x$ with $se = ef$, so in particular ef is an idempotent, $ef \leq s$, and $x \in X_{ef}$. This proves $x \in \Lambda(\theta)$. □

This way, topologically principal partial actions will correspond to topologically principal groupoids of germs, whereas effective partial actions will correspond to effective groupoids of germs.

By a *principal* (or *free*) partial action we mean a topologically principal partial action on a discrete space (that is, a set). Similarly to Proposition 4.2.2, one proves that $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ is a principal partial action if, and only if, the groupoid of germs $S \ltimes X$ is principal.

Proposition 4.2.3. *If $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ is a principal partial action of an inverse semigroup S on a locally compact Hausdorff space X then $S \ltimes X$ is Hausdorff.*

Proof. Suppose $[s, x] \neq [t, y]$ are elements of $S \ltimes X$. If $x \neq y$, then choose disjoint neighborhoods U, V of x and y in X, respectively. Clearly, $[s, U \cap X_{s^*}]$ and $[t, V \cap X_{t^*}]$ are disjoint neighborhoods of $[s, x]$ and $[t, y]$, respectively. Next assume $x = y$. By principality of θ, we have that $\theta_s(x) = \theta_t(x)$ if, and only if, there is $u \in S$, such that $u \leq s, t$ and $x \in X_{u^*}$ if, and only if, $[s, x] = [t, x]$. Hence, if $[s, x] \neq [t, x]$ then $\theta_s(x) \neq \theta_t(x)$. Since X is Hausdorff, there are disjoint neighborhoods U and V of $\theta_s(x)$ and $\theta_t(x)$, respectively. It is easy see that $[s, \theta_{s^*}(U)]$ and $[t, \theta_{t^*}(V)]$ are disjoint neighborhoods of $[s, x]$ and $[t, x]$, respectively. $\square$

Remark 4.2.4. The proof above is a combination of the facts that free partial actions on (discrete) sets correspond to (discrete) principal groupoids of germs, and that every principal topological groupoid with Hausdorff unit space is itself Hausdorff.

It is interesting to note that principality of a partial action implies that the associated groupoid of germs is Hausdorff, however this is not true for topologically principal partial actions, as shown in the example below.

Example 4.2.5. As in Example 4.1.15, let $S = \mathbb{N} \cup \{\infty, z\}$ and θ be the Munn representation of S on $X = E(S) = \mathbb{N} \cup \{\infty\}$, endowed with the same topology as the one-point compactification of $\mathbb{N}$. This is a topologically principal partial action, since $\Lambda(\theta) = \mathbb{N}$ is dense in X, however the associated groupoid of germs $S \ltimes X$ is not Hausdorff.

In the case that G is a group, a partial action of G is principal (or free) if for all $x \in X$ (and for all $g \in G_x$), one has that $\theta_g(x) = x$ implies $g = 1$, where 1 is the identity of G, which is the usual notion of freeness for partial group actions.

4.3 Steinberg algebra of groupoid of germs

Historical notes: In [5], Beuter and Gonçalves showed that any Steinberg algebra, $A_R(G \ltimes_\theta X)$, of a transformation groupoid $G \ltimes X$ given by a partial action $\theta = (\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ of a group G on a locally compact, Hausdorff, and zero-dimensional space X, is isomorphic to the partial skew group algebra $\mathcal{L}_c(\mathrm{X}) \rtimes G$ (see Theorem 2.1.1). In the same paper, they proved that every Steinberg algebra, $A_R(\mathcal{G})$, associated with an ample Hausdorff groupoid $\mathcal{G}$, is isomorphic to the skew inverse semigroup algebra $\mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes \mathcal{G}^a$ (see Theorem 2.3.1). Soon after, Hazrat and Li proved a similar results for graded algebras. More precisely, given a graded ample Hausdorff groupoid, its graded Steinberg algebra can be realized as a partial skew inverse semigroup algebra (see [44, Theorem 2.3]). In sequence, Demeneghi showed that the Steinberg algebra of a Hausdorff groupoid of germs $S \ltimes_\theta X$ associated to an ample action[1] of an inverse semigroup $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ is isomorphic to the skew inverse semigroup algebra $\mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes S$ (see [21, Theorem 2.3.6] version arXiv v1), and as a consequence obtained the latter result presented by the first authors (see [21, Proposition 2.4.3] version arXiv v1). In [2], Beuter and Cordeiro studied orbit equivalence of topologically principal partial actions of inverse semigroups in terms of isomorphisms of the corresponding groupoid of germs, and had the need to generalize [21, Theorem 2.3.6] for partial actions of inverse semigroups. Moreover, they weakened the conditions on each X_s (see [2, Theorem 5.4]). Later, Demeneghi managed to perfect his theorem to groupoid of germs which are not necessarily Hausdorff, however still

[1] An ample action is an action $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ of an inverse semigroup S on a locally compact, Hausdorff and zero-dimensional space X, where each X_s is both open and compact.

under the condition that each X_s is clopen (see [22, Theorem 5.2.4.] version arXiv v3).

Note that the results in [2] and [21] are non-comparable, since [2] deals with partial actions whose corresponding groupoids of germs are Hausdorff, whereas [21] considers actions whose corresponding groupoids of germs are not necessarily Hausdorff. Furthermore, the proof of either of these results does not seem to be easily adaptable to cover the other.

In this section we present the theorem of Beuter and Cordeiro. We will assume that R is a unital commutative ring, and that $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ is a partial action of an inverse semigroup S on a locally compact, Hausdorff and zero-dimensional space X. Moreover, we will consider that $\alpha = (\{D_s\}_{s\in S}, \{\alpha_s\}_{s\in S})$ is the partial action of S on the R-algebra $\mathcal{L}_{\rm c}(\mathrm{X})$ induced by θ, as in Example 1.5.28. Therefore, we will prove that, as long as the groupoid of germs $S \ltimes X$ is Hausdorff, the Steinberg algebra $A_R(S \ltimes X)$ is isomorphic to the partial skew inverse semigroup algebra $\mathcal{L}_{\rm c}(\mathrm{X}) \ltimes S$. In order to obtain such an isomorphism, we need some preliminary lemmas.

Lemma 4.3.1. *Given $s \in S$, any subset B of $[s, X_{s^*}]$ is of the form $B = [s, \mathfrak{s}(B)]$.*

Proof. The inclusion "$\subseteq$" follows trivially, since if $b \in B \subseteq [s, X_{s^*}]$, then $b = [s, \mathfrak{s}(b)] \in [s, \mathfrak{s}(B)]$. On the other hand, if $b \in [s, X_{s^*}] \setminus B$, then

$$b \in [s, \mathfrak{s}([s, X_{s^*}] \setminus B)] = [s, X_{s^*} \setminus \mathfrak{s}(B)].$$

This implies that $\mathfrak{s}(b) \in X_{s^*} \setminus \mathfrak{s}(B)$, that is, $b \notin [s, \mathfrak{s}(B)]$. Therefore, if $b \in [s, \mathfrak{s}(B)]$ then $b \in B$, proving the reverse inclusion. $\square$

Remark 4.3.2. If B is a compact-open bisection of an ample groupoid of germs $S \ltimes X$ (not necessarily Hausdorff), then B has a finite cover $\{[s_i, W_i] \mid i = 1 \cdots, n\}$ of basic compact-open subsets of $S \ltimes X$. Setting $U_1 = W_1$ and $U_i = W_i \setminus \left(\bigcup_{j=1}^{i-1} W_j\right)$, for all $i \in \{1, \cdots, n\}$, then $\{U_i \subseteq X_{s_i^*} \mid i = 1 \cdots, n\}$ is a collection of pairwise disjoint compact-open subsets. By Lemma 4.3.1 and by injectivity of the source map on

B, we obtain

$$B = \bigcup_{i=1}^{n} [s_i, U_i],$$

where $\{[s_i, U_i] \mid i = 1 \cdots, n\}$ is a collection of pairwise disjoint basic compact-open subsets of $S \ltimes X$.

Lemma 4.3.3. *Let $S \ltimes X$ be a Hausdorff groupoid of germs, and consider two finite collections $\{[s_i, U_i] \mid i = 1 \cdots n\}$ and $\{[t_j, V_j] \mid j = 1 \cdots m\}$ of pairwise disjoint basic compact-open subsets of $S \ltimes X$ such that*

$$\bigcup_{i=1}^{n} [s_i, U_i] = \bigcup_{j=1}^{m} [t_j, V_j] .$$

Then, for each pair i, j, there is a finite collection

$$\left\{ \left[u_k^{ij}, W_k^{ij} \right] \mid k = 1, \cdots, l^{ij} \right\}$$

of pairwise disjoint basic compact-open subsets of $S \ltimes X$ such that

$$[s_i, U_i] \cap [t_j, V_j] = \bigcup_{k=1}^{l^{ij}} \left[u_k^{ij}, W_k^{ij} \right],$$

and $u_k^{ij} \leq s_i, t_j$.

Proof. Given a pair i, j, let b be an arbitrary element of $[s_i, U_i] \cap [t_j, V_j]$. Then $[s_i, \mathfrak{s}(b)] = b = [t_j, \mathfrak{s}(b)]$, and there is $u_b \in S$ such that $\mathfrak{s}(b) \in X_{u_b^*}$ and $u_b \leq s_i, t_j$, so

$$b \in \left[u_b, X_{u_b^*} \cap U_i \cap V_j \right] \subseteq [s_i, U_i] \cap [t_j, V_j] .$$

By compactness of $[s_i, U_i] \cap [t_j, V_j]$, we find a finite cover for this set with elements of the form $[u_k^{ij}, W_k^{ij}]$ for certain $u_k^{ij} \leq s_i, t_j$ and $W_{ij} \subseteq U_i \cap V_j$, that is,

$$[s_i, U_i] \cap [t_j, V_j] = \bigcup_{k=1}^{l^{ij}} \left[u_k^{ij}, W_k^{ij} \right] .$$

Since $S \ltimes X$ is Hausdorff, for $k \geq 2$, we can substitute $\left[u_k^{ij}, W_k^{ij} \right]$ by $\left[u_k^{ij}, W_k^{ij} \right] \setminus \bigcup_{p=1}^{k-1} [u_p, W_p]$. By Lemma 4.3.1, we can to rewrite this set as $[u_k^{ij}, \widetilde{W_k^{ij}}]$ for appropriate $\widetilde{W_k^{ij}}$, and to obtain the desired partition of $[s_i, U_i] \cap [t_j, V_j]$. □

Theorem 4.3.4. *Let $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ be a partial action of an inverse semigroup S on a locally compact, Hausdorff and totally disconnected topological space X, and let $\alpha = (\{D_s\}_{s\in S}, \{\alpha_s\}_{s\in S})$ be the partial action of S on $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$ induced by θ.*

If the groupoid of germs $S \ltimes X$ is Hausdorff, then the Steinberg algebra of $S \ltimes_\theta X$ is isomorphic to the partial skew inverse semigroup algebra $\mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\alpha S$.

Proof. Given a generating element $f_s\delta_s$ of $\mathscr{L}(\alpha)$, we define $\phi(f_s\delta_s) : S \ltimes X \to R$ by

$$\phi(f_s\delta_s)(a) = \begin{cases} f_s(\mathfrak{r}(a)), & \text{if } a \in [s, X_{s^*}] \\ 0, & \text{otherwise.} \end{cases}$$

We first need to check that $\phi(f_s\delta_s)$ is well-defined, that is, $\phi(f_s\delta_s)$ is locally constant and has compact support. Notice that the definition of $\phi(f_s\delta_s)$ implies

$$\mathrm{supp}(\phi(f_s\delta_s)) = [s, \theta_s^{-1}(\mathrm{supp}\, f)],$$

which is compact-open, and in particular is clopen because $S \ltimes X$ is Hausdorff.

Of course, $\phi(f_s\delta_s)$ is constant equal to 0 on the complement

$$(S \ltimes X) \setminus \mathrm{supp}(\phi(f_s\delta_s)),$$

and since $\phi(f_s\delta_s)$ coincides with the composition $f_s \circ \mathfrak{r}$ on $\mathrm{supp}(\phi(f_s\delta_s))$ and f_s is locally constant, then $\phi(f_s\delta_s)$ is also locally constant on $\mathrm{supp}(\phi(f_s\delta_s))$. We conclude that $\phi(f_s\delta_s)$ is locally constant on complementary clopen subsets of $S \ltimes X$, so $\phi(f_s\delta_s)$ is locally constant.

Using the presentation of $\mathscr{L}(\alpha)$ in Definition 1.6.1, we extend ϕ linearly to an R-module homomorphism $\phi : \mathscr{L}(\alpha) \to A_R(S \ltimes X)$.

We will show that ϕ is multiplicative. By linearity of ϕ it is enough to verify this map is multiplicative on the generators. Let $f_s\delta_s, f_t\delta_t \in \mathscr{L}(\alpha)$ and $a \in S \ltimes X$. There are two possibilities:

Case 1: $a \notin [s, X_{s^*}][t, X_{t^*}] = [st, \theta_{t^*}(X_t \cap X_{s^*})]$.

Since $\operatorname{supp}(\phi(f_s\delta_s) * \phi(f_t\delta_t)) \subseteq [s, X_{s^*}][t, X_{t^*}]$, then

$$[\phi(f_s\delta_s) * \phi(f_t\delta_t)](a) = 0.$$

On the other hand, $(f_s\delta_s)(f_t\delta_t) = \alpha_s(\alpha_{s^*}(f_s)f_t)\delta_{st}$. Since

$$\operatorname{supp}(\alpha_{s^*}(f_s)f_t) = \theta_{s^*}(\operatorname{supp} f_s) \cap \operatorname{supp}(f_t)$$

then

$$\operatorname{supp}(\alpha_s(\alpha_{s^*}(f_s)f_t)) = \theta_s(\theta_{s^*}(\operatorname{supp}(f_s)) \cap (\operatorname{supp}(f_t)),$$

and this set is contained in $\theta_s(X_{s^*} \cap X_t) = X_s \cap X_{st}$, which is precisely the set on which $\theta_{t^*}\theta_{s^*} = \theta_{(st)^*}$. Thus

$$\theta_{(st)^*}[\operatorname{supp}(\alpha_s(\alpha_{s^*}(f_s)f_t))] = \theta_{t^*}((\operatorname{supp} f_s) \cap \operatorname{supp} f_t)$$

and so

$$\operatorname{supp}[\phi((f_s\delta_s) * (f_t\delta_t))] = [st, \theta_{t^*}(\theta_{s^*}(\operatorname{supp} f_s) \cap \operatorname{supp} f_t)],$$

which is contained in $[st, \theta_{t^*}(X_{s^*} \cap X_t)] = [s, X_{s^*}][t, X_{t^*}]$, and therefore

$$\phi((f_s\delta_s)(f_t\delta_t))(a) = 0 = (\phi(f_s\delta_s) * \phi(f_t\delta_t))(a),$$

as we expected.

Case 2: $a \in [s, X_{s^*}][t, X_{t^*}]$.

In this case, we can write $a = [s, x][t, y]$ for unique $x \in X_{s^*}$ and $y \in X_{t^*}$ with $\theta_t(y) = x$. Since $\operatorname{supp}(\phi(f_s\delta_s)) \subseteq [s, X_{s^*}]$ then

$$\begin{aligned}
(\phi(f_s\delta_s) * \phi(f_t\delta_t))(a) &= \sum_{b \in \mathfrak{r}^{-1}\mathfrak{r}(a)} \phi(f_s\delta_s)(b)\phi(f_t\delta_t)(b^{-1}a) \\
&= \phi(f_s\delta_s)[s, x]\phi(f_t\delta_t)[t, y] \\
&= f_s(\theta_s(x))f_t(\theta_t(y))
\end{aligned}$$

On the other hand, $a \in [s, X_{s^*}][t, X_{t^*}] \subseteq [st, X_{(st)^*}]$, so

$$\begin{aligned}
\phi((f_s\delta_s) * (f_t\delta_t))(a) &= \phi(\alpha_s(\alpha_{s^*}(f_s)f_t)\delta_{st})(a) \\
&= \alpha_s(\alpha_{s^*}(f_s)f_t)(\mathfrak{r}(a)) = \alpha_s(\alpha_{s^*}(f_s)f_t)(\theta_s(x)) \\
&= (\alpha_{s^*}(f_s)f_t)(x) = f_s(\theta_s(x))f_t(x) \\
&= f_s(\theta_s(x))f_t(\theta_t(y)) \\
&= (\phi(f_s\delta_s) * \phi(f_t\delta_t))(a),
\end{aligned}$$

as we desired.

Now let us prove that ϕ vanishes on the ideal $\mathcal{N}(\alpha)$ generated by all elements of form $f\delta_s - f\delta_t$, where $s \leq t$ and $f \in D_s$. Since ϕ is a homomorphism it is enough to show that ϕ is zero on the generators of $\mathcal{N}(\alpha)$, so let $a \in S \ltimes X$. We have that

- if $a \in [s, X_{s^*}]$ then $a \in [t, X_{t^*}]$, and

$$\phi(f\delta_s - f\delta_t)(a) = f(\mathfrak{r}(a)) - f(\mathfrak{r}(a)) = 0;$$

- if $a \in [t, X_{t^*}] \setminus [s, X_{s^*}]$ then $\mathfrak{r}(a) \notin X_s$, because $\mathfrak{r}$ is injective on $[t, X_{t^*}]$, and $f(\mathfrak{r}(a)) = 0$ because $f \in D_s$. Thus

$$\phi(f\delta_s - f\delta_t)(a) = 0 - f(\mathfrak{r}(a)) = 0.$$

- if $a \notin [t, X_{t^*}]$ then $a \notin [s, X_{s^*}]$ as well, so

$$\phi(f\delta_s - f\delta_t)(a) = 0 - 0 = 0;$$

Therefore, ϕ vanishes on the ideal $\mathscr{N}(\alpha)$ and hence factors through the quotient $\mathscr{L}(\alpha)/\mathscr{N}(\alpha) = \mathcal{L}_{\rm c}(\mathrm{X}) \rtimes_\alpha S$ to an R-homomorphism $\Phi : \mathcal{L}_{\rm c}(\mathrm{X}) \rtimes_\alpha S \to A_R(S \ltimes X)$.

In order to prove that Φ is bijective, we will show the existence of a map $\Psi : A_R(S \ltimes X) \to \mathcal{L}_{\rm c}(\mathrm{X}) \rtimes_\alpha S$ which is in fact the inverse map of Φ.

By Remark 4.3.2, any compact-open bisection of $S \ltimes X$ is a disjoint union of basic compact-open subsets of $S \ltimes X$. Hence, any function in $A_R(S \ltimes X)$ can be written as a linear combination of characteristic functions of disjoint basic compact-open subsets.

Given $f = \sum_{i=1}^n c_i 1_{[s_i,U_i]} \in A_R(S \ltimes X)$, where $c_1, \cdots, c_n \in R \setminus \{0\}$ and $[s_1, U_1], \cdots, [s_n, U_n]$ are pairwise disjoints basic compact-open subsets of $S \ltimes X$, define

$$\Psi(f) = \Psi\left(\sum_{i=1}^n c_i 1_{[s_i,U_i]}\right) = \sum_{i=1}^n \overline{c_i 1_{\mathfrak{r}[s_i,U_i]}\delta_{s_i}}.$$

We need to check that Ψ is well-defined. Suppose that also there are other pairwise disjoints basic compact-open subsets $[t_1, V_1], \cdots,$ $[t_m, V_m]$ of $S \ltimes X$ and $b_1, \cdots, b_m \in R \setminus \{0\}$ such that $f = \sum_{j=1}^m b_j 1_{[t_j,V_j]}$. Notice that

$$\bigcup_{i=1}^n [s_i, U_i] = \operatorname{supp}(f) = \bigcup_{j=1}^m [t_j, V_j],$$

and these unions are disjoint. We can then conclude that

$$\sum_{i=1}^n \sum_{j=1}^m c_i 1_{[s_i,U_i]\cap[t_j,V_j]} = f = \sum_{j=1}^m \sum_{i=1}^n b_j 1_{[s_i,U_i]\cap[t_j,V_j]},$$

and the family $\{[s_i, U_i] \cap [t_j, V_j]\}_{i,j}$ is pairwise disjoint, which implies that

$$c_i 1_{[s_i,U_i]\cap[t_j,V_j]} = b_j 1_{[s_i,U_i]\cap[t_j,V_j]}, \tag{4.4}$$

for every pair i, j.

Let i and j be temporarily fixed. By Lemma 4.3.3, there is a finite collection $\left\{\left[u_k^{ij}, W_k^{ij}\right] \middle| k = 1, \cdots, l^{ij}\right\}$ of pairwise disjoint basic compact-open subsets of $S \ltimes X$ such that

$$[s_i, U_i] \cap [t_j, V_j] = \bigcup_{k=1}^{l^{ij}} \left[u_k^{ij}, W_k^{ij}\right],$$

and $u_k^{ij} \leq s_i, t_j$. Hence, for every $k \in \{1, \ldots, l^{ij}\}$, we have that

$$c_i 1_{[u_k^{ij},W_k^{ij}]} = b_j 1_{[u_k^{ij},W_k^{ij}]},$$

and composing both maps on the right with $\mathfrak{r}|_{[s_i,U_i]}^{-1}$, we obtain

$$c_i 1_{\mathfrak{r}[u_k^{ij},W_k^{ij}]} = b_j 1_{\mathfrak{r}[u_k^{ij},W_k^{ij}]}, \tag{4.5}$$

on $\mathfrak{r}[s_i, U_i]$. Of course, $1_{\mathfrak{r}[s_i,U_i]}$ is identically zero on $X \setminus \mathfrak{r}[s_i, U_i]$, so in fact equation (4.5) holds everywhere on X. Then

$$\begin{aligned}
\sum_{i=1}^{n} \overline{c_i 1_{\mathfrak{r}[s_i,U_i]} \delta_{s_i}} &= \sum_{i=1}^{n} \overline{c_i 1_{\bigcup_{j=1}^{m} \bigcup_{k=1}^{l^{ij}} \mathfrak{r}\left([u_k^{ij}, W_k^{ij}]\right)} \delta_{s_i}} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l^{ij}} \overline{c_i 1_{r[u_k^{ij}, W_k^{ij}]} \delta_{s_i}} \\
&\overset{u_k^{ij} \leq s_i}{=} \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l^{ij}} \overline{c_i 1_{r[u_k^{ij}, W_k^{ij}]} \delta_{u_k^{ij}}} \\
&\overset{(4.5)}{=} \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l^{ij}} \overline{b_j 1_{\mathfrak{r}[u_k^{ij}, W_k^{ij}]} \delta_{u_k^{ij}}} \\
&\overset{u_k^{ij} \leq t_j}{=} \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{k=1}^{l^{ij}} \overline{b_j 1_{\mathfrak{r}[u_k^{ij}, W_k^{ij}]} \delta_{t_j}} \\
&= \sum_{j=1}^{m} \overline{b_j 1_{\bigcup_{i=1}^{n} \bigcup_{k=1}^{l^{ij}} \mathfrak{r}\left([u_k^{ij}, W_k^{ij}]\right)} \delta_{t_j}} \\
&= \sum_{j=1}^{m} \overline{b_j 1_{\mathfrak{r}[t_j, V_j]} \delta_{t_j}}.
\end{aligned}$$

This proves that Ψ is well-defined. Moreover, it should be clear that whenever we represent an element $f \in A_R(S \ltimes X)$ as $f = \sum_{i=1}^{n} c_i 1_{[s_i,U_i]}$, where $\{[s_i, U_i] \mid i = 1 \cdots n\}$ is a collection of pairwise disjoint basic compact-open subsets of $S \ltimes X$, then the condition $c_i \neq 0$ in the original definition of Ψ is not necessary, so that we still have

$$\Psi(f) = \sum_{i=1}^{n} \overline{c_i 1_{\mathfrak{r}[s_i,U_i]} \delta_{s_i}}$$

To prove that Ψ is the inverse of Φ, we must first prove that Ψ is additive. Suppose that $f = \sum_{i=1}^{n} c_i 1_{[s_i,U_i]}$ and $g = \sum_{j=1}^{m} b_j 1_{[t_j,V_j]}$, where the collections $\{[s_i, U_i] \mid i = 1 \cdots n\}$ and $\{[t_j, V_j] \mid j = 1 \cdots m\}$ consist of pairwise disjoint basic compact-open subsets of $S \ltimes X$, and $c_i, b_j \in R$.

We can assume that

$$\bigcup_{i=1}^{n}[s_i, U_i] = \bigcup_{j=1}^{m}[t_j, V_j]$$

taking some c_i and b_j equal to zero. Again, take u_k^{ij} satisfying the conditions of Lemma 4.3.3, so

$$f = \sum_{j=1}^{m}\sum_{k=1}^{l^{ij}} c_i 1_{[u_k^{ij}, W_k^{ij}]} \qquad \text{and} \qquad g = \sum_{j=1}^{m}\sum_{k=1}^{l^{ij}} b_j 1_{[u_k^{ij}, W_k^{ij}]}$$

and $f+g = \sum_{j=1}^{m}\sum_{k=1}^{l^{ij}}(c_i+b_j)1_{[u_k^{ij}, W_k^{ij}]}$. Since the basic compact-open subsets $[u_k^{ij}, W_k^{ij}]$ are pairwise disjoint, the definition of Ψ gives us

$$\begin{aligned}
\Psi(f+g) &= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l^{ij}} \overline{(c_i+b_j)1_{\mathfrak{r}[u_k^{ij}, W_k^{ij}]}\delta_{u_k^{ij}}} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l^{ij}} \overline{c_i 1_{\mathfrak{r}[u_k^{ij}, W_k^{ij}]}\delta_{u_k^{ij}}} + \sum_{j=1}^{m}\sum_{i=1}^{n}\sum_{k=1}^{l^{ij}} \overline{b_j 1_{\mathfrak{r}[u_k^{ij}, W_k^{ij}]}\delta_{u_k^{ij}}} \\
&= \Psi(f) + \Psi(g).
\end{aligned}$$

Finally, it remains to be seen that Ψ is the inverse of Φ. Let $f = \sum_{i=1}^{n} c_i 1_{[s_i, U_i]} \in A_R(S \ltimes X)$, where $[s_i, U_i]$ are pairwise disjoint. Then

$$\begin{aligned}
\Phi\circ\Psi(f) &= \Phi\circ\Psi\left(\sum_{i=1}^{n} c_i 1_{[s_i, U_i]}\right) = \Phi\left(\sum_{i=1}^{n}\overline{c_i 1_{\mathfrak{r}[s_i, U_i]}\delta s_i}\right) \\
&= \sum_{i=1}^{n}\phi\left(c_i 1_{\mathfrak{r}[s_i, U_i]}\delta s_i\right) = \sum_{i=1}^{n} c_i 1_{[s_i, U_i]} = f.
\end{aligned}$$

Now, let $f_s = \sum_{j=1}^{m} c_j 1_{L_j} \in D_s$, where $c_j \in R$ and L_j are pairwise disjoint compact-open subsets of X_s. Thus

$$\begin{aligned}
\Psi \circ \Phi\left(\overline{f_s\delta_s}\right) &= \Psi \circ \phi\left(f_s\delta_s\right) = \Psi \circ \phi\left(\sum_{j=1}^{m} c_j 1_{L_j}\delta_s\right) \\
&= \Psi\left(\sum_{j=1}^{m} \phi(c_j 1_{L_j}\delta_s)\right) = \Psi\left(\sum_{j=1}^{m} c_j 1_{[s,\theta_{s^*}(L_j)]}\right) \\
&= \sum_{j=1}^{m} \overline{c_j 1_{L_j}\delta_s} = \overline{f_s\delta_s}.
\end{aligned}$$

By additivity of Ψ and Φ, we have that $\Psi \circ \Phi(f) = f$, for all $f \in \mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\alpha S$. Therefore Φ is an R-isomorphism. $\square$

Remark 4.3.5. If each X_s is compact, then each D_s is unital, and so we can use the universal property of partial skew inverse semigroup algebras described in Theorem 1.6.19. In this case, one can prove that the pair (π, u), where $\pi : \mathcal{L}_{\mathrm{c}}(\mathrm{X}) \to A_R(S \ltimes X)$ is the embedding of $\mathcal{L}_{\mathrm{c}}(\mathrm{X}) = A_R(X)$ in $A_R(S \ltimes X)$, and $u : S \to A_R(S \ltimes X)$ is defined by $u(s) = 1_{[s, X_{s^*}]}$, is a covariant representation of α. Moreover,

$$(\pi \times u)(\overline{f_s\delta_s})(b) = \begin{cases} f_s(\mathfrak{r}(b)), & \text{if } b \in [s, X_{s^*}] \\ 0, & \text{if otherwise} \end{cases}$$

coincides with the map Φ in the proof of the above theorem.

On the other hand, we can construct the inverse of Φ using the universal property of Steinberg algebras as follows: Given $U \in (S \ltimes X)^a$, decompose U as a disjoint union $U = \bigcup_i [s_i, U_i]$ for certain $s_i \in S$ and $U_i \subseteq X_{s_i^*}$, and define

$$t_U = \sum_i \overline{1_{\theta_{s_i}(U_i)}\delta_{s_i}} = \sum_i \overline{1_{\mathfrak{r}([s_i, U_i]}\delta_{s_i}}.$$

The collection $\{t_U \mid U \in (S \ltimes X)^a\}$ is a representation of $(S \ltimes X)^a$ on $A_R(S \ltimes X)$ (see Definition 1.3.6). Then, by the universal property of $S \ltimes X$ (see Theorem 1.3.7), there exists a unique R-homomorphism $\Psi : A_R(S \ltimes X) \to \mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes S$ satisfying $\Psi(1_U) = t_U$, for all $U \in (S \ltimes X)^a$ (see the second part of the proof of Theorem 4.4.32 in [19]).

Remark 4.3.6. Notice that the isomorphism $\Phi : \mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes S \to A_R(S \ltimes X)$ obtained in the proof of Theorem 4.3.4 above maps the diagonal subalgebra $\mathcal{D}$ of $\mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes S$ (as in Definition 1.3.8) to the diagonal subalgebra $D_R(S \ltimes X)$ of $A_R(S \ltimes X)$ (as in Definition 1.6.15). Under the usual identifications of both of these diagonal subalgebras as $\mathcal{L}_{\mathrm{c}}(\mathrm{X}) = A_R(X)$, the isomorphism Φ restricts to the identity on $\mathcal{L}_{\mathrm{c}}(\mathrm{X}) = A_R(X)$.

Corollary 4.3.7. *Let $\mathcal{G}$ be an ample Hausdorff groupoid. Then the Steinberg Algebra $A_R(\mathcal{G})$ is isomorphic to the skew inverse semigroup algebras $\mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes_\mu \mathcal{G}^{op}$ and $\mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes_\eta \mathcal{G}^a$, where μ and η are the induced actions of the natural actions of $\mathcal{G}^{op}$ and $\mathcal{G}^a$ on $\mathcal{G}^{(0)}$ (as in Example 1.5.9).*

Proof. By Example 4.1.16, $\mathcal{G}$ is isomorphic to the groupoids of germs $\mathcal{G}^{op} \ltimes \mathcal{G}^{(0)}$ and $\mathcal{G}^a \ltimes \mathcal{G}^{(0)}$, given by the respective natural actions of $\mathcal{G}^{op}$ and $\mathcal{G}^a$ on $\mathcal{G}^{(0)}$. Then the desired result follows from Theorem 4.3.4. □

It is interesting to note that the skew algebras $\mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes \mathcal{G}^{op}$ and $\mathcal{L}_c(\mathcal{G}^{(0)}) \rtimes \mathcal{G}^a$ arise from *actions* and not simply partial action as in the previous theorem. Further, using Theorem 4.3.4 and Corollary 4.3.7 for the groupoid of germs of a partial action, we obtain

$$\mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\alpha S \simeq A_R(S \ltimes X) \simeq \mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\eta (S \ltimes X)^a,$$

where η is the induced action of the natural action of $(S \ltimes X)^a$ on $(S \ltimes X)^{(0)} \simeq X$.

4.4 Constructing a Steinberg algebra from a partial skew inverse semigroup algebra

In Section 4.3 we saw that the Steinberg algebra of an ample Hausdorff groupoid of germs can be seen as a partial skew inverse semigroup algebra. In this section we will be interested in the opposite direction, that is, to characterize partial skew inverse semigroup algebras of the form $\mathcal{L}_{\mathrm{c}}(\mathrm{X}) \rtimes_\alpha S$ as Steinberg algebras $A_R(S \ltimes_\theta S)$ in such a way that α is induced by θ. To do this, we will prove that under certain conditions we can obtain a topological partial action of S on X from the action α.

Some results of this nature are already known. For example, on the C*-algebras level, if X is a locally compact Hausdorff topological space then every closed ideal of $C_0(X)$ is the form $C_0(U) = \{f \in C_0(X) \mid \mathrm{supp}(f) \subseteq U\}$ for some (unique) open set U of X, and the Gelfand-Naimark Theorem implies that every $\mathbb{C}$-isomorphism $T : C_0(U) \to C_0(V)$ between two such ideals is of the form $T(f) = f \circ \phi$ for some (unique) homeomorphism $\phi : V \to U$. This gives a one-to-one correspondence between algebraic partial actions of a group G on $C_0(X)$ and topological partial actions of G on X.

In [4], a similar relation is shown at the purely algebraic level. More precisely, let $\mathbb{K}$ be a field and denote by $\mathcal{F}_0(X)$ the algebra of all functions $X \to \mathbb{K}$ with finite support, endowed with pointwise operations. Then there is a bijection between the non-zero ideals of $\mathcal{F}_0(X)$ and the non-empty subsets of X, and, moreover, there is a one-to-one correspondence between the partial actions of a group G on X and the partial actions of G on $\mathcal{F}_0(X)$.

In order to find a one-to-one correspondence between topological partial actions $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in X_s})$ of S on X and algebraic partial actions $\alpha = (\{D_s\}_{s\in S}, \{\alpha_s\}_{s\in S})$ of S on $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$, we will need a few preliminary results.

A unital commutative ring that has only trivial idempotents will be called an *indecomposable ring.*

Proposition 4.4.1. *Let R be an indecomposable ring.* $\Gamma : \mathcal{L}_c(Y) \to \mathcal{L}_{\mathrm{c}}(\mathrm{X})$ *is an R-isomorphism if, and only if, there exists a unique homeomorphism $\varphi : X \to Y$ such that* $\Gamma(f) = f \circ \varphi$*, for all* $f \in \mathcal{L}_{\mathrm{c}}(\mathrm{X})$*.*

Proof. The "if" part is straightforward, and for the converse we will use [18, Theorem 1.19]. For both $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$ or $\mathcal{L}(Y)$ we consider the "disjointness" relation $\perp$, given by

$$f \perp g \quad \text{if, and only if,} \quad \mathrm{supp}(f) \cap \mathrm{supp}(g) = \varnothing,$$

which in this case coincides with the "strong disjointness" relation of [18, Definition 1.1(2)] since the supports of all functions that we consider are clopen.

Suppose then that $\Gamma : \mathcal{L}_c(Y) \to \mathcal{L}_{\mathrm{c}}(\mathrm{X})$ is an R-isomorphism. Notice that, since R has only trivial idempotents, the idempotents of $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$ and $\mathcal{L}_c(Y)$ are precisely the characteristic functions of compact-open subsets of X, respectively. This implies that if f and g are idempotent, then

$$f \perp g \quad \text{if, and only if,} \quad fg = 0.$$

Therefore, if f and g are idempotents of $\mathcal{L}_c(Y)$, then

$$f \perp g \quad \text{if, and only if,} \quad \Gamma(f) \perp \Gamma(g).$$

For general elements $f, g \in \mathcal{L}_c(Y)$, we have $f \perp g$ if, and only if, there are idempotent elements $f_1, \ldots, f_n$, $g_1, \ldots, g_m$, and $a_1, \ldots, a_n$, $b_1, \ldots, b_m \in R$ such that

$$f = \sum_{i=1}^{n} a_i f_i, \quad g = \sum_{j=1}^{m} b_j g_j, \quad \text{and } f_i \perp g_j, \text{ for all pair } i, j. \tag{4.6}$$

Indeed, if condition (4.6) is satisfied, then

$$\operatorname{supp}(f) \cap \operatorname{supp}(g) \subseteq \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} \operatorname{supp}(f_i) \cap \operatorname{supp}(g_j) = \varnothing.$$

In the converse direction we assume $f \neq 0$ and $g \neq 0$, and take an enumeration $\{a_1, \ldots, a_n\} = f(Y) \setminus \{0\}$ and $f_i = 1_{f^{-1}(a_i)}$, and construct b_j and g_j similarly, so that the conditions in (4.6) are satisfied.

The same type of condition as in (4.6) describes disjointness of elements of $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$, and so we can conclude that

$$f \perp g \quad \text{if, and only if,} \quad \Gamma(f) \perp \Gamma(g).$$

By [18, Theorem 1.19] there is a unique Γ-homeomorphism $\varphi : X \to Y$ that

$$\varphi(\operatorname{supp} \Gamma(f)) = \operatorname{supp} f, \qquad \text{for all } f \in \mathcal{L}(Y). \tag{4.7}$$

Now, if $f, g \in \mathcal{L}(Y)$ and $x \in X$ then

$$\begin{aligned} f(\varphi(x)) \neq 0 &\iff \varphi(x) \in \operatorname{supp}(f) = \varphi(\operatorname{supp}(\Gamma(f))) \\ &\iff x \in \operatorname{supp}(\Gamma(f)) \iff \Gamma(f)(x) \neq 0. \end{aligned}$$

Hence, using additivity of Γ,

$$f(\varphi(x)) = g(\varphi(x)) \iff \Gamma(f(x)) = \Gamma(g(x)).$$

By [18, Proposition 2.6], Γ is φ-basic and there is a unique (φ, Γ)-transform $\chi : X \times R \to R$ such that

$$\Gamma(f(x)) = \chi(x, f(\varphi(x))), \quad \text{for all } f \in \mathcal{L}(Y) \text{ and } x \in X.$$

Now note that since Γ is an R-isomorphism and the operations are pointwise, then for every fixed element $x \in X$ the map $\chi(x, \cdot) : R \to R$ defined by $a \mapsto \chi(x, a)$ is an R-automorphism (by [18, Proposition 2.9]). Since the identity map is the unique R-automorphism of R, we can conclude that $\chi(x, a) = a$, for any $x \in X$ and any $a \in R$. Therefore,

$$\Gamma(f(x)) = \chi(x, f(\varphi(x))) = f(\varphi(x)),$$

for all $x \in X$, which is what we desired. Uniqueness of φ for which this formula holds follows from the uniqueness of φ with the property described in equation 4.7 (see [18, Theorem 1.19]). ☐

From the above propositions, we conclude that there is a bijective anti-homorphism between the group of all homeomorphism from X to Y, and the group of all R-isomorphisms from $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$ to $\mathcal{L}_c(Y)$, given by

$$\begin{aligned} T : Homeo(X, Y) &\longrightarrow Iso(\mathcal{L}_c(Y), \mathcal{L}_{\mathrm{c}}(\mathrm{X})) \\ \varphi &\longmapsto T_\varphi \end{aligned}$$

where $T_\varphi(f) = f \circ \varphi$.

We will prove that, when R is indecomposable, there is a bijection between ideals with local units of $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$ and open subsets of X. On one hand, if U is an open subset of X, then

$$\mathbf{I}(U) := \{f \in \mathcal{L}_{\mathrm{c}}(\mathrm{X}) : \operatorname{supp}(f) \subseteq U\} \simeq \mathcal{L}_c(U) \tag{4.8}$$

is an ideal of $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$ with local units. Indeed, if $f_1, \cdots, f_n \in \mathbf{I}(U)$ then the characteristic function 1_K, where $K = \bigcup_{i=1}^n \operatorname{supp}(f_i)$, is a local unit for these functions. Moreover, U is compact if, and only if, $\mathbf{I}(U)$ has identity, namely, the characteristic function 1_U is its identity.

Proposition 4.4.2. *Suppose that R is an indecomposable ring and X a locally compact, Hausdorff, and zero-dimensional space. Then the map*

$$U \mapsto \mathbf{I}(U)$$

is an order isomorphism between the lattices of open subsets of X and of ideals with local units of $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$. The inverse map is given by

$$I \mapsto \mathbf{U}(I) = \bigcup_{f \in I} \operatorname{supp}(f).$$

Proof. Given an ideal with local units I of $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$, let us show that $I = \mathbf{I}(\mathbf{U}(I))$. The inclusion $I \subseteq \mathbf{I}(\mathbf{U}(I))$ follows immediately from the definitions of $\mathbf{I}$ and $\mathbf{U}$. For the converse inclusion, suppose $f \in \mathcal{L}_{\mathrm{c}}(\mathrm{X})$ and

$$\operatorname{supp}(f) \subseteq \mathbf{U}(I) = \bigcup_{g \in I} \operatorname{supp}(g).$$

By compactness of $\operatorname{supp}(f)$, there are $g_1, \ldots, g_n \in I$ with $\operatorname{supp}(f) \subseteq \bigcup_{i=1}^n \operatorname{supp}(g_i)$. Notice that if $e \in I$ is a local unit for $g_1, \ldots, g_n$, then $e = 1_C$, for some open-compact subset C of X because e is idempotent and R only has trivial idempotents. Since e is a local unit for $g_1, \ldots, g_n$, we get that

$$\operatorname{supp}(f) \subseteq \bigcup_{i=1}^n \operatorname{supp}(g_i) \subseteq C,$$

and so, $f = f 1_C = fe \in I$. This proves that $\mathbf{I}(\mathbf{U}(I)) = I$.

For the converse, given an open subset U of X we need to check that $U = \mathbf{U}(\mathbf{I}(U))$. The inclusion $\mathbf{U}(\mathbf{I}(U)) \subseteq U$ is also immediate from the definitions of $\mathbf{I}$ and $\mathbf{U}$. If $x \in U$, simply take any compact-open subset V with $x \in V \subseteq U$, so $1_V \in \mathbf{I}(U)$ and

$$x \in \operatorname{supp}(1_V) \subseteq \mathbf{U}(\mathbf{I}(U)),$$

which proves that $U = \mathbf{U}(\mathbf{I}(U))$. $\square$

Proposition 4.4.3. *Suppose that R is an indecomposable ring and X is a locally compact, Hausdorff, and zero-dimensional space. If $\alpha = (\{D_s\}_{s\in G}, \{\alpha_s\}_{s\in S})$ is a partial action of S on the R-algebra $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$ for which each ideal D_s has local units, then there is a partial action $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ of S on X which induces α.*

Proof. Let $\alpha = (\{D_s\}_{s\in G}, \{\alpha_s\}_{s\in S})$ be a partial action of S in $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$ satisfying the hypotheses above. By Proposition 4.4.2, for each $s \in S$ there is an open subset $X_s \subseteq X$ such that

$$D_s = \mathbf{I}(X_s) = \{f \in \mathcal{L}_{\mathrm{c}}(\mathrm{X}) \mid \mathrm{supp}(f) \subseteq X_s\}\,.$$

By Proposition 4.4.1, for each isomorphism

$$\alpha_s : \mathcal{L}_c(X_{s^*}) \simeq D_{s^*} \to D_s = \mathcal{L}_c(X_s),$$

there is a unique homeomorphism $\theta_{s^*} : X_s \to X_{s^*}$ such that

$$\alpha_s(f) = f \circ \theta_{s^*}, \quad \text{for all } f \in D_{s^*} \simeq \mathcal{L}_c(X_{s^*}).$$

So we simply let $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$, and it is clear that, as long as θ is indeed a partial action, then α is induced by θ.

To finish the proof we need to we show that θ is indeed a partial action. By its very definition, each X_s is open in X and $\theta_s : X_{s^*} \to X_s$ is a homeomorphism. Non-degeneracy of θ can be proven as follows:

Let $x \in X$ and $f \in \mathcal{L}_{\mathrm{c}}(\mathrm{X})$ such that $x \in \mathrm{supp}(f)$. Since we can write f as $f = \sum_{i=1}^n f_i$ for certain elements $s_i \in S$ and $f_i \in D_{s_i}$, we get that

$$\mathrm{supp}(f) \subseteq \bigcup_{i=1}^n \mathrm{supp}(f_i) \subseteq \bigcup_{i=1}^n X_{s_i},$$

and so $x \in X_{s_i}$ for some i. This proves that $X = \bigcup_{s\in S} X_s$.

To conclude that θ is a partial action we will check the conditions of Proposition 1.5.2.

(a) Given $s \in S$, $\alpha_{s^*} \circ \alpha_s$ is the identity on D_{s^*}, however for all $f \in D_{s^*} \simeq \mathcal{L}(X_{s^*})$,

$$f \circ \mathrm{id}_{X_{s^*}} = f = \alpha_{s^*}(\alpha_s(f)) = \alpha_{s^*}(f \circ \theta_{s^*}) = f \circ (\theta_{s^*} \circ \theta_s),$$

so the uniqueness part of Proposition 4.4.1 implies that $\theta_{s^*} \circ \theta_s = \mathrm{id}_{X_{s^*}}$, that is, $\theta_{s^*} = \theta_s^{-1}$.

(b) Let $s, t \in S$. Notice that,

$$\begin{aligned} \mathcal{L}_c(\theta_{t^*}(X_t \cap X_{s^*})) = \alpha_{t^*}(\mathcal{L}_c(X_t \cap X_{s^*})) &= \alpha_{t^*}(\mathcal{L}_c(X_t) \cap \mathcal{L}_c(X_{s^*})) \\ &\overset{(*)}{=} \mathcal{L}_c(X_{t^*}) \cap \mathcal{L}_c(X_{(st)^*}) = \mathcal{L}_c(X_{t^*} \cap X_{(st)^*}), \end{aligned}$$

where the equalities marked by $(*)$ follow because α is a partial action (under the usual identification $\mathcal{L}_c(U) \simeq \mathbf{I}(U)$). Therefore,

$$\theta_{t^*}(X_t \cap X_{s^*}) = X_{t^*} \cap X_{(st)^*}.$$

(c) Since α is a partial action we get that $\alpha_{s^*}(\alpha_{t^*}(f)) = \alpha_{s^*t^*}(f)$, for all $f \in \mathcal{L}_c(X_t \cap X_{ts})$. But this implies that

$$f(\theta_t \circ \theta_s(x)) = (\alpha_{s^*} \circ \alpha_{t^*}(f))(x) = (\alpha_{(ts)^*}(f))(x) = f(\theta_{ts}(x)),$$

for all $x \in X_{s^*} \cap X_{(ts)^*}$. The uniqueness part of Proposition 4.4.1 implies that $\theta_t \circ \theta_s = \theta_{st}$, for all $x \in X_{s^*} \cap X_{(ts)^*}$. □

So, from an algebraic partial action α on $\mathcal{L}_c(\mathrm{X})$ we managed to obtain an appropriate topological partial action θ on X. In order to identify the skew inverse semigroup algebra of α with the Steinberg algebra of the groupoid of germs of θ, we need to guarantee that the conditions of Theorem 4.3.4 are satisfied.

First we describe the ideals associated with clopen sets algebraically. The following definition is an algebraic version of [8, Definition 1.5.9].

Definition 4.4.4. A *conditional expectation* of an R-algebra A onto a subalgebra B is an R-module map $E : A \to B$ such that

(i) $E(b) = b$, for all $b \in B$ (i.e., E is a projection onto B),

(ii) For all $a \in A$ and $b \in B$, $E(ba) = bE(a)$ and $E(ab) = E(a)b$ (i.e., E is a B-bimodule morphism).

Remark 4.4.5. If B is s-unital then condition (ii) above can be substituted by

(ii') For all $a \in A$ and $b, b' \in B$, $E(bab') = bE(a)b'$.

Indeed, let $a \in A$, $b \in B$ and let $u \in B$ such that

$uE(ab) = E(ab) = E(ab)u$, $\;bu = b = ub\;$ and $\;uE(a) = E(a) = E(a)u$.

Then

$$E(ab) = uE(ab)u \overset{(ii')}{=} E(uabu) = E(uab) \overset{(ii')}{=} uE(a)b = E(a)b$$

and similarly $E(ba) = bE(a)$, so (ii) is satisfied.

Lemma 4.4.6. *Let R be an indecomposable ring, X a locally compact, Hausdorff, and zero-dimensional space and U an open subset of X. Then the following are equivalent:*

(i) U is clopen,

(ii) There exists a conditional expectation of $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$ onto $I(U)$,

(iii) There exists an ideal J of $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$ such that $\mathcal{L}_{\mathrm{c}}(\mathrm{X}) = \mathbf{I}(U) \oplus J$ (as R-modules).

Proof. (i) $\Rightarrow$ (ii): If U is clopen then 1_U is continuous, and it is easy to see that the map $E : \mathcal{L}_{\mathrm{c}}(\mathrm{X}) \to \mathbf{I}(U)$, defined by $E(f) = f1_U$ is a conditional expectation.

(ii) $\Rightarrow$ (iii): Suppose $E : \mathcal{L}_{\mathrm{c}}(\mathrm{X}) \to \mathbf{I}(U)$ is a conditional expectation, and define $J = \{f - E(f) \mid f \in \mathcal{L}_{\mathrm{c}}(\mathrm{X})\}$. Then J is an R-submodule of $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$ such that $\mathcal{L}_{\mathrm{c}}(\mathrm{X}) = \mathbf{I}(U) \oplus J$, since E is a projection onto $\mathbf{I}(U)$. In order to prove that J is an ideal, take $f, g \in \mathcal{L}_{\mathrm{c}}(\mathrm{X})$. Since $\mathbf{I}(U)$ is an ideal of $\mathcal{L}_{\mathrm{c}}(\mathrm{X})$ with local unit, there is $u \in \mathbf{I}(U)$ such that $E(fg)u = E(fg)$ and $E(f)g = [E(f)g]u$. Then

$$E(fg) = E(fg)u = E(fgu) = E(f)gu = E(f)g$$

so

$$(f - E(f))g = fg - E(f)g = fg - E(fg) \in J$$

and therefore J is a right ideal. Similarly it is a left ideal.

(iii) $\Rightarrow$ (iii): Suppose that $\mathcal{L}_c(\mathrm{X}) = \mathbf{I}(U) \oplus J$ for some ideal J. Let us show that $J \subseteq \mathbf{I}(X \setminus \overline{U})$. Indeed, if $f \in J$ and $x \in \overline{U}$, then f is constant on a neighbourhood V of x. Choose $y \in U \cap V$, and W a compact-open neighbourhood satisfying $y \in W \subseteq U \cap V$. Since $\mathbf{I}(U)$ and J are complementary ideals, then $f1_W = 0$ and in particular

$$f(x) = f(y) = (f1_W)(y) = 0.$$

Therefore $J \subseteq \mathbf{I}(X \setminus \overline{U})$.

We can now prove that U is clopen. Given $x \in \overline{U}$, let V be any compact-open neighbourhood of x. By hypothesis, we may write $1_V = f + g$ for some $f \in \mathbf{I}(U)$ and $g \in J \subseteq \mathbf{I}(X \setminus \overline{U})$, so $1 = 1_V(x) = f(x) + g(x) = f(x)$. In particular $x \in \mathrm{supp}(f) \subseteq U$. Therefore U is clopen. □

By Lemma 4.4.6 and Theorems 4.1.19 and 4.3.4, we conclude the following:

Theorem 4.4.7. *Let S be an inverse semigroup which is a weak semilattice, R be an indecomposable ring and X a zero-dimensional, locally compact Hausdorff space. Let $\alpha = (\{D_s\}_{s\in S}, \{\alpha_s\}_{s\in S})$ be an algebraic partial action of S on $\mathcal{L}_c(\mathrm{X})$ where each ideal D_s has local units and satisfies one of the equivalent conditions of Lemma 4.4.6.*

Then $\mathcal{L}_c(\mathrm{X}) \ltimes_\alpha X$ is isomorphic to a Steinberg algebra $A_R(S \ltimes_\theta X)$, where θ is a topological partial action of S on X which induces α.

4.5 Continuous orbit equivalence

In [52], Li characterized continuous orbit equivalence of topologically free partial group actions in terms of diagonal-preserving isomorphisms of the associated C*-crossed products. In Section 2.2, we characterized diagonal-preserving isomorphisms of the associated skew group algebras. Now, we will extend the notion of continuous orbit equivalence to partial actions of inverse semigroups and characterize orbit equivalence of topologically principal systems in terms of diagonal-preserving isomorphisms of the associated skew inverse semigroup algebras.

Recall that given a partial action $\theta = \left(\{X_s\}_{s\in S}, \{\alpha_s\}_{s\in S}\right)$ of an inverse semigroup S on a topological space X, we denote

$$S * X = \{(s,x) \in S \times X \mid x \in X_{s^*}\},$$

and for $x \in X$ fixed,

$$S_x = \{s \in S \mid x \in X_{s^*}\}.$$

Definition 4.5.1. Let X, Y be topological spaces and let S, T be inverse semigroups. We say that two partial actions $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ and $\gamma = (\{Y_t\}_{t\in T}, \{\gamma_t\}_{t\in T})$ on X and Y, respectively, are *continuously orbit equivalent* if there are a homeomorphism

$$\varphi : X \longrightarrow Y$$

and continuous maps

$$a : S * X \longrightarrow T \qquad \text{and} \qquad b : T * Y \longrightarrow S$$

such that for all $x \in X$, $s \in S_x$, $y \in Y$ and $t \in T_y$,

(i) $\varphi(\theta_s(x)) = \gamma_{a(s,x)}(\varphi(x))$,

(ii) $\varphi^{-1}(\gamma_t(y)) = \theta_{b(t,y)}(\varphi^{-1}(y))$.

Implicitly, we require that $a(g,x) \in T_{\varphi(x)}$ and $b(t,y) \in S_{\varphi^{-1}(y)}$. We will call the triple (φ, a, b) a continuous orbit equivalence between θ and γ.

Our next goal is to prove that continuous orbit equivalence of topological principal actions is equivalent to isomorphism of the respective groupoids. This is a generalization of the analogous results for groups ([53, Theorem 1.2] and [52, Theorem 2.7]).

To this end, we need to prove some identities related to how the functions a and b above preserve the structure of S and T.

Lemma 4.5.2. *Let $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ and $\gamma = (\{Y_t\}_{t\in T}, \{\gamma_t\}_{t\in T})$ be topologically principal partial actions, and (φ, a, b) be a continuous orbit equivalence from θ to γ. Assume that X and Y are Hausdorff. Then the following implications hold:*

(a) $[s_1, x] = [s_2, x]$ implies that $[a(s_1, x), \varphi(x)] = [a(s_2, x), \varphi(x)]$, for all $x \in X$ and $s_1, s_2 \in S_x$.

(b) $[a(s_1 s_2, x), \varphi(x)] = [a(s_1, \theta_{s_2}(x))a(s_2, x), \varphi(x)]$, for all $x \in X$ and $s_2 \in S_x$ and $s_1 \in S_{\theta_{s_2}(x)}$.

(c) $[b(a(s, x), \varphi(x)), x] = [s, x]$, for all $x \in X$ and $s \in S_x$.

Analogous statements hold with (φ^{-1}, a, b) in place of (φ, b, a).

Proof. (a) Let $x \in X$ and $s_1, s_2 \in S_x$. Suppose that $[s_1, x] = [s_2, x]$. First, choose $s \leq s_1, s_2$ such that $x \in X_{s^*}$. Then choose an open neighbourhood $U \subseteq X_{s^*}$ of $x \in X$ such that

$$a(s_1, \tilde{x}) = a(s_1, x) \qquad \text{and} \qquad a(s_2, \tilde{x}) = a(s_2, x).$$

Then for all $\tilde{x} \in U \cap \varphi^{-1}(\Lambda(\gamma))$ and for $i = 1, 2$, we have $[s_i, \tilde{x}] = [s, \tilde{x}]$, so

$$\begin{aligned}\gamma_{a(s_i,x)}(\varphi(\tilde{x}) &= \gamma_{a(s_i,\widetilde{x})}(\varphi(\tilde{x})) = \varphi(\theta_{s_i}(\tilde{x})) \\ &= \varphi(\mathfrak{r}[s_i, \tilde{x}]) = \varphi(\mathfrak{r}[s, \tilde{x}]).\end{aligned}$$

It follows that $\gamma_{a(s_1,x)}(\varphi(\tilde{x})) = \gamma_{a(s_2,x)}(\varphi(\tilde{x}))$. As $\varphi(\tilde{x}) \in \Lambda(\gamma)$, the description of $\Lambda(\gamma)$ as in Lemma 3.2.8 implies that

$$[a(s_1, x), \varphi(\tilde{x})] = [a(s_2, x), \varphi(\tilde{x})], \quad \text{for all } x \in U \cap \varphi^{-1}(\Lambda(\gamma)). \tag{4.9}$$

In particular, $[a(s_i, x), \varphi(\tilde{x})]$ and $[a(s_i, x), \varphi(x)]$ belong to the bisection $[a(s_1, x), \varphi(U)]$, which is Hausdorff.

Since γ is topologically principal, $\Lambda(\gamma)$ is dense in Y, so $U \cap \varphi^{-1}(\Lambda(\gamma))$ is dense in U and therefore we may take the limit $\widetilde{x} \to x$ in Equation (4.9) and conclude that $[a(s_1, x), \varphi(x)] = [a(s_2, x), \varphi(x)]$, limits are unique in Hausdorff spaces.

(b) Choose an open neighbourhood U of $x \in X$ such that

$$a(s_1 s_2, \tilde{x}) = a(s_1 s_2, x), \quad a(s_1, \theta_{s_2}(\tilde{x})) = a(s_1, \theta_{s_2}(x))$$

$$\text{and} \quad a(s_2, \tilde{x}) = a(s_2, x), \qquad \text{for all } \tilde{x} \in U.$$

Then, for every $\tilde{x} \in U \cap \varphi^{-1}(\Lambda(\gamma))$,

$$\begin{aligned}\gamma_{a(s_1 s_2, \tilde{x})}(\varphi(\tilde{x})) &= \varphi(\theta_{s_1 s_2}(\tilde{x})) = \varphi(\theta_{s_1}(\theta_{s_2}(\tilde{x})) \\ &= \gamma_{a(s_1, \theta_{s_2}(\tilde{x}))}(\varphi(\theta_{s_2}(\tilde{x}))) \\ &= \gamma_{a(s_1, \theta_{s_2}(\tilde{x}))}(\gamma_{a(s_2, \tilde{x})}(\varphi(\tilde{x}))) \\ &= \gamma_{a(s_1, \theta_{s_2}(\tilde{x}))a(s_2, \tilde{x})}(\varphi(\tilde{x}))\end{aligned}$$

so, the same way as in item (a), the given property of U and the definition of $\Lambda(\gamma)$ imply that

$$[a(s_1 s_2, x), \varphi(\tilde{x})] = [a(s_1, \theta_{s_2}(x))a(s_2, x), \varphi(\tilde{x})].$$

Since $\varphi^{-1}(\Lambda(\gamma)) \cap U$ is dense in the Hausdorff space U, we conclude that $[a(s_1 s_2, x), \varphi(x)] = [a(s_1, \theta_{s_2}(x))a(s_2, x), \varphi(x)]$ by taking the limit $\tilde{x} \to x$.

(c) Similarly to the previous items, take neighbourhoods U of x and V of $\varphi(x)$ such that

$$a(s, \tilde{x}) = a(s, x) \qquad \text{and} \qquad b(a(s, x), \tilde{y}) = b(a(s, x), \varphi(x))$$

whenever $\tilde{x} \in U$ and $\tilde{y} \in V$. Then for all $\tilde{x} \in U \cap \varphi^{-1}(V) \cap \Lambda(\theta)$,

$$\theta_{b(a(s,\tilde{x}), \varphi(\tilde{x}))}(\tilde{x}) = \varphi^{-1}(\gamma_{a(s,\tilde{x})}(\varphi(\tilde{x}))) = \varphi^{-1}(\varphi(\theta_s(\tilde{x}))) = \theta_s(x)$$

so the properties of U, V and $\Lambda(\theta)$ yield $[b(a(s, x), \varphi(x)), \tilde{x}] = [s, \tilde{x}]$ and again taking $\tilde{x} \to x$ gives us the desired result. $\square$

Theorem 4.5.3. *Let* $\theta = (\{X_s\}_{s \in S}, \{\theta_s\}_{s \in S})$ *and* $\gamma = (\{Y_t\}_{t \in T}, \{\gamma_t\}_{t \in T})$ *be topologically principal, continuously orbit equivalent partial actions, and suppose that* X *and* Y *are Hausdorff. Then* $S \ltimes X$ *and* $T \ltimes Y$ *are isomorphic as topological groupoids.*

Proof. Let (φ, a, b) be e a continuous orbit equivalence between θ and γ. By Lemma 4.5.2 (a), the map

$$\Phi\colon S \ltimes X \to T \ltimes Y, \qquad \Phi[s,x] = [a(s,x), \varphi(x)]$$

is well-defined, and by Lemma 4.5.2 (a) it is a groupoid homomorphism. Since a and φ are continuous it follows that Φ is continuous. Similarly, the map

$$\Psi\colon T \ltimes Y \to S \ltimes X, \qquad \Psi[t,y] = [b(t,y), \varphi^{-1}(y)]$$

is a continuous groupoid morphism. Φ and Ψ are inverses of each other due to Lemma 4.5.2 (c). $\square$

We will now be interested in constructing an orbit equivalence for two actions from an isomorphism of the corresponding groupoids of germs. Note that in general the continuous maps a and b in the definition of continuous orbit equivalence take values in discrete spaces (namely, the corresponding semigroups), and so X and Y are required to have sufficiently many sets for a continuous orbit equivalence between the corresponding partial actions to exist. Since we will now be interested in constructing an orbit equivalence for two actions from an isomorphism of the corresponding groupoids of germs, we will need to concentrate on spaces which have sufficiently many clopen sets and partial actions which respect this structure.

Definition 4.5.4 ([73, Definition 5.2])**.** We say that a partial action $\theta = \left(\{X_s\}_{s\in S}, \{\alpha_s\}_{s\in S}\right)$ of an inverse semigroup S on a topological space X is *ample* if

(i) X is locally compact, Hausdorff and totally disconnected;

(ii) each X_s is a compact-open subset of X.

Lemma 4.5.5. *Let* $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ *and* $\gamma = (\{Y_t\}_{t\in T}, \{\gamma_t\}_{t\in T})$ *be ample partial actions. Let* $\Phi : S \ltimes X \to T \ltimes Y$ *be a topological isomorphism, and consider* $\varphi = \Phi|_X : X \to Y$*. Then, for every* s *in* S*,* *there are elements $t_1, \ldots, t_n$ in T and there are disjoint compact-open subsets $K_1, \ldots K_n$ of Y such that:*

(a) $K_i \subseteq Y_{t_i{}^*}$,

(b) $\varphi(X_{s^*}) = \bigcup_{i=1}^n K_i$,

(c) $\{\varphi^{-1}(K_i) \mid i = 1, \cdots, n\}$ *is a partition to* X,

(d) for every i and for every $x \in \varphi^{-1}(K_i)$, one has that $\Phi([s,x]) = [t_i, \varphi(x)]$.

Proof. Since $[s, X_{s^*}]$ is a compact-open bisection, then $\Phi([s, X_{s^*}])$ is a compact-open bisection in $T \ltimes Y$, so there are elements $t_1, \ldots, t_n$ of T and disjoint compact-open subsets $K_1, \ldots, K_n$ of Y with $K_i \in Y_{t_i^*}$ such that

$$\Phi([s, X_{s^*}]) = \bigcup_{i=1}^{n} [t_i, K_i]. \tag{4.10}$$

Then item (a) is trivially satisfied. Taking sources on both sides of (4.10) yields

$$\varphi(X_{s_*}) = \Phi(\mathfrak{s}([s, X_{s^*}])) = \mathfrak{s}(\Phi([s, X_{s^*}])) = \mathfrak{s}\left(\bigcup_i [t_i, K_i]\right) = \bigcup_i K_i,$$

and item (b) is proved. Item (c) follows by (b) and the fact that φ is injective.

In order to prove (d), consider $i \in \{1, \cdots, n\}$ and $x \in \varphi^{-1}(K_i)$. From Equation 4.10, $\Phi([s,x]) \in [t_j, K_j]$, for some $j \in \{1, \cdots, n\}$, and in particular, $\mathfrak{s}([s,x]) \in K_j$. As $K_1, \cdots, K_n$ are pairwise disjoint, we have

$$\mathfrak{s}(\Phi([s,x])) = \Phi(\mathfrak{s}([s,x])) = \varphi(x) \in K_i,$$

and hence $K_i = K_j$. Therefore, $\Phi([s,x]) = [t_i, \varphi(x)]$. $\square$

We are now ready to prove that topological isomorphisms between Hausdorff groupoids of germs yield a continuous orbit equivalence between the respective partial actions.

Theorem 4.5.6. *Let* $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ *and* $\gamma = (\{Y_t\}_{t\in T}, \{\gamma_t\}_{t\in T})$ *be ample partial actions. Then* θ *and* γ *are continuously orbit equivalent.*

Proof. Let $\Phi : S \ltimes X \to T \ltimes Y$ be an isomorphism of topological groupoids. Then

$$\varphi := \Phi|_X : X \to Y$$

is a homeomorphism.

Given $s \in S$, choose $t_1, \dots, t_n \in T$ and compact-open subsets $K_1, \dots, K_n \subseteq Y$ satisfying properties (a) - (d) of Lemma 4.5.5. Define $a(s,x) = t_i$ whenever $x \in K_i$, so that a is a continuous map on $\{s\} \times X_{s^*}$. This way, we define a continuous function a on all of $S * X = \bigcup_{s\in S} \{s\} \times X_{s^*}$. Let's show that a satisfies the desired property for a continuous orbit equivalence between θ and γ: Given $(s,x) \in S * X$, let $t = a(s,x)$. Then the definition of $a(s,x)$ implies

$$\begin{aligned}\gamma_{a(s,x)}(\varphi(x)) &= \mathfrak{r}[a(s,x), \varphi(x)] = \mathfrak{r}[t, \varphi(x)] = \mathfrak{r}(\Phi[s,x]) = \Phi(\mathfrak{r}[s,x]) \\ &= \varphi(\theta_s(x))\end{aligned}$$

as desired.

Proceeding similarly with Φ^{-1} in place of Φ, we construct a function $b : T * Y \to S$ with analogous properties, so that a and b describe a continuous orbit equivalence between θ and γ. □

By Proposition 4.1.19, Example 4.1.16 and Theorem 4.5.6 we conclude the following:

Corollary 4.5.7. *Let* $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ *be an ample partial action of* S *on* X. *Let* $\tau = (\{\mathfrak{r}(B)\}_{B\in(S\ltimes X)^a}, \{\tau_B\}_{B\in(S\ltimes X)^a})$ *be the canonical action of* $(S \ltimes_\theta X)^a$ *on* X *(see Example 1.5.9). Then* θ *and* τ *are continuously orbit equivalent.*

We want to connect the equivalence between continuously orbit equivalent partial actions, isomorphisms of groupoids of germs and diagonal-preserving isomorphisms of certain algebras. However, first, let us consider another semigroup associated to a groupoid, which we will call topological full pseudogroup.

Topological full groups, which were introduced in [35, 36], are defined as follows: If G is a group acting by homeomorphisms on the Cantor set X, then the full group of this action consists of all self-homeomorphisms of X which locally act as some element of G. Topological full groups can then be generalized to the setting of étale groupoids $\mathcal{G}$ on Cantor sets [55], on which the following isomorphism theorem holds: two étale groupoids over the Cantor set are isomorphic if, and only if, they have isomorphic topological full groups ([56, Theorem 5.1].

We will use a similar nomenclature to that of [55]. For each compact-open bisection U of an ample groupoid $\mathcal{G}$, we denote by τ_U the homeomorphism given by the canonical action of $\mathcal{G}^a$ on $\mathcal{G}^{(0)}$, namely

$$\tau_U = \mathfrak{r} \circ (\mathfrak{s}|_U^{-1}) : \mathfrak{s}(U) \to \mathfrak{r}(U).$$

In particular, we have that $U \mapsto \tau_U$ is an inverse semigroup homomorphism from $\mathcal{G}^a$ to $\mathcal{I}(\mathcal{G}^{(0)})$.

Definition 4.5.8. The *topological full pseudogroup* of an étale groupoid is the semigroup

$$[[\mathcal{G}]] = \{\tau_U \mid U \text{ compact-open bisection of } \mathcal{G}\}.$$

Example 4.5.9. Let $\theta = (\{X_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ be a partial action of an inverse semigroup S on a locally compact, Hausdorff, and zero-dimensional space X. The topological full pseudogroup $[[S \ltimes_\theta X]]$ is the set of all partial homeomorphisms $\varphi : U \to V$ $(U, V \subseteq X)$ for which there are $s_1, \ldots, s_n \in S$ and compact-open $U_1, \ldots, U_n$ such that

(i) $U = \bigcup_{i=1}^n U_i$,

(ii) $U_i \subseteq X_{s_i^*}$, for all $i \in \{1, \cdots, n\}$,

(iii) $\varphi|_{U_i} = \theta_{s_i}|_{U_i}$, for all $i \in \{1, \cdots, n\}$.

The theorem below was proven in [66, Corollary 3.3] when one considers open bisections instead of compact-open ones. In any case, we provide a short and direct proof of it.

Proposition 4.5.10. *Suppose $\mathcal{G}$ is an ample (possibly non-Hausdorff) groupoid. Then the homomorphism $\tau:\mathcal{G}^a\to[[\mathcal{G}]]$ is an isomorphism if, and only if, $\mathcal{G}$ is effective.*

Proof. First assume that $\mathcal{G}$ is effective, that is, $\mathcal{G}^{(0)}=\operatorname{int}(\operatorname{Iso}(\mathcal{G}))$. Let $U,V\in\mathcal{G}^a$ such that $\tau_U=\tau_V$. Then

$$\tau_{V^*U}=\tau_{V^*}\circ\tau_U=\tau_{V^*}\circ\tau_V=\mathrm{id}_{\mathfrak{s}(V)},$$

which means that $V^*U\subseteq\operatorname{Iso}(\mathcal{G})$. Since V^*U is open, we obtain $V^*U\subseteq\mathcal{G}^{(0)}$. Thus the domain of τ_{V^*U} is $\mathfrak{s}(V)=\mathfrak{s}(V^*U)=V^*U$, which implies $V=VV^*U\subseteq U$, and symetrically we obtain $U\subseteq V$. Thus τ is injective.

Conversely, suppose that $\mathcal{G}$ is not effective. Take any nonempty compact-open bisection $U\subseteq\operatorname{int}(\operatorname{Iso}(\mathcal{G}))$ which is not contained in $\mathcal{G}^{(0)}$. Then $U\neq\mathfrak{s}(U)$, but $\tau_U=\tau_{\mathfrak{s}(U)}$, and so, τ is not injective. □

Let us now summarize the connections between continuous orbit equivalence of partial actions, isomorphisms of groupoids of germs, isomorphisms of topological full pseudogroups, diagonal-preserving isomorphisms of Steinberg algebras, and consequently diagonal-preserving isomorphisms of the associated crossed products. To do so, we will use [70, Corollary 5.8], which is an improvement of [12, Theorem 3.1].

Note that each individual implication in the next theorem is valid under weaker hypotheses.

Theorem 4.5.11. *Let R be an indecomposable ring and let θ and γ be ample, topologically principal partial actions of inverse semigroups S and T on spaces X and Y, respectively, and suppose that the groupoids of germs $S\ltimes X$ and $T\ltimes Y$ are Hausdorff. Then the following are equivalent:*

(i) the partial actions θ and γ are continuously orbit equivalent;

(ii) the groupoids of germs $S \ltimes X$ and $T \ltimes Y$ are isomorphic;

(iii) the inverse semigroups $(S \ltimes X)^a$ and $(T \ltimes Y)^a$ are isomorphic;

(iv) the inverse semigroups $[[S \ltimes X]]$ and $[[T \ltimes Y]]$ are isomorphic;

(v) there exists a diagonal-preserving isomorphism between the Steinberg algebras $A_R(S \ltimes X)$ and $A_R(T \ltimes Y)$;

(vi) there exists a diagonal-preserving isomorphism between the partial skew inverse semigroup rings $\mathcal{L}_\mathrm{c}(\mathrm{X}) \rtimes S$ and $\mathcal{L}(Y) \rtimes T$.

Proof. *(i)* $\iff$ *(ii)* follows from Theorems 4.5.3 and 4.5.6,

(ii) $\iff$ *(iii)* follows from non-commutative Stone duality[2] (see [50, Theorem 3.23]),

(iii) $\iff$ *(iv)* follows from Proposition 4.5.10,

(i) $\iff$ *(v)* follows from [70, Corollary 5.8.],

(v) $\iff$ *(vi)* follows from Theorem 4.3.4. □

4.6 Aplication to Leavitt path algebras

In [9], the notion of continuous orbit equivalence for directed graphs was introduced, following Matsumoto's notion of continuous orbit equivalence for topological Markov shifts (see [54]). We will compare this notion with continuous orbit equivalence of the canonical actions of the inverse semigroups associated to graphs (see Example 1.5.13). A similar study was made by Li in [52], who considered the case of partial actions of free groups generated by edges of a graph. We reiterate that we do not make any assumptions on the second-countability of topological spaces, or countability of graphs.

In this section we will make constant use of the properties of the Examples 1.2.16, 1.3.10 and 1.5.13, as well as their notation.

[2] Note that Hausdorff Boolean groupoids of [50] correspond to ample Hausdorff groupoids.

First, we will show that the boundary path groupoid (see Example 1.2.16) of a directed graph is isomorphic to the groupoid of germs given by the action of Example 1.5.13.

Lemma 4.6.1. *Let $E = (E^0, E^1, r, s)$ be a directed graph. Then the groupoid of germs $\mathcal{S}_E \ltimes \partial E$, associated the action of the inverse semigroup $\mathcal{S}_E$ on the boundary path space ∂E, and the boundary path groupoid $\mathcal{G}_E$ are isomorphic as topological groupoids.*

Proof. Consider the map $\phi : \mathcal{G}_E \to \mathcal{S}_E \ltimes \partial_E$, defined by

$$\phi(\mu x, n, \nu x) = [(\mu, \nu), \nu x].$$

We need to check that ϕ is well-defined. Suppose that

$$(\mu x, n, \nu x) = (\zeta y, n, \eta y).$$

It follows that μ and ζ are comparable, as are ν and η. Moreover, either both μ and ν are subpaths of ζ and η, respectively, or the reverse is true.

By symmetry, let us assume that either both μ and ν are subpaths of ζ and η, respectively, say $\zeta = \mu p$ and $\eta = \nu q$. Since $\mu x = \zeta y$ and $\nu x = \eta y$ we obtain $x = py$ and hence $\mu p y = \mu q y$, and therefore $p = q$. In other words

$$(\mu, \nu) \leq (\zeta, \eta) \quad \text{or} \quad (\zeta, \eta) \leq (\mu, \nu),$$

and then ϕ is well-defined.

It is straightforward to check that ϕ is a homomorphism between groupoids. Notice that

$$\begin{aligned}
&\phi^{-1}\left([(\mu, \nu), Z(\mu, F)]\right) \\
&\quad = \phi^{-1}\left(\{[(\mu, \nu), \nu x] \mid x \in \partial E, s(x) = r(\mu) \text{ and } x_1 \notin F\}\right) \\
&\quad = \{(\mu x, n, \nu x) \mid x \in \partial E, s(x) = r(\mu) \text{ and } x_1 \notin F\} \\
&\quad = Z(\mu, \nu, F),
\end{aligned}$$

that is, every preimage of a basic open subset of $\mathcal{S}_E \ltimes \partial_E$ is a basic open subset of $\mathcal{G}_E$. Similarly, the image of a basic open $Z(\mu, \nu, F)$ of

$\mathcal{G}_E$ by ϕ is $[(\mu,\nu), Z(\mu,F)]$, which is also a basic open subset of $\mathcal{S}_E \ltimes \partial_E$. Therefore, ϕ is a continuous and open map.

In order to prove that ϕ is bijective, consider $\psi : \mathcal{S}_E \ltimes \partial E \to \mathcal{G}_E$ defined by

$$\psi([(\mu,\nu),\nu x]) = (\mu x, |\mu| - |\nu|, \nu x).$$

We need to check that ψ is well-defined. Suppose that

$$[(\mu,\nu),\nu x] = [(\zeta,\eta),\eta y] \in \mathcal{S}_E \ltimes \partial E.$$

Then $\nu x = \eta y$ and there is $(\alpha,\beta) \in \mathcal{S}_E$ such that $(\alpha,\beta) \leq (\mu,\nu),(\zeta,\eta)$ with $\nu x \in Z(\beta)$. From $(\alpha,\beta) \leq (\mu,\nu),(\zeta,\eta)$ there are $b,c \in E^\star$ such that $\alpha = \mu b = \zeta c$ and $\beta = \nu b = \eta c$. Thus

$$|\mu| - |\nu| = |\mu b| - |\nu b| = |\alpha| - |\beta| = |\zeta c| - |\eta c| = |\zeta| - |\eta|.$$

It remains to show that $\mu x = \zeta y$. From of $\nu x = \eta y$, there is $a \in E^\star$ such that $\nu = \eta a$ or $\eta = \nu a$. In the case that $\nu = \eta a$, we have that $\eta c = \nu b = \eta a b$ implies that $c = ab$, and so, $\eta a x = \nu x = \eta y$ implies that $ax = y$. Hence

$$\mu b = \eta c = \zeta a b \quad \Rightarrow \quad \mu = \zeta a \quad \Rightarrow \quad \mu x = \zeta a x = \zeta y.$$

The case $\eta = \nu a$ is similar. Therefore

$$(\mu x, |\mu| - |\nu|, \nu x) = (\zeta y, |\zeta| - |\eta|, \eta y),$$

and ψ is well-defined as required. Clearly ϕ is the inverse map of ψ. □

Recall that given a directed graph $E = (E^0, E^1, r, s)$ we denote by $\sigma_E : \partial E^{\geq 1} \to \partial E$ the one-sided shift map (see Equation 1.1).

Definition 4.6.2. [9, Definition 3.1] Two countable directed graphs $E = (E^0, E^1, r, s)$ and $F = (F^0, F^1, r, s)$ are *continuously orbit equivalent* if there exists a homeomorphism $\varphi : \partial E \to \partial F$ together with continuous maps $k, l : \partial E^{\geq 1} \to \mathbb{N}$ and $k', l' : \partial F^{\geq 1} \to \mathbb{N}$ such that

$$\sigma_F^{k(x)}(\varphi(\sigma_E(x))) = \sigma_F^{l(x)}(\varphi(x)), \text{ for all } x \in \partial E^{\geq 1}, \tag{4.11}$$

and

$$\sigma_E^{k'(y)}(\varphi^{-1}(\sigma_F(x))) = \sigma_E^{l'(y)}(\varphi^{-1}(y)), \text{ for all } y \in \partial F^{\geq 1}. \tag{4.12}$$

Recall that a *loop* or *cycle* in a graph E is a path $\mu \in E^\star$ such that $|\mu| \geq 1$ and $s(\mu) = r(\mu)$. An edge e is an exit to the cycle μ if there exists i such that $s(e) = s(\mu_i)$ and $e \neq \mu_i$. A graph is said to satisfy *condition (L)* if every loop has an exit.

The following is an analogue of [9, Proposition 2.3]. We provide a simple proof for completeness.

Proposition 4.6.3. *Let $E = (E^0, E^1, r, s)$ be a directed graph. Then E satisfies Condition (L) if and only if the canonical action θ of $\mathcal{S}_E$ on ∂E is topologically principal (or equivalently, $\mathcal{G}_E$ is topologically principal).*

Proof. Let us say that an element $x \in \partial E$ is *cyclic* if there exists $x' \in E^\star$ with $|x'| \geq 1$ such that $x = x'x$, or equivalently $x = x'x'x' \cdots$, and that x is *periodic* if $x = \nu y$ for some $\nu \in E^\star$ and some cyclic y.

First suppose that E satisfies Condition (L). Consider the set $X = (E^\star \cap \partial E) \cup \{x \in E^\infty : x \text{ is not periodic}\}$. Condition (L) implies that X is dense in ∂E. We are done by proving that $X \subseteq \Lambda(\theta)$. Suppose $(\mu, \nu) \in \mathcal{S}_E$ and $x = \nu y \in Z(\nu)$ is such that $\theta_{(\mu,\nu)}(x) = x$. Let us prove that $\mu = \nu$. We have

$$\mu y = \theta_{(\mu,\nu)}(x) = x = \nu y. \tag{4.13}$$

It follows that μ and ν are comparable, so to prove that $\mu = \nu$ it suffices to prove that $|\mu| = |\nu|$. Without loss of generality, let us assume that $\mu = \nu\mu'$ for some μ'. From (4.13) we obtain $y = \mu' y$. However, y is not cyclic, since x is not periodic, so $|\mu'| = 0$, and $|\mu| = |\nu\mu'| = |\nu|$. We conclude that θ is topologically principal.

Conversely, suppose E does not satisfy Condition (L), and let y be any loop in E without exit. The element $x = yyy \cdots$ is isolated in ∂E, because $Z(y) = \{x\}$, and $\theta_{(y,yy)}(x) = x$. However, the only idempotent in $\mathcal{S}_E$ which is smaller than (y, yy) is the zero, and θ_0 is the empty function, thus $Z(y) \cap \Lambda(\theta) = \varnothing$. This proves that $\Lambda(\theta)$ is not dense in ∂E, therefore θ is not topologically principal. $\square$

Remark 4.6.4. Let $E = (E^0, E^1, s, r)$ and $F = (F^0, F^1, s, r)$ be directed graphs, and let $\varphi\colon \partial E \to \partial F$ be a continuous function. Suppose that for every $(\mu,\nu) \in S_E$ and every $x \in Z(\nu)$, there exists a neighbourhood $U \subseteq Z(\nu)$ of x and $(\alpha,\beta) \in S_F$ such that for all $\tilde{x} \in U$

$$\varphi(\theta_{(\nu,\mu)}(\tilde{x})) = \theta^F_{(\alpha,\beta)}(\varphi(\tilde{x})).$$

Thus we can find a clopen partition $\mathscr{U}_s$ of $Z(\mu)$, and a family

$$\{(\alpha,\beta)_U : U \in \mathscr{U}_s\} \subseteq S_F$$

such that for all $U \in \mathscr{U}_s$ and all $x \in U$,

$$\varphi(\theta^E_{(\mu,\nu)}(x)) = \theta^F_{(\alpha,\beta)_U}(\varphi(x)).$$

We define $a\colon S_E * X \to S_F$, by setting $a((\mu,\nu),x) = (\alpha,\beta)_U$, where U is chosen as the unique element of $\mathscr{U}_s$ such that $x \in U$. Then a continuous function such that for every $(\mu,\nu) \in S_E$ and $x \in Z(\nu)$,

$$\varphi(\theta^E_{(\mu,\nu)}(x)) = \theta^F_{a((\mu,\nu),x)}(\varphi(x)).$$

We will now compare continuous orbit equivalence of graphs and continuous orbit equivalence of the canonical action of the associated semigroups. The following is analogue to [52, Lemma 3.8], but we do not require that the graphs satisfy Condition (L).

Proposition 4.6.5. *Let $E = (E^0, E^1, s, r)$ and $F = (F^0, F^1, s, r)$ be directed graphs. Then E and F are continuously orbit equivalent if and only if the canonical actions θ^E and θ^F associated to E and F are continuously orbit equivalent.*

Proof. Assume that (φ, a, b) is a continuous orbit equivalence between θ^E and θ^F. Given $x \in \partial E^{\geq 1}$, let us denote by $x_1 \in E^1$ the first edge of x (i.e., $x = x_1 y$ for some $y \in \partial E$). The map $x \mapsto x_1$ is locally constant on $\partial E^{\geq 1}$ – namely, it is the constant map $x \mapsto e$ on $Z(e)$ for each $e \in E^1$, and $\left\{Z(e) : e \in E^1\right\}$ is a partition of $\partial E^{\geq 1}$.

Let $\alpha, \beta\colon \partial E^{\geq 1} \to \mathcal{S}_E$ be functions such that $a((r(x_1), x_1), x) = (\alpha(x), \beta(x))$ for all $x \in \partial E^{\geq 1}$, and define $k(x) = |\alpha(x)|$ and $l(x) =$

$|\beta(x)|$. As a is continuous, then k and l are continuous. Moreover, we have

$$\varphi(\sigma_E(x)) = \varphi(\theta^E_{(r(x_1),x_1)}(x)) = \theta^F_{(\alpha(x),\beta(x))}(\varphi(x)),$$

which means that $\varphi(\sigma_E(x)) = \alpha(x)y$ and $\varphi(x) = \beta(x)y$, for some $y \in \partial F$. Thus

$$\sigma_F^{k(x)}(\varphi(\sigma_E(x))) = \sigma_F^{|\alpha(x)|}(\alpha(x)y) = y = \sigma_F^{|\beta(x)|}(\beta(x)y) = \sigma_F^{l(x)}(\varphi(x))$$

and so (4.11) holds. To prove (4.12), k' and l' are defined in a similar way, using b.

Conversely, suppose $\varphi : \partial E \to \partial F$ is a homeomorphism and that there are maps $k, l \colon \partial E^{\geq 1} \to \mathbb{N}$ satisfying, for all $x \in \partial E^{\geq 1}$,

$$\sigma_F^{k(x)}(\varphi(\sigma_E(x))) = \sigma_F^{l(x)}(\varphi(x)). \tag{4.14}$$

We must show that there is a continuous function $a \colon \mathcal{S}_E * \partial E \to \mathcal{S}_F$ such that

$$\varphi(\theta^E_{(\mu,\nu)}(x)) = \theta^F_{a(\mu,\nu,x)}(\varphi(x)), \tag{4.15}$$

for all $(\mu,\nu) \in \mathcal{S}_E$ and $x \in Z^E(\nu)$.

By Remark 4.6.4, it is sufficient to prove that for all $(\mu,\nu) \in \mathcal{S}_E$ and for all $x \in Z(\nu)$, there exists an open set U containing x and $(\alpha,\beta) \in \mathcal{S}_F$ such that for all $\widetilde{x} \in U$,

$$\varphi(\theta^E_{(\mu,\nu)}(\widetilde{x})) = \theta^F_{(\alpha,\beta)}(\varphi(\widetilde{x})).$$

Let us separate the proof in cases:

1. Assume that $|\mu| = |\nu| = 0$ (which implies that $\mu = \nu$).

   In this case, we simply take $U = Z^E(\nu) \cap \varphi{-}1(Z^F(s(\varphi(x))))$. Then for all $\widetilde{x} \in U$,

   $$\varphi(\theta^E_{(\mu,\nu)}(\widetilde{x})) = \varphi(\widetilde{x}) = \theta^F_{(s(\varphi(x)),s(\varphi(x)))}(\varphi(\widetilde{x})),$$

   so we are done.

2. Assume that $|\mu| = 0$ and $|\nu| = 1$.

   Let $K = k(x)$ and $L = l(x)$. For all $\widetilde{x} \in Z^E(\nu) \subseteq \partial E^{\geq 1}$, we have

$$\theta^E_{(\mu,\nu)}(\widetilde{x}) = \sigma_E(\widetilde{x})$$

   Let $U_1 = Z^E(\nu) \cap k^{-1}(K) \cap l^{-1}(L)$. Then for all $\widetilde{x} \in U_1$, Equation (4.14) implies that

$$\sigma_F^K(\varphi(\theta^E_{(\mu,\nu)}(\widetilde{x})) = \sigma_F^L(\varphi(\widetilde{x})). \tag{4.16}$$

   Equation (4.16) with $\widetilde{x} = x$ implies that there exist $(\alpha, \beta) \in \mathcal{S}_F$, with $|\alpha| = K$ and $|\beta| = L$, such that

$$\varphi(\theta^E_{(\mu,\nu)}(x)) = \theta^F_{(\alpha,\beta)}(\varphi(x)).$$

   Thus setting $U = U_1 \cap \varphi^{-1}(Z^F(\nu)) \cap (\varphi \circ \theta^E_{(\mu,\nu)})^{-1}(Z^F(\mu))$, we obtain that Equation (4.15) holds on U.

3. Assume that $|\mu| = 0$ and $|\nu| \geq 1$.

   Write $\nu = \nu_1 \cdots \nu_{|\nu|}$, where $\nu_i \in E^1$. Notice that

$$(\mu, \nu) = (\mu, \nu_{|\nu|})(s(\nu_{|\nu|}), \nu_{|\nu|-1}) \cdots (s(\nu_3), \nu_2)(s(\nu_2), \nu_1)$$

   In other words, there are elements $e_1, \ldots, e_{|\nu|}$ of the form considered in the previous case, such that $(\mu, \nu) = e_{|\nu|} \cdots e_1$. Applying the previous case, for each $k \geq 1$ we may find a neighbourhood U_k of $\theta_{e_{k-1}\cdots e_1}(x)$ (or simply x in the case $k = 1$) and an element $f_k \in \mathcal{S}_F$ such that

$$\varphi \circ \theta^E_{e_k} = \theta^F_{f_k} \circ \varphi$$

   on U_k. Then $U = U_1 \cap \bigcap_{k=2}^{|\nu|} \theta^{-1}_{e_{k-1}\cdots e_1}(U_k)$ is a neighbourhood of x such that

$$\varphi \circ \theta^E_{(\mu,\nu)} = \varphi \circ \theta^E_{e_{|\nu|}} \circ \cdots \circ \theta^E_{e_1} = \theta^F_{f_{|\nu|}} \circ \cdots \circ \theta^F_{f_1} \circ \varphi = \theta^F_{f_{|\nu|}\cdots f_1} \circ \varphi,$$

   since θ^E and θ^F are actions.

4. Assume that $|\mu| \geq 1$ and $|\nu| = 0$.

   Applying the case 3 to (ν, μ), there exists a neighbourhood V of $\theta^E_{(\mu,\nu)}(x)$ and $(\beta, \alpha) \in \mathcal{S}_F$ such that $\varphi \circ \theta^E_{(\nu,\mu)} = \theta^F_{(\beta,\alpha)} \circ \varphi$ on V. In other words, $\varphi \circ \theta^E_{(\mu,\nu)} = \theta^F_{(\alpha,\beta)} \circ \varphi$ on the neighbourhood $U = \theta^E_{(\nu,\mu)}(V)$ of x, as we wanted.

5. Assume that $|\mu|, |\nu| \geq 1$. In this case, $(\mu, \nu) = (\mu, r(\mu))(r(\mu), \nu)$, so we may apply cases 3 and 4, and proceed in a manner similar to that of case 3.

Since we have exhausted all possibilities for (μ, ν), the theorem is proved. $\square$

We have seen that the Steinberg algebra $A_R(\mathcal{G}_E)$ of the boundary path groupoid $\mathcal{G}_E$ is isomorphic to the Leavitt path algebra $L_R(E)$ of graph E (see Example 1.3.10). By Example 4.6.1 the groupoids $\mathcal{G}_E$ and $\mathcal{S}_E \ltimes \partial E$ are isomorphic. Thus, there is an isomorphism between the Steinberg algebras $A_R(\mathcal{G}_E)$ and $A_R(\mathcal{S}_E \ltimes \partial E)$. Moreover, by Theorem 4.3.4, the Steinberg algebra $A_R(\mathcal{S}_E \ltimes \partial E)$ is isomorphic to the skew inverse semigroup algebra $\mathcal{L}(\partial E) \rtimes \mathcal{S}_E$. Then we can conclude the following:

Proposition 4.6.6. *Let $E = (E^0, E^1, r, s)$ be a directed graph. Then*

$$L_R(E) \simeq A_R(\mathcal{G}_E) \simeq A_R(\mathcal{S}_E \ltimes \partial E) \simeq \mathcal{L}_c(\partial E) \rtimes \mathcal{S}_E.$$

Finally, from Lemma 4.6.5, Example 4.6.1, Theorem 4.5.11 and 4.5.10 and Proposition 4.6.6, we obtain the following Theorem:

Theorem 4.6.7. *Let E and F be directed graphs that satisfy Condition (L) and let R be an indecomposable ring. Then the following are equivalent:*

(i) the graphs E and F are continuously orbit equivalent,

(ii) θ_E and θ_F are continuously orbit equivalent,

(iii) $\mathcal{S}_E \ltimes X_E$ and $S_F \ltimes X_F$ are isomorphic as topological groupoids,

(iv) $\mathcal{G}_E$ *and* $\mathcal{G}_F$ *are isomorphic as topological groupoids,*

(v) the topological full pseudogroups $[[\mathcal{G}_E]]$ *and* $[[\mathcal{G}_F]]$ *are isomorphic*

(vi) there exists a diagonal-preserving isomorphism between the Steinberg algebras $A_R(\mathcal{G}_E)$ *and* $A_R(\mathcal{G}_F)$*,*

(vii) there exists a diagonal-preserving isomorphism between the partial skew inverse semigroup rings $\mathcal{L}(X_E) \rtimes \mathcal{S}_E$ *and* $\mathcal{L}(X_F) \rtimes S_F$*,*

(viii) there exists a diagonal-preserving isomorphism between the Leavitt path algebras $L_R(E)$ *and* $L_R(F)$*.*

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