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Cohomology for partial actions of Hopf algebras

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Resumo

Neste trabalho, formulamos uma teoria de cohomologia para ações parciais de álgebras de Hopf cocomutativas sobre álgebras comutativas. Ela generaliza tanto a teoria de cohomologia para álgebras de Hopf introduzidas por M. E. Sweedler como também a teoria da cohomologia para ações parciais de grupo, introduzidas por M. Dokuchaev e M. Khrypchenko. Alguns exemplos não triviais, ou seja, não provenientes de grupos, são construídos.

Ainda, dada uma ação parcial de uma álgebra Hopf cocomutativa H sobre uma álgebra comutativa A, definimos uma nova estrutura, nomeada $\widetilde{A}$, que possui os mesmos grupos de cohomologia da álgebra original A. Essa estrutura é interessante pois possui estrutura de álgebra de Hopf sobre o anel comutativo $E(A)$ e H permanece agindo parcialmente sobre $\widetilde{A}$. Por fim, verificamos a relação do segundo grupo de cohomologia, $H^2(H, A)$, com extensões cleft parciais de álgebras comutativas por ações parciais de álgebras de Hopf cocomutativas e provamos que extensões cleft parciais podem ser vistas como extensões cleft de Hopf algebroides.

Palavras-chave Álgebras de Hopf; ações parciais; Cohomologia parcial; produto cruzado parcial; extensões cleft parciais.

Resumo Expandido

Introdução:

A história das álgebras de Hopf tem início no contexto da topologia algébrica com o artigo de H. Hopf, publicado em 1941, descrevendo as propriedades algébricas do anel de cohomologia de uma variedade de grupo [23]. O assunto de cohomologia de grupo rapidamente se tornou independente do seu contexto topológico, assumindo uma formulação mais algébrica [1, 31]. A primeira formulação de uma teoria de cohomologia para álgebras de Hopf foi feita por M. Sweedler em 1968 [30], em que o mesmo considerou álgebras de Hopf cocomutativas atuando sobre álgebras comutativas. Este trabalho se tornou paradigmático para desenvolvimentos futuros nessa área.

Neste trabalho a cohomologia foi definida por meio de um complexo explícito que surge como um complexo de cocadeia e os grupos neste complexo consistem no grupo multiplicativo de elementos invertíveis em $Hom(C, A)$, em que C é a coálgebra advinda da álgebra de Hopf, tensorizada sobre si mesma um número de vezes.

Neste mesmo artigo, M. Sweedler dedicou-se também ao estudo da classificação de extensões de álgebra por álgebras de Hopf. Uma extensão de álgebra por uma álgebra de Hopf, possui estrutura de álgebra e outras propriedades. Assim, definindo o que são equivalência e produto de extensões, foi possível a Sweedler demonstrar que a 2-cohomologia $H^2(H, A)$, com A um H-módulo álgebra, é isomórfica ao grupo das classes de equivalência de extensões cleft. Parte desta teoria consiste na definição de certas álgebras, denominadas produto cruzado.

Essa noção de produto cruzado foi generalizada, de forma independente por Y. Doi e M. Takeuchi [15] e R. Blattner, M. Cohen, S. Montgomery [28] em 1986. Eles também introduziram condições para caracterizar produtos cruzados como extensões cleft. Então, S. Montgomery em [28] introduziu um critério de equivalência para isomorfismos entre produtos cruzados. Tal resultado tinha por expectativa obter uma teoria geral de cohomologia para álgebras sobre álgebras de Hopf, mas até hoje ainda não foi possível.

A teoria de ações parciais de grupo teve início com o trabalho de R. Exel na classificação de certas classes de C^*-álgebras que não podiam ser descritas como um produto cruzado usual [21]. Uma formulação mais algébrica para ações parciais foi feita por M. Dokuchaev e R. Exel em [16] e então, ações parciais passaram a chamar a atenção de algebristas, gerando desenvolvimento em diversas direções. Neste sentido, para este trabalho é relevante a noção de ações parciais torcidas de um grupo [17] e sua globalização [18]. Nelas, a noção de 2-cociclo parcial é utilizada para se definir ações parciais torcidas e produtos cruzados parciais, sugerindo assim a existência de uma teoria geral de cohomologia em que os 2-cociclos estariam inseridos. Esta teoria cohomológica para ações parciais de grupo foi introduzida por M. Dokuchaev e M. Khrypchenko em [19], e é construída em cima de ações parciais de grupos sobre monóides comutativos. Ainda, em [20], os autores relacionam sua teoria cohomológica para ações parciais de grupo com o contexto de cohomologia de semigrupos inversos desenvolvida por H. Lausch [25], uma vez que a noção de ação parcial de grupo está profundamente relacionada com ações de semigrupos inversos [22],

Ações parciais entraram no contexto de álgebras de Hopf pelo trabalho de S. Caenepeel e K. Janssen em [14]. Este trabalho permitiu a generalização de vários resultados clássicos na teoria de álgebra de Hopf e de várias ideias desenvolvidas para ações parciais de grupo, como o teorema de globalização [2], equivalência de Morita entre produto smash parcial e a subalgebra dos invariantes [3], dualidade para ações parciais [4], representações parciais [7], etc. Indicamos [8] para mais detalhes sobre o desenvolvimento recente de ações parciais de grupos e álgebras de Hopf.

Para esta tese, nosso interesse são as noções de ações parciais torcidas de álgebras de Hopf. produtos cruzados parciais e extensões parcialmente cleft de álgebras por álgebras de Hopf introduzidas por M. Alves, E. Batista, A. Paques e M. Dokuchaev em [5]. Neste trabalho, em certo sentido, os autores generalizam as noções introduzidas em [15] e [10] e demonstram um análogo do Teorema de Isomorfismo entre produtos cruzados introduzido por S. Montgomery [28].

Objetivos:

Baseado nos artigos de M. Dokuchaev e M. Khrypchenko [19] e [20], surge a seguinte questão: *será que os resultados obtidos por eles para ações parciais de grupo poderiam ser extendidos para ações parciais de* *álgebras de Hopf sobre uma álgebra? Quais caracterizações poderiam ser obtidas?*

Neste sentido, nosso objetivo com este trabalho é estabelecer uma teoria cohomológica para ações parciais de álgebras de Hopf a partir da cohomologia do algebróide de Hopf H_{par}. E a partir disso, caracterizar produtos cruzados parciais a partir de extensões.

Metodologia: Como citado anteriormente, a teoria de cohomologia para álgebras de Hopf surgiu com M. Sweedler em [30] no contexto de álgebras de Hopf cocomutativas sobre álgebras comutativas. Desde então, busca-se uma generalização para álgebras de Hopf arbitrárias, porém, não houveram muitos avanços até então.

A partir do trabalho desenvolvido por M. Dokuchaev e M. Khryp chenko em [19], em que os mesmos introduzem as noções de cohomologia para ações parciais de grupo, observamos que era possível extender esses resultados para álgebras de Hopf, considerando as noções de ações parcias definidas em [14].

Obtendo os resultados iniciais e avançando no entendimento de [19], definimos uma “nova" álgebra de Hopf, denominada $\widetilde{A}$ e dada pelo quociente da álgebra comutativa livre gerada pela imagem de todas as cocadeias $f \in \widetilde{C}^n_{par}(H, A)$ (em que $\widetilde{C}^n_{par}(H, A)$ representa uma n-cocadeia parcial reduzida, ou seja, o quociente de $C^n_{par}(H, A)$ por $A^{\times}$). Esta álgebra se torna importante pois possui os mesmos grupos de cohomologia que os do complexo de cocadeia original $C^n_{par}(H, A)$.

Por estarmos trabalhando com álgebras de Hopf cocomutativas e álgebras comutativas, a teoria de ações parciais torcidas e produtos cruzados parciais introduzidas em [5] se tornam importantes aqui e conseguimos dar uma noção cohomológica para estes produtos cruzados parciais, classificando-os pelo segundo grupo de cohomologia parcial $H^2_{par}(H, A)$.

Ainda, se considerarmos o produto cruzado $\underline{\widetilde{A}\#_{\omega}H}$, temos sobre o mesmo uma estrutura de Hopf algebróide, sugerindo assim que nossa teoria de cohomologia para ações parciais pode ser vista do ponto de vista da teoria cohomologica para Hopf algebróides.

Por fim, em [30] é mostrado que produtos cruzados estão diretamente relacionados com extensões cleft, analogamente, ao considerarmos a teoria de extensões cleft parciais introduzidas em [5] conseguimos resultados similares. E mais, devido a estrutura de Hopf algebróide do produto cruzado parcial, conseguimos um resultado completamente inesperado que consiste em relacionar extensões cleft parciais com extensões de álgebras sobre Hopf algebróides introduzidas em [12].

Resultados Obtidos:

Apresentamos aqui os principais resultados obtidos neste trabalho. No Capítulo 1 apresentamos a teoria de cohomologia para álgebras de Hopf apresentada por Sweedler em [30], a teoria de cohomologia parcial de grupos introduzida por Dokuchaev e Khrypchenko em [19] e finalizamos com as noções de ações parciais de álgebras de Hopf de Caenepeel e Janssen em [14].

No Capítulo 2, após introduzirmos as noções sobre idempotentes, definimos o complexo de cocadeias por:

Definição 2.7 *Sejam A uma álgebra comutativa, H uma biálgebra cocomutativa, n um inteiro positivo e $\alpha : H \otimes A \to A$ uma ação parcial, então uma n-cocadeia "parcial" (cocadeia de ordem n) de H com valores em A ($C^n_{par}(H, A)$) é uma aplicação invertível em um ideal de $Hom_k(H^{\otimes n}, A)$, ou seja,*

$$C^n_{par}(H, A) = (I(H^{\otimes n}, A))^{\times},$$

em que

$$\begin{aligned} I(H^{\otimes n}, A) &= e_n * Hom_k(H^{\otimes n}, A) \\ &= \{e_n * g : g \in Hom_k(H^{\otimes n}, A)\} \trianglelefteq Hom_k(H^{\otimes n}, A). \end{aligned}$$

Por uma 0-cocadeia entendemos $C^0_{par}(H, A) = (I(H^{\otimes 0}, A))^{\times} = A^{\times}$, com a multiplicação de A e $e_0 = 1_A$.

Definimos então o operador cobordo parcial por:

Definição 2.8 *Para quaisquer $f \in C^n_{par}(H, A)$, $(h^1 \otimes \cdots \otimes h^{n+1}) \in H^{\otimes n+1}$, definimos*

$$\begin{aligned} (\delta_n f)(h^1, \ldots, h^{n+1}) &= (h^1_{(1)} \cdot f(h^2_{(1)}, \ldots, h^n_{(1)})) * \\ * & \prod_{i=1}^{n} f^{(-1)^i}(h^1_{(i+1)}, \ldots, h^i_{(i+1)} h^{i+1}_{(i+1)}, \ldots, h^{n+1}_{(i+1)}) * \\ * & f^{(-1)^{n+1}}(h^1_{(n+2)}, \ldots, h^n_{(n+2)}) \end{aligned}$$

Se $n = 0$ e a é um elemento invertível de A, temos que

$$(\delta_0 a)(h) = (h \cdot a)a^{-1}.$$

Após a demonstração de alguns resultados auxiliares, concluímos que:

Teorema 2.11/2.13 *O operador cobordo δ_n definido acima é um homomorfismo de $C^n_{par}(H,A) \to C^{n+1}_{par}(H,A)$, tal que*

$$\delta_{n+1} \circ \delta_n(f) = e_{n+2}$$

para qualquer $f \in C^n_{par}(H,A)$.

Definimos então os grupos abelianos dos n-cociclos parciais, n-cobordos parciais e n-cohomologias parciais de H com valores em A por $Z^n(H,A) = \ker \delta_n$, $B^n(H,A) = Im\ \delta_{n-1}$ e $H^n(H,A) = \ker \delta_n / Im\ \delta_{n-1}$, respectivamente, $n \geq 1$ (para $n = 0$, definimos $H^0(H,A) = Z^0(H,A) = \ker \delta_0$).

O Capítulo 3 é destinado ao estudo da álgebra de Hopf $\widetilde{A}$ e tem como principais resultados, após a construção da álgebra $\widetilde{A} := \dfrac{\widehat{A}}{\mathfrak{J}}$, em que $\widehat{A}$ é uma álgebra comutativa livre unital, o fato de que as cohomologias geradas anteriormente por uma álgebra A e as geradas por $\widetilde{A}$ são as mesmas, ou seja, $H^n_{par}(H,\widetilde{A}) \cong H^n_{par}(H,A)$ e também, que $\widetilde{A}$ tem estrutura de álgebra de Hopf, conforme o Teorema 4.5.

No Capítulo 4 trabalhamos com a noção de produto cruzado torcido dada em [5] e obtemos como principais resultados:

Teorema 4.7 *Seja H uma álgebra de Hopf cocomutativa e A um H-módulo álgebra parcial. Então, dados dois 2-cociclos parciais ω, $\sigma \in Z^2_{par}(H,A)$, os produtos cruzados parciais associados $\underline{A\#_\omega H}$ e $\underline{A\#_\sigma H}$ são isomorfos se, e somente se, ω e σ são cohomólogos, ou seja, pertencem a mesma classe no grupo de cohomologia $H^2_{par}(H,A)$.*

Ainda, provamos que toda classe de cociclos em $H^2_{par}(H,A)$ contém um 2-cociclo normalizado, ou seja, dado um 2-cociclo $\omega \in Z^2_{par}(H,A)$, existe um 2-cociclo normalizado $\widetilde{\omega} \in Z^2(H,A)$ que é cohomologo a ω. Esses dois resultados nos permitem então concluir que o segundo grupo parcial de cohomologia $H^2_{par}(H,A)$ classifica todas as classes de isomorfismos de produtos cruzados parciais.

O segundo resultado importante deste capítulo e talvez um dos mais surpreendentes desta tese consiste em conseguirmos determinar uma estrutura de Hopf algebróide para este produto cruzado.

Teorema 4.10 *Seja H uma álgebra de Hopf cocomutativa e A um H-módulo álgebra parcial. Tomando a álgebra de Hopf comutativa e cocomutativa $\widetilde{A}$, construída no Teorema 4.5, sobre a álgebra comutativa $E(A)$, que é também um H-módulo álgebra parcial. Então, o produto* *cruzado* $\widetilde{A}\#_{\omega}H$*, em que* ω *é um 2-cociclo parcial em* $H^2_{par}(H,A) \cong H^2_{par}(H,\widetilde{A})$ *é um Hopf algebróide sobre a álgebra base* $E(A)$*.*

O Capítulo 5 é dedicado ao estudo de extensões cleft parciais e é nele que surge o resultado mais interessante desta tese, pois conseguimos relacionar estas extensões com extensões cleft de álgebras por Hopf algebróides, criando um paralelo entre estas duas teorias introduzidas em [5] e [12].

Teorema 5.11 *Seja* H *uma álgebra de Hopf cocomutativa agindo parcialmente sobre uma álgebra de Hopf comutativa e cocomutativa* $\widetilde{A}$ *e seja* ω *um 2-cociclo parcial em* $Hom^2_{par}(H,\widetilde{A})$*. Então o produto cruzado parcial* $\underline{\widetilde{A}\#_{\omega}H}$ *é um* $\mathcal{H} = \underline{E(A)\#H}$*-modulo álgebra à direita, com* $\widetilde{A} \cong (\underline{\widetilde{A}\#_{\omega}H})^{co\mathcal{H}}$*. E mais, a extensão* $\widetilde{A} \subset \underline{\widetilde{A}\#_{\omega}H}$ *é* $\mathcal{H}$*-cleft no sentido de [12].*

Considerações Finais Como podemos ver, não só conseguimos uma teoria de cohomologia parcial para álgebras de Hopf, estendendo os resultados de [19], como também fomos capazes de dar uma noção cohomológica para o produto cruzado parcial introduzido em [5], desde que tenhamos H uma álgebra de Hopf cocomutativa e A uma álgebra comutativa. Além disso, inesperadamente, demonstramos que a teoria de extensões cleft parciais para álgebras de Hopf [5] pode ser entendida no contexto da teoria de extensões cleft para Hopf algebróides em [12]. Observamos que toda a teoria de cohomologia aqui feita pode ser generalizada para objetos álgebra de Hopf cocomutativa e objetos álgebra comutativa na categoria de monóides trançados.

Além disso, em associação com o professor J. Vercruysse (ULB), buscamos investigar se haveria uma teoria cohomológica geral, com H e A arbitrários, porém, assim como para a teoria cohomológica para álgebras de Hopf, também aqui não obtivemos resultados.

Porém, ainda há muitas perguntas a serem solucionadas nessa área, como estabelecer uma ponte entre a teoria desenvolvida aqui e a teoria clássica desenvolvida por M. Sweedler. Também, por conta do último teorema, talvez possamos entender toda a nossa teoria cohomológia como uma teoria cohomologica para Hopf algebrídes. Um outro ponto a ser explorado pode ser a teoria de obstruções para a existência de extensões cleft parciais e sua relação com o terceiro grupo de cohomologia, na mesma direção de [29]

Palavras-chave álgebras de Hopf; ações parciais; cohomologia parcial; produto cruzado parcial; extensões cleft parciais.

Abstract

In this work, the cohomology theory for partial actions of co-commutative Hopf algebras over commutative algebras is formulated. This theory generalizes the cohomology theory for Hopf algebras introduced by Sweedler and the cohomology theory for partial group actions, introduced by Dokuchaev and Khrypchenko. Some nontrivial examples, not coming from groups are constructed. Given a partial action of a co-commutative Hopf algebra H over a commutative algebra A, we prove that there exists a new Hopf algebra $\widetilde{A}$, over a commutative ring $E(A)$, upon which H still acts partially and which gives rise to the same co-homologies as the original algebra A. We also study the partially cleft extensions of commutative algebras by partial actions of cocommutative Hopf algebras and prove that these partially cleft extensions can be viewed as cleft extensions by Hopf algebroids.

Keywords Hopf algebras; partial action; partial cohomology; partial crossed product; partial cleft extension.

Contents

Chapter 1

Introduction

The history of Hopf algebras began within the context of algebraic topology with the seminal paper by H. Hopf, published in 1941, describing the algebraic properties of the cohomology ring of a group manifold [23]. The subject of group cohomology soon became increasingly more independent of its topological background assuming a more algebraic formulation [1, 31]. The first formulation of a cohomology theory of cocommutative Hopf algebras acting over commutative algebras was done by M. Sweedler in 1968 [30], which, in certain sense, became paradigmatic for further developments in this area.

In his work, M. Sweedler extended the notions of group cohomology for Hopf algebras by defining an explicit complex which arises as the cochain complex and the groups in this complex consist of the multiplicative group of invertible elements in $Hom(C, A)$, where C is the underlying coalgebra of the Hopf algebra tensored with itself a number of times. In the same paper, M. Sweedler also devoted some time the study of extensions and proved that $H^2(H, A)$, in which H is a Hopf algebra and A an H-module algebra, is isomorphic to the group of equivalence classes of extensions defining certain algebras which we call crossed products.

This notion of crossed product was generalized, independently, by Y. Doi and M. Takeuchi [15] and R. Blattner, M. Cohen, S. Montgomery [28] in 1986. Conditions to characterize crossed products as cleft extensions were estabilished in those works. Then, S. Montgomery in [28], introduced an equivalence criteria for isomorphism between crossed product, the hope was, with this result, to get a general cohomology theory for algebras over Hopf algebras but, unfortunately, we are still looking for it.

The theory of partial group actions, in its turn, had its beginning with the work of Ruy Exel in the classification of certain class of C*-algebras with an action of the unit circle but which cannot be described as a usual crossed product [21]. A more algebraic formulation for partial actions was done by Mikhailo Dokuchaev and Ruy Exel in [16] and then, partial actions drew the attention of algebraists and allowed further developments in several directions. One of the developments particularly relevant for our discussion here is the notion of a twisted partial actions of a group [17] and its globalization [18]. There, one can see the definition of partial 2-cocycles in order to define twisted actions and partial crossed products, this suggested the existence of a general cohomology theory in which these partial 2-cocycles could be placed. This cohomological theory for partial group actions was achieved by Mikhailo Dokuchaev and Mykola Khrypchenko in [19]. This theory is constructed upon partial actions of groups over commutative monoid. As the notion of a partial action of a group is itself deeply related with actions of inverse semigroups [22], in reference [20], the authors could place their cohomology theory for partial group actions within the context of cohomology of inverse semigroups, developed by Hans Lausch [25].

Partial actions came into the Hopf algebra context by the work of Stefaan Caenepeel and Kris Janssen in [14]. This work allowed the generalization of classical results in Hopf algebra theory and of several ideas developed for partial group actions, as the globalization theorem [2], Morita equivalence between the partial smash product and the invariant subalgebra [3], duality for partial actions [4], partial representations [7], etc. For a more detailed account on recent developments of partial actions of groups and Hopf algebras, see [8] and references therein.

Of particular interest for the present thesis are the notions of twisted partial actions of Hopf algebras, partial crossed products and partially cleft extensions of algebras by Hopf algebras introduced by M. Alves, E. Batista, A. Paques and M. Dokuchaev in [5]. There, the authors generalized “in some sense" the notions introduced in [15] and [10] and prove an analogous theorem of Montgomery [28] on isomorphic crossed product.

The aim of this thesis, is exactly to formulate a cohomology theory for partial actions of Hopf algebras, in the same spirit of [19], such that the partial 2-cocycles defined in [5] can be placed properly. This cohomological theory is obtained for the case of partial actions of a cocommutative Hopf algebra H acting partially over a commutative algebra A. Moreover, one can, without loss of generality, replace the original algebra A by a commutative and cocommutative Hopf algebra $\widetilde{A}$ over a base algebra $E(A) \subseteq A$ and yet obtain the same cohomology theory. This is a surprising result, we can replace the crossed product $\underline{A\#_\omega H}$ by the crossed product $\underline{\widetilde{A}\#_\omega H}$, which has a a structure of Hopf algebroid over the base algebra $E(A)$, this leads to interesting consequences in the analysis of cleft extensions.

As we have already learned in [7], the theory of partial actions of Hopf algebras in fact is deeply related to the theory of representations of Hopf algebroids. This opens a totally new landscape to be explored, for example, in this work we prove that partially cleft extensions can be understood as cleft extensions by Hopf algebroids in the sense of Gabriella Böhm and Tomasz Brzezinski [12] and then one can raise new questions on how to put this cohomological theory for partial actions in the context of cohomology for Hopf algebroids [13, 24].

This thesis is organized in the following way:

In Chapter 1, we review the main results of the cohomology theory for algebras over Hopf algebras developed by Sweedler in [30] and of the theory of cohomology developed by M. Dokuchaev and M. Khrypchenko [19]. We conclude this chapter recalling the notion of a partial action of a Hopf algebra over a unital algebra and giving some examples of such partial actions. Special attention is required for examples 2.22 and 2.23, which will serve as basis for our specific examples of cohomologies given in Section 3.4.

Chapter 2 is dedicated to the construction of our cohomological theory for partial actions of a cocommutative Hopf algebra H over a commutative algebra A. We start with the study of a system of idempotents in the commutative convolution algebras $\mathrm{Hom}_k(H^{\otimes n}, A)$, for a natural n. In Section 3.2 we define the cochain complex, $C^\bullet_{par}(H, A)$ associated to the partial action of H upon A. The involved ideals follow the same principles of the classical construction due to M.E. Sweedler [30] but in order to overcome the complexities coming from partial actions we define some auxiliary operators which help us to prove that the coboudary operator is a morphism of abelian groups (Theorem 3.10) and it is nilpotent in this context (Theorem 3.12). Example 3.15 considers the case of $H = kG$, for G a given group, in this case, we recover the cohomology theory for partial group actions developed by M. Dokuchaev and M. Khrypchenko in [19]. Other examples, such as the cohomology theory for partial group actions and partial group gradings over the base field are given in Section 3.4. More specifically, in Example 3.18 we calculate the first cohomology groups of a partial grading of the base field by the Klein four group, this extends what has been done in [6].

In Chapter 3, we define a new Hopf algebra $\widetilde{A}$ given by a quotient of the free commutative algebra generated by the images of all cochains $f \in \widetilde{C}^n_{par}(H, A)$. This Hopf algebra is interesting because it has the same cohomology groups as the original cochain complex $C^n_{par}(H, A)$. In fact, given a partial action of H on A, we define the reduced cochain complex $\widetilde{C}^n_{par}(H, A) \cong C^n_{par}(H, A)/A^{\times}$ and this new cochain complex produces the same cohomology groups as the original cochain complex $C^n_{par}(H, A)$. Next, we consider the algebra $\widetilde{A}$, which is a quotient of the free commutative algebra generated by the images of all cochains $f \in \widetilde{C}^n_{par}(H, A)$. This defines a commutative and cocommutative Hopf algebra over the commutative algebra $E(A)$, which is the subalgebra of A generated by elements of the form $h \cdot 1_A$. One proves that $\widetilde{A}$ generates the same cohomologies as the original algebra A, that is, for any $n \in \mathbb{N}$, we have the isomorphisms $H^n_{par}(H, \widetilde{A}) \cong H^n_{par}(H, A)$. Then, without loss of generality, one can consider only the cohomological theory for partial actions of a cocommutative Hopf algebra H over a commutative and cocommutative Hopf algebra $\widetilde{A}$.

Chapter 4 is devoted an analysis of twisted partial actions and partial crossed products [5]. For the case of a cocommutative Hopf algebra H and a commutative algebra A, all twisted partial actions are indeed partial actions. Nonetheless, we still can get nontrivial partial crossed products by means of choosing a partial 2-cocycle. In fact, the partial crossed products are classified by the second partial cohomology group $H^2_{par}(H, A)$. The new feature which appears in the context of partial actions is that the crossed product $\underline{\widetilde{A} \#_\omega H}$ has a structure of a Hopf algebroid over the same base algebra $\overline{E(A)}$ (Theorem 5.10). This suggests that the cohomology theory for partial actions can be viewed as a cohomological theory for Hopf algebroids.

Chapter 5 is devoted to study partially cleft extensions introduced in [5]. The most interesting result in this chapter comes from the Hopf algebroid structure of the crossed product $\underline{\widetilde{A} \#_\omega H}$, because using this fact, we are able to show that the theory of partially cleft extension is related to the theory of cleft extensions of algebras by Hopf algebroids developed in [12]. In fact, by Theorem 6.12 given a partially cleft extension B of a commutative cocommutative Hopf algebra $\widetilde{A}$ by a cocommutative Hopf algebra H, there exists a Hopf algebroid over the base subalgebra $E(A)$, namely, the partial smash product $\underline{E(A) \# H}$, such that B is an $\underline{E(A)\#H}$-cleft extension of $\widetilde{A}$ in the sense of G. Böhm and T. Brzezinski [12].

Chapter 2

Mathematical preliminaries

In this chapter, we introduce some concepts that motivated this work. We begin with the notion of cohomology for algebras which are modules over a given Hopf algebra, as developed by M. E. Sweedler in [30]. After, we present the initial results about cohomology theory of groups based on partial actions, as developed by M. Dokuchaev and M. Khrypchenko in [19]. Finally, we introduce the notion of partial action of Hopf algebras on algebras, as introduced by S. Caenepeel and K. Janssen in [14]. We suggest to the interested reader the works above for further details.

2.1 Cohomology of algebras over Hopf algebras

In 1967, M. E. Sweedler, in his paper entitled Cohomology of Algebras over Hopf Algebras, presented a cohomology theory for algebras which are modules over a given Hopf algebra. In that work, the Hopf algebras are cocommutative and the module algebras are commutative.

He defined this cohomology by means of an explicit complex. The groups in this complex are the multiplicative group of invertible elements in $Hom(C, A)$, where A is an algebra and C is the underlying coalgebra of the Hopf algebra tensored with itself a number of times. The complex arises as the chain complex associated with a semicosimplicial complex whose face operators are induced by the maps

$\mu_i : \otimes^{n+1} \to \otimes^n$, given by $\mu_i(h^0 \otimes \ldots \otimes h^n) = h^0 \otimes \ldots \otimes h^i h^{i+1} \otimes \ldots \otimes h^n$.

Then, Sweedler used familiar examples of Hopf algebras, like the group algebra kG and the universal enveloping algebra UL of the Lie algebra L, to show that the Hopf algebra cohomologies $H^i(kG, A)$ and $H^i(UL, A)$ are canonically isomorphic to group cohomology $H^i(G, A)$ and the Lie cohomology $H^i(L, A^+)$, provided that the commutative algebra A is an admissible kG or UL-module respectively.

The last half of his paper, Sweedler devoted to study extensions and proved the usual result that $H^2(H, A)$ is isomorphic to the group of equivalence classes of extensions, where part of the theory involves the definition of certain algebras which are called crossed products.

These results are generalized for non-commutative Hopf algebras and non-commutative algebras by Susan Montgomery.

In the following, we present the most important results developed by Sweedler and Montgomery. We indicate [30] and [28] for more details.

2.1.1 Cohomology Definition

Let H be a cocommutative Hopf algebra and A a commutative algebra. To construct a cochain complex, we must to dualize a chain complex, whose objects are the H-module coalgebras $\{H^{\otimes^{q+1}}\}_{q \geq 0}$, the face operators are

$$\begin{array}{lcl} \partial_i : & H^{\otimes^{q+1}} & \longrightarrow H^{\otimes^q} \\ & (x_0 \otimes \ldots \otimes x_i \otimes x_{i+1} \otimes \ldots \otimes x_q) & \mapsto x_0 \otimes \ldots \otimes x_i x_{i+1} \otimes \ldots \otimes x_q \end{array},$$

for $i = 0, 1, \ldots, q-1$ and $\partial_q(x_0 \otimes \ldots \otimes x_q) = (x_0 \otimes \ldots \otimes x_{q-1})\varepsilon(x_q)$.

For $i = 0, \ldots, q$ the degeneracy operators are given by

$$\begin{array}{lccl} s_i : & H^{\otimes^{q+1}} & \longrightarrow & H^{\otimes^{q+2}} \\ & (x_0 \otimes \ldots \otimes x_q) & \mapsto & x_0 \otimes \ldots \otimes x_i \otimes 1_H \otimes x_{i+1} \otimes \ldots \otimes x_q \end{array}.$$

It's easy to see that all face and degeneracy operators are H-module coalgebra morphisms and satisfy the face-degeneracy operators identities.

Then, to obtain the cochain complex, suppose that A is an H-module algebra and apply the contravariant functor $Hom_H(\_, A)^\times$ (where $Hom_H(\_, A)^\times$ means the convolution-invertible-elements in $Hom_H(\_, A)$) from cocommutative H-module coalgebras to abelian groups in the chain complex above.

In fact, we recall that given a field k, k-bialgebra or a Hopf algebra H and a k-algebra A, for each $n \geq 0$ we have the convolution algebras

$\mathrm{Hom}_k(H^{\otimes n}, A)$, with convolution product given by

$$f * g(h^1 \otimes \cdots \otimes h^n) = f(h^1_{(1)} \otimes \cdots \otimes h^n_{(1)})g(h^1_{(2)} \otimes \cdots \otimes h^n_{(2)}),$$

and unit

$$\mathbf{1}(h^1 \otimes \cdots \otimes h^n) = \varepsilon_H(h^1) \ldots \varepsilon_H(h^n)\mathbf{1}_A.$$

In particular, for $n = 0$, we have

$$\mathrm{Hom}_k(H^{\otimes 0}, A) = \mathrm{Hom}_k(k, A) \cong A.$$

The following result can be easily obtained, we leave the details of the proof to the reader.

Proposition 2.1 *Let H be a cocommutative bialgebra, or Hopf algebra and A a commutative algebra. Then, for each $n \geq 0$ the convolution algebras $Hom_k(H^{\otimes n}, A)$ are commutative.*

□

So, the objects of cochain complex are $\{Hom_H(H^{\otimes^{q+1}}, A)^{\times}\}_{q \geq 0}$, the coface operators, for $i = 0, \ldots, q$, are denoted by ∂^i and given by

$$\partial^i : Hom_H(H^{\otimes^q}, A)^{\times} \to Hom_H(H^{\otimes^{q+1}}, A)^{\times}, \text{ for } i = 0, \ldots, q.$$

Definition 2.2 *The homology of the cochain complex is defined by means of the differential d^{q-1}: $Hom_H(H^{\otimes^q}, A)^{\times} \to Hom_H(H^{\otimes^{q+1}}, A)^{\times}$ where*

$$d^{q-1} = (\partial^0) * (\partial^1)^{-1} * \ldots * (\partial^q)^{\pm 1}.$$

Thus, we have

$$Hom_H(H^{\otimes^1}, A)^{\times} \xrightarrow{d^0} Hom_H(H^{\otimes^2}, A)^{\times} \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} Hom_H(H^{\otimes^{n+1}}, A)^{\times} \xrightarrow{d^n} \cdots.$$

Therefore, the cohomology of H in A is defined as the homology of the above complex and the q-th group is given by

$$H^q(H, A) := ker\ d^q / Im(d^{q-1}),$$

for $q > 0$ and $ker\ d^0$, for $q = 0$.

The theory introduced here comes from [27], where the homology of a chain complex is defined. The Sweedler complex is obtained from a contravariant functor applied to a chain complex, hence, it is a cochain complex. Dualizing the theory in [27] one obtains the above homology of $\{Hom_H(H^{\otimes^{q+1}}, A)^{\times}\}$.

Remark 2.3 *There is a natural algebra isomorphism between* $Hom_H(H^{\otimes^q}, A)$ *and* $Hom_H(H^{\otimes^{q-1}}, A)$ *induced by* $\otimes^{q-1} \to \otimes^q$, $x \mapsto 1 \otimes x$. *This induces an isomorphism*

$$\iota : Hom_H(H^{\otimes^q}, A)^\times \to Hom_H(H^{\otimes^{q-1}}, A)^\times.$$

Let $\psi : H \otimes A \to A$, given by $\psi(h \otimes a) := h \cdot a$ (remember that A is an H-module algebra), then, with respect to ι, the coface operator $\partial^0 : Hom_H(H^{\otimes^q}, A)^\times \to Hom_H(H^{\otimes^{q+1}}, A)^\times$ corresponds to the map

$$\begin{array}{rccc} \delta^0 : & Hom_H(H^{\otimes^{q-1}}, A)^\times & \longrightarrow & Hom_H(H^{\otimes^q}, A)^\times \\ & f & \mapsto & \psi(I \otimes f) \end{array},$$

where $\psi(I \otimes f)(h^1 \otimes \ldots \otimes h^q) = h^1 \cdot f(h^2 \otimes \ldots \otimes h^q)$

For $i = 1, \ldots, q-1$, the coface operator ∂^i corresponds to

$$\begin{array}{rccc} \delta^i : & Hom_H(H^{\otimes^{q-1}}, A)^\times & \longrightarrow & Hom_H(H^{\otimes^q}, A)^\times \\ & f & \mapsto & f(I \otimes \ldots \otimes I \otimes m \otimes I \otimes \ldots \otimes I) \end{array},$$

where m is the multiplication in the $i-th$ position, which means

$$f(I \otimes \ldots \otimes I \otimes m \otimes I \otimes \ldots \otimes I)(h^1 \otimes \ldots \otimes h^q) = f(h^1 \otimes \ldots \otimes h^i h^{i+1} \otimes \ldots \otimes h^q).$$

And the coface operator ∂^q corresponds to the map

$$\begin{array}{rccc} \delta^q : & Hom_H(H^{\otimes^{q-1}}, A)^\times & \longrightarrow & Hom_H(H^{\otimes^q}, A)^\times \\ & f & \mapsto & f \otimes \varepsilon \end{array},$$

given by $f \otimes \varepsilon(h^1 \otimes \ldots \otimes h^q) = f(h^1 \otimes \ldots \otimes h^{q-1})\varepsilon(h^q)$.

Thus, if we define the differential $D^{q-1} : Hom_H(H^{\otimes^{q-1}}, A)^\times \to Hom_H(H^{\otimes^q}, A)^\times$ by

$$D^{q-1}(f) = \psi(I \otimes f) * \prod_{i=1}^{q-1} \delta^i(f^{(-1)^i}) * \delta^q(f^{(-1)^q}),$$

the chain complex $\{Hom_H(H^{\otimes^q}, A)^\times, D^q\}_{q \geq 0}$ is isomorphic to the chain complex $\{Hom_H(H^{\otimes^{q+1}}, A)^\times, d^q\}_{q \geq 0}$ which defines the cohomology $H^q(H, A)$, $q \geq 0$.

Let us look at the groups of cohomology $H^i(H, A)$, for $i = 0, 1$. In fact, for $i = 0$, $Hom_H(H^{\otimes^0}, A)^\times \simeq A^\times$ ($A^\times$ means the invertible-elements in A) and if $a \in H^0(H, A)$, then $(h \cdot a)a^{-1} = \varepsilon(h)$, for all $h \in H$. Thus, $h \cdot a = \varepsilon(h)a$.

If we denote by A^H the set of invariants $\{ a \in A \mid h \cdot a = \varepsilon(h)a, \forall h \in H\}$, this is a subalgebra of A since A is an H-module algebra.

Suppose $a \in A^{\times} \bigcap A^H$, for all $h \in H$,

$$\begin{aligned} \varepsilon(h) &= h \cdot 1 = h \cdot (aa^{-1}) \\ &= \sum (h_{(1)} \cdot a)(h_{(2)} \cdot a^{-1}) \\ &= \sum \varepsilon(h_{(1)}) a (h_{(2)} \cdot a^{-1}) \\ &= a(h \cdot a^{-1}) \end{aligned}$$

which implies $a^{-1} \in A^{\times} \bigcap A^H$ and then, $H^0(H, A) = A^{\times}$.

For $i = 1$, if $f : H \to A$ is a 1-cocycle then

$$\mu(\varepsilon \otimes \varepsilon) = D^1(f) = \psi(I \otimes f) * (f^{-1} \circ m) * (f \otimes \varepsilon),$$

which implies for all $g,\ h \in H$, that

$$f(gh) = \sum (g_{(1)} \cdot f(h_{(1)}))(f(g_{(2)})\varepsilon(h_{(2)})) = \sum (g_{(1)} \cdot f(h)) f(g_{(2)}).$$

Consider now $A = A^H$, so, this reduces the equation above to

$$f(gh) = f(h)f(g)$$

and f is a homomorphism. In general f is a "crossed" homomorphism and $H^1(H, A)$ is the group of regular crossed homomorphism modulo the subgroup of regular inner crossed homomorphisms (that is, one of the form $D^1(a)$ for $a \in A$).

One way to validate these ideas is to do a comparison with the theory of group cohomology. Suppose G is a group and kG the cocommutative group algebra of G. Let A be a kG-module algebra. The elements of G act as automorphisms of A, so they carry $A^{\times}$ into itself. By restricting the module action, the multiplicative abelian group $A^{\times}$ becomes a G-module and one can consider the group cohomology $H^q(G, A^{\times})$. Then, $H^q(kG, A)$ are canonically isomorphic to $H^q(G, A^{\times})$ for all q.

In fact, this isomorphism is induced by a canonical isomorphism between the standard complex to compute $H^q(kG, A)$ and the standard complex to compute $H^q(G, A^{\times})$. For $g_1, \ldots, g_q \in G$, consider the element $\delta_{g_1} \otimes \ldots \otimes \delta_{g_q} \in kG \otimes \ldots \otimes kG$ such that

$$\Delta(\delta_{g_1} \otimes \ldots \otimes \delta_{g_q}) = (\delta_{g_1} \otimes \ldots \otimes \delta_{g_q}) \otimes (\delta_{g_1} \otimes \ldots \otimes \delta_{g_q}).$$

Thus $f^{-1}(\delta_{g_1} \otimes \ldots \otimes \delta_{g_q}) = (f(\delta_{g_1} \otimes \ldots \otimes \delta_{g_q}))^{-1}$ and $f(\delta_{g_1} \otimes \ldots \otimes \delta_{g_q}) \in A^{\times}$ for all $f \in Hom_{kG}(kG^{\otimes^q}, A)^{\times}$. Note that the map $G \times \ldots \times G \to$

$kG \otimes \ldots \otimes kG$ given by $g_1 \times \ldots \times g_q \mapsto \delta_{g_1} \otimes \ldots \otimes \delta_{g_q}$ induces the group homomorphism $Hom_{kG}(kG^{\otimes^q}, A)^{\times} \to Hom_{set}(G \times \ldots \times G, A^{\times})$, because $\{\delta_{g_1} \otimes \ldots \otimes \delta_{g_q} | g_1 \times \ldots \times g_q \in G \times \ldots \times G\}$ is a basis for $kG \otimes \ldots \otimes kG$.

When $q = 0$, $Hom_{kG}(kG^{\otimes^0}, A)^{\times} = Hom(k, A)^{\times}$ which is canonically isomorphic to $A^{\times}$, the 0-th group in the standard group cohomology complex.

Then, the group morphism $Hom_{kG}(kG^{\otimes^q}, A) \to Hom_{set}(G \times \ldots \times G, A^{\times})$ and $Hom_{kG}(kG^{\otimes^0}, A) = Hom(k, A)$ form a morphism of complexes and $H^q(kG, A) \simeq H^q(G, A^{\times})$.

2.1.2 Crossed products, equivalences classes of cleft extensions and $H^2(H, A)$

In [10], 1986, R. Blattner, M. Cohen e S. Montgomery extended the notions of Crossed Product and Inner (weak) actions of arbitrary Hopf algebras on noncommutative algebras. Then, in [28], 1992, S. Montgomery characterized a Hopf crossed product as a cleft extension and gave necessary and sufficient conditions for two crossed products to be isomorphic.

First of all, we say that an arbitrary Hopf algebra H measures an algebra A if there is a k-linear map $H \otimes A \to A$ given by $h \otimes a \mapsto h \cdot a$, such that $h \cdot 1_A = \varepsilon(h) 1_A$ and $h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b)$, for all $h \in H$, a, $b \in A$.

Definition 2.4 *Let H be a Hopf algebra and A an algebra. Assume that H measures A and that σ is an invertible map in $Hom_k(H \otimes H, A)$. The crossed product $A \#_{\sigma} H$ of A with H is the set $A \otimes H$ as a vector space, with multiplication*

$$(a\#h)(b\#k) = \sum a(h_{(1)} \cdot b)\sigma(h_{(2)}, k_{(1)})\#h_{(3)}k_{(2)}$$

for all h, $k \in H$, a, $b \in A$. Here we write $a\#h$ for the tensor $a \otimes h$.

Lemma 2.5 *$A\#_{\sigma}H$ is an associative algebra with identity element $1\#1$ if and only if the following two conditions are satisfied:*

(1) A is a twisted H-module, that is, $1_H \cdot a = a$, for all $a \in A$ and

$$h \cdot (k \cdot a) = \sum \sigma(h_{(1)}, k_{(1)})(h_{(2)}k_{(2)} \cdot a)\sigma^{-1}(h_{(3)}, k_{(3)}), \qquad (2.1)$$

all h, $k \in H$, $a \in A$.

(2) σ is a cocycle, that is, $\sigma(h, 1_H) = \sigma(1_H, h) = \varepsilon(h)1_A$, all $h \in H$, and

$$\sum(h_{(1)} \cdot \sigma(k_{(1)}, l_{(1)}))\sigma(h_{(2)}, k_{(2)}l_{(2)}) = \sum \sigma(h_{(1)}, k_{(1)})\sigma(h_{(2)}k_{(2)}, l_{(2)}) \quad (2.2)$$

for all $h,\ k,\ l \in H$.

□

Note that A need not be an H-module and that σ does not necessarily have values in the center of A. In the case where A is commutative and H is cocommutative we always have that A is an H-module and then, (2.1) is not needed. The next proposition gives a necessary and sufficient condition for A to be an H-module when H is cocommutative.

Proposition 2.6 *Let H be cocommutative and A a twisted H-module which is measured by H. Then, A is an H-module if, and only if, $\sigma(H \otimes H) \subseteq Z(A)$, the center of A.*

□

To characterize crossed products $B = A\#_\sigma H$ as special kind of extensions $A \subset B$, we recall

Definition 2.7 *Let $A \subset B$ be k-algebras, and H a Hopf algebra.*

(1) $A \subset B$ is a (right) H-extension if B is a right H-comodule algebra with $B^{coH} = A$.

(2) The H-extension $A \subset B$ is H-cleft if there exists a right H-comodule map $\gamma : H \to B$ which is (convolution) invertible.

Observe that we always can assume $\gamma(1_H) = 1_B$. In fact, if not, we replace γ by $\overline{\gamma} = \gamma(1)^{-1}\gamma$.

To prove that cleft extensions are related with crossed products, we need the follows results:

Proposition 2.8 *Let $A \subset B$ be a right H-extension, which is H-cleft via $\gamma : H \to B$ such that $\gamma(1_H) = 1_B$. Then, there is an action of H on A, given by*

$$h \cdot a = \sum \gamma(h_{(1)})a\gamma^{-1}(h_{(2)}), \qquad \forall\ a \in A, h \in H \quad (2.3)$$

and a convolution invertible map $\sigma : H \otimes H \to A$ given by

$$\sigma(h, k) = \sum \gamma(h_{(1)})\gamma(k_{(1)})\gamma^{-1}(h_{(2)}k_{(2)}), \qquad \forall\ h,\ k \in H. \quad (2.4)$$

This action endows B with a structure of an H-crossed product over A. Moreover, the algebra isomorphism $\Phi : A\#_\sigma H \to B$ given *by* $a\#h \mapsto a\gamma(h)$ *is both a left* A*-module and right* H*-comodule map, where* $A\#_\sigma H$ *is a right* H*-comodule via* $a\#h \mapsto \sum a\#h_{(1)} \otimes h_{(2)}$.

□

We require a technical lemma.

Lemma 2.9 *Assume that* $A \subset B$ *is a right* H*-extension, via* $\rho : B \to B \otimes H$*, and that* $A \subset B$ *is* H*-cleft via* γ *with* $\gamma(1_H) = 1_B$*. Then,*

(1) $\rho \circ \gamma^{-1} = (\gamma^{-1} \otimes S) \circ \tau \circ \Delta$;

(2) for any $b \in B$, $\sum b_{(0)}\gamma^{-1}(b_{(1)}) \in A = B^{coH}$.

□

The lemma enables us to define an inverse of Φ. In fact, we can define

$$\Psi : B \to A\#_\sigma H \qquad \text{by} \qquad b \mapsto \sum b_{(0)}\gamma^{-1}(b_{(1)})\#b_{(2)}.$$

Proposition 2.10 *Let* $A\#_\sigma H$ *be a crossed product, and define the map* $\gamma : H \to A\#_\sigma H$ *by* $\gamma(h) = 1_A\#h$*. Then,* γ *is convolution-invertible, with inverse*

$$\gamma^{-1}(h) = \sum \sigma^{-1}(S(h_{(2)}), h_{(3)})\#S(h_{(1)}).$$

In particular, $A \hookrightarrow A\#_\sigma H$ *is* H*-cleft.*

□

Then, we conclude

Theorem 2.11 *An* H*-extension* $A \subset B$ *is* H*-cleft if, and only if,* $B \simeq A\#_\sigma H$.

□

Another important result about crossed products is to stablish necessary and sufficient conditions for two crossed products to be isomorphic.

Theorem 2.12 *Let* A *be an algebra and* H *be a Hopf algebra, with two crossed product actions* $h \otimes a \mapsto h \cdot a$*,* $h \otimes a \mapsto h \bullet a$ *with respect to two cocycles* σ*,* $\sigma' : H \otimes H \to A$*, respectively. Assume that*

$$\phi : A\#_\sigma H \to A\#^{\bullet}_{\sigma'} H$$

is an algebra isomorphism, which is also a left A*-module, right* H*-comodule map. Then, there exists an invertible map* $u \in Hom(H, A)$ *such that, for all* $a \in A$*,* h*,* $k \in H$*,*

(i) $\phi(a\#h) = \sum au(h_{(1)})\#^{\bullet} h_{(2)}$;
(ii) $h \bullet a = \sum u^{-1}(h_{(1)})(h_{(2)} \cdot a)u(h_{(3)})$;
(iii) $\sigma'(h,k) = \sum u^{-1}(h_{(1)})(h_{(2)} \cdot u^{-1}(k_{(1)}))\sigma(h_{(3)}, k_{(2)})u(h_{(4)}k_{(3)})$.

Conversely given a map $u \in Hom(H, A)$ such that (ii) and (iii) hold, then, the map ϕ in (i) is an isomorphism.

□

The theorem above suggests the following definition:

Definition 2.13 *Let H be a Hopf algebra and A an algebra. Two crossed products $A\#_{\sigma}H$ and $A\#^{\bullet}_{\sigma'}H$ are equivalent if there exists an algebra isomorphism $\phi : A\#_{\sigma}H \to A\#^{\bullet}_{\sigma'}H$ which is a left A-module, right H-comodule morphism.*

These ideas shown here were intended to generate a general cohomology theory for algebras over Hopf algebras, however, in the last two decades it has not yet been possible to make much progress in this regard. We recall that in [30], it was proved that for H cocommutative and A commutative, there is a bijective correspondence between the second cohomology group $H^2(H, A)$ and the equivalence classes of H-cleft extensions B of A. Note that in this case A is an H-module, and in addition all the crossed products in a given equivalence class have the same H-action, by 2.12, (ii). Only the cocycle may be differ.

2.2 Partial cohomology of groups

In this section, we present the begining of the theory developed by M. Dokuchaev and M. Khrypchenko who inspirate us to think in cohomology for partial actions of Hopf algebras. In their paper, they introduced a new kind of cohomology theory of groups where partial actions of groups over commutative monoids are considered. Their ideas consisted to consider a unital twisted partial action of a group G on a commutative ring A, then, they can derive the concept of a partial 2-cocycle (the twisting) whose values belong to groups of invertible elements of appropriate ideals of A. By an equivalence of twisted partial actions introduced in [18], the concept of a partial 2-coboundary follows and then, replacing A by a commutative multiplicative monoid, the second cohomology group $H^2(G, A)$ is defined. In a similar way, we are able to define the n-th groups cohomology $H^n(G, A)$ with arbitrary n. For more details, we suggest [19].

Let G a group and A a semigroup. A partial action θ of G on A is a collection of semigroup isomorphisms $\theta_x : A_{x^{-1}} \to A_x$, where A_x is an ideal of A, $x \in G$, such that

(i) $A_1 = A$ and $\theta_1 = Id_A$;

(ii) $\theta_x(A_{x^{-1}} \bigcap A_y) = A_x \bigcap A_{xy}$;

(iii) $\theta_x \circ \theta_y = \theta_{xy}$ on $A_{y^{-1}} \bigcap A_{y^{-1}x^{-1}}$.

When A is a commutative monoid and each ideal A_x is unital, i.e., A_x is generated by an idempotent $1_x = 1_x^A$, which is central in A, we shall say that θ is a unital partial action. Then $A_x \bigcap A_y = A_x A_y$, so the properties (ii) and (iii) from the above definition can be replaced by

(ii') $\theta_x(A_{x^{-1}} A_y) = A_x A_{xy}$;

(iii') $\theta_x \circ \theta_y = \theta_{xy}$ on $A_{y^{-1}} A_{y^{-1}x^{-1}}$.

We observe that (ii') implies a more general equality

$$\theta_x(A_{x^{-1}} A_{y_1} \dots A_{y_n}) = A_x A_{xy_1} \dots A_{xy_n}, \tag{2.5}$$

which follows because $A_{x^{-1}} A_{y_1} \dots A_{y_n} = (A_{x^{-1}} A_{y_1}) \dots (A_{x^{-1}} A_{y_n})$.

Definition 2.14 *A commutative monoid A with a unital partial action θ of G on A will be called a (unital) partial G-module.*

A morphism of partial actions $(A, \theta) \to (A', \theta')$ of G is a homomorphism of semigroups $\phi : A \to A'$ such that $\phi(A_x) \subseteq A'_x$ and $\phi \circ \theta_x = \theta'_x \circ \phi$ on $A_{x^{-1}}$.

We denote by $pMod(G)$ the category of (unital) partial G-modules and their homomorphisms. Sometimes (A, θ) will be simplified to A.

Definition 2.15 *Let $A \in pMod(G)$ and n be a positive integer. An n-cochain of G with values in A is a function $f : G^n \to A$, such that $f(x_1, \dots, x_n)$ is an invertible element of the ideal $A_{(x_1,\dots,x_n)} = A_{x_1} A_{x_1 x_2} \dots A_{x_1 \dots x_n}$. By a 0-cochain we shall mean an invertible element of A.*

Denote the set of n-cochains by $C^n(G, A)$. It is an abelian group under the pointwise multiplication. Indeed, its identity is

$$e_n(x_1, \dots, x_n) = 1_{x_1} 1_{x_1 x_2} \dots 1_{x_1 \dots x_n}$$

and the inverse of $f \in C^n(G, A)$ is $f^{-1}(x_1, \dots, x_n) = f(x_1, \dots, x_n)^{-1}$, where $f(x_1, \dots, x_n)^{-1}$ means the inverse of $f(x_1, \dots, x_n)$ in $A_{(x_1,\dots,x_n)}$.

Definition 2.16 *Let $(A, \theta) \in pMod(G)$ and n be a positive integer. For any $f \in C^n(G, A)$ and $x_1, \ldots, x_{n+1} \in G$ define*

$$\begin{aligned}(\delta^n f)(x_1, \ldots, x_{n+1}) &= \theta_{x_1}(1_{x_1^{-1}} f(x_2, \ldots, x_{n+1})) \\ &\prod_{i=1}^{n} f(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1})^{(-1)^i} \\ &f(x_1, \ldots, x_n)^{(-1)^{n+1}}. \qquad (2.6)\end{aligned}$$

Here the inverse elements are taken in the corresponding ideals. If $n = 0$ and a is an invertible element of A, we set $(\delta^0 a)(x) = \theta_x(1_{x^{-1}} a) a^{-1}$.

The next result shows us that δ^n is a homomorphism such that $\delta^{n+1}\delta^n f = e_{n+2}$. We present the full prove for the reader to compare with the results obtained later for partial cohomology of Hopf algebras.

Proposition 2.17 *[19] The map $\delta^n : C^n(G, A) \to C^{n+1}(G, A)$ is a homomorphism such that*

$$\delta^{n+1}\delta^n f = e_{n+2} \qquad (2.7)$$

for any $f \in C^n(G, A)$.

Proof: Let $f \in C^n(G, A)$. We check first that $\delta^n f \in C^{n+1}(G, A)$. Indeed, for $x_1, \ldots, x_{n+1} \in G$ the element $f(x_2, \ldots, x_{n+1})$ is invertible in $A_{(x_2, \ldots, x_{n+1})}$. Then, being multiplied by $1_{x_1^{-1}}$, it becomes an invertible element of $A_{x_1^{-1}} A_{(x_2, \ldots, x_{n+1})}$. Therefore, $\theta_{x_1}(x_1^{-1} f(x_2, \ldots, x_{n+1}))$ is invertible in $A_{(x_1, \ldots, x_{n+1})}$ because θ_{x_1} maps isomorphically $A_{x_1^{-1}} A_{(x_2, \ldots, x_{n+1})}$ onto $A_{(x_1, \ldots, x_{n+1})}$ by 2.5. Since the product of invertible elements of some ideals is invertible in the product of these ideals, then by 2.6 the image $(\delta^n f)(x_1, \ldots, x_{n+1})$ is invertible in

$$A_{(x_1, \ldots, x_{n+1})} \left(\prod_{i=1}^{n} A_{(x_1, \ldots, x_i x_{i+1}), \ldots, x_{n+1})} \right) A_{(x_1, \ldots, x_n)} = A_{(x_1, \ldots, x_{n+1})}.$$

As A is a commutative, to see that δ^n is a homomorphism, it suffices to note that

$$\begin{aligned} &\theta_{x_1}(1_{x_1^{-1}} fg(x_2, \ldots, x_{n+1})) \\ =\ &\theta_{x_1}(1_{x_1^{-1}} f(x_2, \ldots, x_{n+1}) 1_{x_1^{-1}} g(x_2, \ldots, x_{n+1}) \\ =\ &\theta_{x_1}(1_{x_1^{-1}} f(x_2, \ldots, x_{n+1}))\theta_{x_1}(1_{x_1^{-1}} g(x_2, \ldots, x_{n+1}))\end{aligned}$$

It remains to prove that $\delta^{n+1}\delta^n f = e_{n+2}$. Take arbitrary $x_1, \ldots, x_{n+2} \in G$. The factors in the product $(\delta^{n+1}\delta^n f)(x_1, \ldots, x_{n+2})$ to which the partial actions is applied are as follows:

$$\theta_{x_1}(1_{x_1^{-1}}\theta_{x_2}(1_{x_2^{-1}}f(x_3, \ldots, x_{n+2}))),$$
$$\theta_{x_1x_2}(1_{x_2^{-1}x_1^{-1}}f(x_3, \ldots, x_{n+2})^{-1}),$$
$$\theta_{x_1}(1_{x_1^{-1}}f(x_2, \ldots, x_{n+1})^{(-1)^{n+1}}),$$
$$\theta_{x_1}(1_{x_1^{-1}}f(x_2, \ldots, x_{n+1})^{(-1)^{n+2}}),$$
$$\theta_{x_1}(1_{x_1^{-1}}f(x_2, \ldots, x_ix_{i+1}, \ldots, x_{n+2})^{(-1)^{i-1}}),\ 2 \geq i \geq n+1,$$
$$\theta_{x_1}(1_{x_1^{-1}}f(x_2, \ldots, x_ix_{i+1}, \ldots, x_{n+2})^{(-1)^{i}}),\ 2 \geq i \geq n+1.$$

The product of all the factors, except the first two, is $e_{n+2}(x_1, \ldots, x_{n+2})$ for $n \leq 1$. For $n = 0$ the product is $e_1(x_1)$. Furthermore,

$$\begin{aligned} & \theta_{x_1}(1_{x_1^{-1}}\theta_{x_2}(1_{x_2^{-1}}f(x_3, \ldots, x_{n+2}))) = \\ = \ & \theta_{x_1}(\theta_{x_2}(1_{x_2^{-1}}1_{x_2^{-1}x_1^{-1}}f(x_3, \ldots, x_{n+2}))) \\ = \ & \theta_{x_1x_2}(1_{x_2^{-1}}1_{x_2^{-1}x_1^{-1}}f(x_3, \ldots, x_{n+2})) \end{aligned}$$

By the property (iii') from the definition of a partial action. After multiplying this by the second factor we shall obtain

$$\begin{aligned} \theta_{x_1x_2}(1_{x_2^{-1}}1_{x_2^{-1}x_1^{-1}}e_n(x_3, \ldots, x_{n+2})) &= 1_{x_1}1_{x_1x_2}e_n(x_1x_2x_3, x_4, \ldots, x_{n+2}) \\ &= e_{n+2}(x_1, \ldots, x_{n+2}). \end{aligned}$$

Any other factor in $(\delta^{n+1}\delta^n f)(x_1, \ldots, x_{n+2})$ appears together with its inverse, as in the classical case, and multiplying such a pair we obtain a product of some of the idempotents $1_{x_1}, 1_{x_1x_2}, \ldots$. Thus, $(\delta^{n+1}\delta^n f)(x_1, \ldots, x_{n+2}) = e_{n+2}(x_1, \ldots, x_{n+2})$ as desired.

■

Definition 2.18 *The map δ^n is called a coboundary homomorphism. As in the classical case, we define the abelian grups $Z^n(G, A) = ker\delta^n$, $B^n(G, A) = Im\delta^{n-1}$ and $H^n(G, A) = ker\delta^n/Im\delta^{n-1}$ of partial n-cocycles, n-coboundaries and n-cohomologies of G with values in A, $n \leq 1$ ($H^0(G, A) = Z^0(G, A) = ker\delta^0$).*

For example,

$$H^0(G, A) = Z^0(G, A) = \{a \in A^\times | \theta_x(1_{x^{-1}}a) = 1_x a, \forall x \in G\}$$

$$B^1(G,A) = \{f \in C^1(G,A) | f(x) = \theta_x(1_{x^{-1}} a a^{-1}),\ \text{for some } a \in A^\times\}$$

(here and below $A^\times$ denotes the group of invertible elements of A). Notice that $H^0(G,A)$ is the subgroup of 0-invariants of $A^\times$. Furthermore, for $f \in C^1(G,A)$

$$(\delta^1 f)(x,y) = \theta_x(1_{x^{-1}} f(y)) f(xy)^{-1} f(x),$$

so

$$Z^1(G,A) = \{f \in C^1(G,A) | 1_x f(xy) = f(x)\theta_x(1_{x^{-1}} f(y)), \forall x,y \in G\},$$

and, for some $f \in C^1(G,A)$

$$B^2(G,A) = \{g \in C^2(G,A) | g(x,y) = \theta_x(1_{x^{-1}} f(y)) f(xy)^{-1} f(x)\}$$

For $n = 2$ we have

$$\delta^2 f(x,y,z) = \theta_x(1_{x^{-1}} f(y,z)) f(x,yz)^{-1} f(x,y)^{-1},$$

with $f \in C^2(G,A)$, and $\forall\ x,y,z \in G$,

$$Z^2(G,A) = \{f \in C^2(G,A) | \theta_x(1_{x^{-1}} f(y,z)) f(x,yz) = f(xy,z) f(x,y)\}.$$

Observe that if one takes a unital twisted partial action (see [[17], Def. 2.1]) of G on a commutative ring A, then it is readily seen that the twisting is a 2-cocycle with values in the partial G module A, and the concept of equivalent unital twisted partial actions from [[18], Def 6.1] is exactly the notion of cohomologous 2-cocycles from Definition2.18.

2.3 Partial Actions of Hopf Algebras

The theory of partial actions appeared for the first time in [21], where R. Exel introduced a new and successful method to study C^*-algebras. In this paper, Exel defined the notion of partial action of a group G on a set X to calculate the K-theory of some C^*-algebras which have an action by automorphisms of the circle $\mathbb{S}^1$. In [16], M. Dokuchaev e R. Exel defined partial group actions on algebras and partial skew group algebras, giving an algebraic context for partial actions and arousing the interest of algebraists. The algebraic theory of partial actions and partial representations of groups urderwent several advances and one of interest consists to extend Galois theory for commutative algebras. Them, S. Caenepeel and K. Janssen [14], defined a partial Hopf-Galois theory and introduced what is a partial action of a Hopf algebra H over an algebra A.

Definition 2.19 *[14] A partial action of a Hopf algebra H over an algebra A is a linear map $\cdot : H \otimes A \to A$, such that, for every $a, b \in A$ and $h, l \in H$, we have*

(PA1) $1_H \cdot a = a$;

(PA2) $h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b)$;

(PA3) $h \cdot (l \cdot a) = (h_{(1)} \cdot 1_A)(h_{(2)} l \cdot a)$.

We say that the partial action is symmetric if, in addition, we have

(PA3') $h \cdot (l \cdot a) = (h_{(1)} l \cdot a)(h_{(2)} \cdot 1_A)$.

The algebra A with a partial action of H on A is said to be a partial H-module algebra.

Note that, if H is a cocommutative Hopf algebra and A is a commutative algebra, then every partial action of H on A is automatically symmetric.

Example 2.20 *[2] Let G be a group, recall that a partial action of G over an algebra A is a pair $(\{A_g\}_{g \in G}, \{\alpha_g : A_{g^{-1}} \to A_g\}_{g \in G})$, where A_g is an ideal of A for each $g \in G$ and α_g is an isomorphism of (not necessarily unital) algebras. We say that the partial action of G on A is unital if, for each $g \in G$, $A_g = 1_g A$, where 1_g is a central idempotent in A and α_g is a unital isomorphism between $A_{g^{-1}}$ and A_g. For the case where $H = kG$, the group algebra of G, symmetric partial actions of kG are in one to one correspondence with unital partial actions of G. This correspondence can be easily seen: Given a unital partial action $(\{A_g = 1_g A\}_{g \in G}, \{\alpha_g : A_{g^{-1}} \to A_g\}_{g \in G})$, one defines $\cdot : kG \otimes A \to A$, by $\delta_g \cdot a = \alpha_g(1_{g^{-1}} a)$. Conversely, given a symmetric partial action of kG over A, define, for each $g \in G$, the idempotents $1_g = \delta_g \cdot 1_A$, by them, construct the ideals $A_g = 1_g A$ and the isomorphisms $\alpha_g = \delta_g \cdot \_\_|_{A_{g^{-1}}}$.*

Example 2.21 *[2] Given a Hopf algebra H, a left H-module algebra B and a central idempotent $e \in B$, one can define a partial action of H on $A = eB$. Denoting by $\triangleright$ the action of H over B, the induced partial action is given by $h \cdot ea = e(h \triangleright (ea))$, for every $a \in B$ and $h \in H$.*

The next two examples will be explored with more details throughout this paper for giving examples of cohomologies.

Example 2.22 *Consider a group G, let us see the partial actions of the Hopf algebra $H = kG$ over $A = k$, the base field. A partial action $\cdot : kG \otimes k \to k$, associates to each $g \in G$ the linear transformation $\delta_g \cdot \_\_ : K \to k$, this is the same as defining a linear functional*

$\lambda : kG \to k$. Denoting $\lambda(\delta_g)$ simply by λ_g, one can write $\delta_g \cdot a = \lambda_g a$, for every $a \in k$. Using the functional λ, the axiom (PA1) says that $\lambda_e = 1$, where e is the neutral element of the group G. Axiom (PA2), in its turn, implies that $\lambda_g = \lambda_g \lambda_g$, for every $g \in G$, and consequently $\lambda_g = 1$ or $\lambda_g = 0$. Define

$$H = \{g \in G \mid \lambda_g = 1\},$$

it is clear that $e \in H$. Axiom (PA3) says that $\lambda_g \lambda_h = \lambda_g \lambda_{gh}$, this implies that for $g, h \in H$, we have $gh \in H$. Finally, putting $h = g^{-1}$ in the previous identity, we conclude that $g \in H$ implies that $g^{-1} \in H$, therefore H is a subgroup of G. It is easy to see that the action is global if, and only if $H = G$. Then, we can label the partial actions of kG over k by the subgroups of G.

Example 2.23 *Let G be a finite abelian group and consider $H = (kG)^* = \langle p_g |\ g \in G\rangle$, the dual of the group algebra, with bialgebra structure given by*

$$p_g p_h = \delta_{g,h} p_g, \quad \mathbf{1} = \sum_{g \in G} p_g, \quad \Delta(p_g) = \sum_{g \in G} p_h \otimes p_{h^{-1}g}, \quad \varepsilon(p_g) = \delta_{g,e}.$$

As in the previous example, partial actions of $(kG)^*$ over k are associated to a linear functional $\lambda : (kG)^* \to k$ defined by $\lambda(p_g) = \lambda_{p_g} = p_g \cdot 1$. In this case, the axioms for a partial action (PA1), (PA2) and (PA3) become, respectively

$$\sum_{g \in G} \lambda_{p_g} = 1; \quad \lambda_{p_g} = \sum_{h \in G} \lambda_{p_h} \lambda_{p_{h^{-1}g}}; \quad \lambda_{p_g} \lambda_{p_h} = \lambda_{p_{gh^{-1}}} \lambda_{p_h} = \lambda_{p_{gh^{-1}}} \lambda_{p_g}.$$

Defining $L = \{g \in G |\ \lambda_{p_g} \neq 0\}$, one can see that L is a subgroup of G: First, as $\sum_{g \in G} \lambda_{p_g} = 1$ then there exists some $g \in G$ such that $\lambda_{p_g} \neq 0$, and therefore $L \neq \emptyset$. Moreover, given $g,\ h \in L$, (PA3) says that $0 \neq \lambda_{p_g} \lambda_{p_h} = \lambda_{p_{gh^{-1}}} \lambda_{p_g}$, which implies that $\lambda_{p_{gh^{-1}}} \neq 0$, and therefore $gh^{-1} \in L$.

In order to analyse the possible values of λ_{p_g}, for $g \in L$, take $g = h$ in the third equation, then $\lambda_{p_g} \lambda_{p_g} = \lambda_{p_{gg^{-1}}} \lambda_{p_g} = \lambda_{p_e} \lambda_{p_g}$. This implies that, $\lambda_{p_g} = \lambda_{p_e}$, $\forall\ g \in L$. Finally, from the first equation,

$$1 = \sum_{g \in G} \lambda_{p_g} = \sum_{g \in L} \lambda_{p_e} = |L| \lambda_{p_e},$$

and therefore $\lambda_{p_g} = \lambda_{p_e} = \frac{1}{|L|}$, for all $g \in L$. We leave to the reader the verification that the action is global if, and only if, $L = G$.

We conclude that the partial actions of $(kG)^*$ over the base field k are classified by subgroups of G and given by

$$\lambda_{p_g} = \begin{cases} \frac{1}{|L|} & , \ g \in L \\ 0 & , \text{otherwise.} \end{cases}$$

Chapter 3

Cohomology for partial actions

In his 1968's seminal article, M.E. Sweeder presented a cohomology theory for commutative algebras which are modules over a given cocommutative Hopf algebra. Basically, the cochain complexes $C^n(H, A)$ are defined as the abelian groups of the invertible elements of the commutative convolution algebras $\mathrm{Hom}_k(H^{\otimes n}, A)$. The main difference between the cohomology theory of global and partial actions is that in the partial case one needs to find appropriated unital ideals in the convolution algebras in order to define correctly the cochain complexes. These ideals are constructed upon a system of idempotents for the convolution algebras.

Henceforth, for sake of simplicity, we will denote $f(h^1 \otimes \cdots \otimes h^n)$ just by $f(h^1, \ldots, h^n)$ and H is always a cocommutative Hopf algebra acting partially on a commutative algebra A.

3.1 A system of idempotents for the convolution algebras

We start introducing some idempotent elements of $\mathrm{Hom}_k(H^{\otimes n}, A)$ which are important throughout this thesis. As the convolution algebras are commutative, for each $n \geq 0$, an idempotent is automatically a central idempotent. Moreover, the convolution product of a finite number of idempotents is also an idempotent. In what follows, we introduce a nested system of idempotents in the convolution algebra related to the partial action.

Proposition 3.1 *For each $n \geq 1$, the linear maps*

$$\begin{array}{cccc} \widetilde{e}_n : & H^{\otimes n} & \longrightarrow & A \\ & (h^1 \otimes \cdots \otimes h^n) & \mapsto & (h^1 \cdots h^n) \cdot 1_A \end{array}$$

are idempotent in the corresponding convolution algebras $Hom_k(H^{\otimes n}, A)$.

Proof: Indeed, consider $n \geq 1$ and $(h^1 \otimes \cdots \otimes h^n) \in H^{\otimes n}$, then

$$\begin{aligned} \widetilde{e}_n * \widetilde{e}_n(h^1, \ldots, h^n) &= \widetilde{e}_n(h^1_{(1)}, \ldots, h^n_{(1)})\widetilde{e}_n(h^1_{(2)}, \ldots, h^n_{(2)}) \\ &= ((h^1_{(1)} \cdots h^n_{(1)}) \cdot 1_A)((h^1_{(2)} \cdots h^n_{(2)}) \cdot 1_A) \\ &= ((h^1 \cdots h^n)_{(1)} \cdot 1_A)((h^1 \cdots h^n)_{(2)} \cdot 1_A) \\ &\overset{(PA2)}{=} (h^1 \cdots h^n) \cdot 1_A \\ &= \widetilde{e}_n(h^1, \ldots, h^n). \end{aligned}$$

This proves our statement.

■

Proposition 3.2 *Let $n < m$ and $e \in Hom_k(H^{\otimes n}, A)$ an idempotent, then*

$$e_{n,m} = e \otimes \underbrace{\varepsilon_H \otimes \cdots \otimes \varepsilon_H}_{m-n} \in Hom_k(H^{\otimes m}, A)$$

is an idempotent in $Hom_k(H^{\otimes m}, A)$.

Proof: Take any $n < m$ and $(h^1 \otimes \cdots \otimes h^m) \in H^{\otimes m}$, then

$$\begin{aligned} & e_{n,m} * e_{n,m}(h^1, \ldots, h^m) = \\ &= e_{n,m}(h^1_{(1)}, \ldots, h^m_{(1)})e_{n,m}(h^1_{(2)}, \ldots, h^m_{(2)}) \\ &= e(h^1_{(1)}, \ldots, h^n_{(1)})\varepsilon(h^{n+1}_{(1)}) \ldots \varepsilon(h^m_{(1)})e(h^1_{(2)}, \ldots, h^n_{(2)})\varepsilon(h^{n+1}_{(2)}) \cdots \varepsilon(h^m_{(2)}) \\ &= e(h^1_{(1)}, \ldots, h^n_{(1)})e(h^1_{(2)}, \ldots, h^n_{(2)})\varepsilon(h^{n+1}_{(1)}) \cdots \varepsilon(h^m_{(1)})\varepsilon(h^{n+1}_{(2)}) \cdots \varepsilon(h^m_{(2)}) \\ &\overset{(*)}{=} e * e(h^1, \ldots, h^n)\varepsilon(h^{n+1}) \cdots \varepsilon(h^m) \\ &= e(h^1, \ldots, h^n)\varepsilon(h^{n+1}) \cdots \varepsilon(h^m) \\ &= e_{n,m}(h^1, \ldots, h^m). \end{aligned}$$

The identity $(*)$ follows from $\varepsilon(h_{(1)})\varepsilon(h_{(2)}) = \varepsilon(h_{(1)}\varepsilon(h_{(2)})) = \varepsilon(h)$, and therefore $e_{n,m}$ is idempotent.

■

Definition 3.3 *For arbitrary $n \geq 1$ and $1 \leq l \leq n$, we define:*

$$\widetilde{e}_{l,n} := (\widetilde{e}_l)_{l,n} = \widetilde{e}_l \otimes \underbrace{\varepsilon_H \otimes \cdots \otimes \varepsilon_H}_{n-l}.$$

Note that, for $l = n$ in the above definition $\widetilde{e}_{n,n} = \widetilde{e}_n$.

Definition 3.4 *For $n \geq 1$, we define*

$$e_n := \widetilde{e}_{1,n} * \widetilde{e}_{2,n} * \cdots * \widetilde{e}_n \in Hom_k(H^{\otimes n}, A).$$

The next proposition gives us a useful characterization of the idempotents $e_n \in Hom_k(H^{\otimes n}, A)$.

Proposition 3.5 *For any $(h^1 \otimes \ldots \otimes h^n) \in H^{\otimes n}$ we have that*

$$e_n(h^1, \ldots, h^n) = h^1 \cdot (h^2 \cdot (\ldots \cdot (h^n \cdot 1_A) \ldots)).$$

Proof: Take $(h^1 \otimes \ldots \otimes h^n) \in H^{\otimes n}$, then

$$\begin{aligned}
& e_n(h^1, \ldots, h^n) = \widetilde{e}_{1,n} * \widetilde{e}_{2,n} * \cdots * \widetilde{e}_n(h^1, \ldots, h^n) \\
= \;& \widetilde{e}_{1,n}(h^1_{(1)}, \ldots, h^n_{(1)})\widetilde{e}_{2,n}(h^1_{(2)}, \ldots, h^n_{(2)}) \cdots \widetilde{e}_n(h^1_{(n)}, \ldots, h^n_{(n)}) \\
= \;& \widetilde{e}_1(h^1_{(1)})\varepsilon(h^2_{(1)}) \cdots \varepsilon(h^n_{(1)})\widetilde{e}_2(h^1_{(2)}, h^2_{(2)})\varepsilon(h^3_{(2)}) \cdots \varepsilon(h^n_{(2)}) \\
& \cdots \widetilde{e}_{n-1}(h^1_{(n-1)}, \ldots, h^{n-1}_{(n-1)})\varepsilon(h^n_{(n-1)})\widetilde{e}_n(h^1_{(n)}, \ldots, h^n_{(n)}) \\
= \;& (h^1_{(1)} \cdot 1_A)\varepsilon(h^2_{(1)}) \cdots \varepsilon(h^n_{(1)})(h^1_{(2)}h^2_{(2)} \cdot 1_A)\varepsilon(h^3_{(2)}) \ldots \varepsilon(h^n_{(2)}) \\
& \ldots (h^1_{(n)} \cdots h^n_{(n)} \cdot 1_A) \\
= \;& (h^1_{(1)} \cdot 1_A)(h^1_{(2)}h^2_{(1)} \cdot 1_A) \ldots (h^1_{(n-1)}h^2_{(n-2)} \ldots h^{n-1}_{(1)} \cdot 1_A) \\
& (h^1_{(n)}h^2_{(n-1)} \ldots h^{n-1}_{(2)}h^n \cdot 1_A) \\
= \;& (h^1_{(1)} \cdot 1_A)(h^1_{(2)}h^2_{(1)} \cdot 1_A) \ldots ((h^1_{(n-1)}h^2_{(n-2)} \ldots h^{n-1})_{(1)} \cdot 1_A) \\
& ((h^1_{(n-1)}h^2_{(n-2)} \ldots h^{n-1})_{(2)}h^n \cdot 1_A) \\
\overset{(PA3)}{=} & (h^1_{(1)} \cdot 1_A)(h^1_{(2)}h^2_{(1)} \cdot 1_A) \ldots ((h^1_{(n-1)} \ldots h^{n-2}_{(2)}h^{n-1} \cdot (h^n \cdot 1_A)) \\
= \;& (h^1_{(1)} \cdot 1_A)(h^1_{(2)}h^2_{(1)} \cdot 1_A) \ldots (h^1_{(n-2)} \ldots h^{n-2}_{(1)} \cdot 1_A) \\
& (h^1_{(n-1)} \ldots h^{n-2}_{(2)}h^{n-1} \cdot (h^n \cdot 1_A)) \\
= \;& (h^1_{(1)} \cdot 1_A)(h^1_{(2)}h^2_{(1)} \cdot 1_A) \ldots ((h^1_{(n-2)} \ldots h^{n-3}_{(1)}h^{n-2})_{(1)} \cdot 1_A) \\
& ((h^1_{(n-2)} \ldots h^{n-3}_{(2)}h^{n-2})_{(2)}h^{n-1} \cdot (h^n \cdot 1_A)) \\
\overset{(PA3)}{=} & (h^1_{(1)} \cdot 1_A)(h^1_{(2)}h^2_{(1)} \cdot 1_A) \ldots (h^1_{(n-1)} \ldots h^{n-3}_{(1)}h^{n-2} \cdot (h^{n-1} \cdot (h^n \cdot 1_A))) \\
= \;& \cdots = h^1 \cdot (h^2 \cdot (\ldots \cdot (h^{n-1} \cdot (h^n \cdot 1_A)) \ldots)).
\end{aligned}$$

in which $(\cdots)$ between the last two equalities means applying repeatedly the process using (PA3) until we obtain the result.

■

3.2 Cochain complexes

Based on the idempotents defined in the previous section, one can define the cochain complexes for the partial action of H on A. For each $n > 0$ define the following ideals of $\mathrm{Hom}_k(H^{\otimes n}, A)$:

$$I(H^{\otimes n}, A) = e_n * \mathrm{Hom}_k(H^{\otimes n}, A) = \{e_n * g : g \in Hom_k(H^{\otimes n}, A)\}.$$

As e_n is a central idempotent, the ideal $I(H^{\otimes n}, A)$ can be considered as a unital algebra with unit e_n. An element $f \in I(H^{\otimes n}, A)$ is said to be (convolution) invertible in this ideal if there is another element $g \in I(H^{\otimes n}, A)$ such that $f * g = g * f = e_n$.

Definition 3.6 *Let H be a cocommutative bialgebra and A be a partial H-module algebra with partial action $\cdot : H \otimes A \to A$. For $n > 0$, a "partial" n-cochain of H with values in A is an invertible element in the ideal $I(H^{\otimes n}, A)$. Denoting by $C^n_{par}(H, A)$ the set of n-cochains, we have that $C^n_{par}(H, A) = (I(H^{\otimes n}, A))^{\times}$. For $n = 0$ we say that a 0-cochain is an invertible element in the agebra A, that is $C^0_{par}(H, A) = A^{\times}$.*

Note that $C^n_{par}(H, A)$ is an abelian group with respect to the convolution product, while $C^0_{par}(H, A) = A^{\times}$, is an abelian group with the ordinary multiplication in A and the unit $e_0 = 1_A$.

Definition 3.7 *For an arbitrary $f \in C^n_{par}(H, A)$, $(h^1 \otimes \cdots \otimes h^{n+1}) \in H^{\otimes n+1}$, we define the "partial" coboundary operator*

$$\begin{aligned}(\delta_n f)(h^1, \ldots, h^{n+1}) &= (h^1_{(1)} \cdot f(h^2_{(1)}, \ldots, h^{n+1}_{(1)})) * \\ &* \prod_{i=1}^{n} f^{(-1)^i}(h^1_{(i+1)}, \ldots, h^i_{(i+1)} h^{i+1}_{(i+1)}, \ldots, h^{n+1}_{(i+1)}) \\ &* f^{(-1)^{n+1}}(h^1_{(n+2)}, \ldots, h^n_{(n+2)}).\end{aligned}$$

If $n = 0$ and $a \in A^{\times}$, we have $(\delta_0 a)(h) = (h \cdot a)a^{-1}$.

The challenge is to prove that the coboundary operators are well defined, that is, for every $f \in C^n_{par}(H, A)$, the map $\delta_n f$ is indeed in $C^{n+1}_{par}(H, A)$. Moreover, one needs to prove that the sequence

$$C^0_{par}(H, A) \stackrel{\delta_0}{\to} C^1_{par}(H, A) \stackrel{\delta_1}{\to} \cdots \stackrel{\delta_{n-1}}{\to} C^n_{par}(H, A) \stackrel{\delta_n}{\to} C^{n+1}_{par}(H, A) \stackrel{\delta_{n+1}}{\to} \cdots$$

is a cochain complex, that is, each δ_n is a homomorphism of abelian groups between $C^n_{par}(H,A)$ and $C^{n+1}_{par}(H,A)$ satisfying $\delta_{n+1}\circ\delta_n(f)=e_{n+2}$, for each $f\in C^n_{par}(H,A)$. For this purpose, we introduce some auxiliary operators which will help us to describe the coboundary operators in a more intrinsic way and whose properties will lead us to the desired results.

Definition 3.8 (1) For each $n\geq 0$ define the map $E^n : C^n_{par}(H,A)\to \mathrm{Hom}_k(H^{\otimes n+1},A)$, given by

$$E^n(f)(h^1,\dots,h^{n+1}) := h^1\cdot f(h^2,\dots,h^{n+1}).$$

(2) For $n<m$ define the map, $i_{n,m} : C^n_{par}(H,A)\to \mathrm{Hom}_k(H^{\otimes m},A)$, given by

$$i_{n,m}(f)(h^1,\dots,h^m) := f(h^1,\dots,h^n)\varepsilon(h^{n+1})\dots\varepsilon(h^m).$$

(3) For $i\in\{1,\dots,n\}$, define the map $\mu_i : H^{\otimes n+1}\to H^{\otimes n}$, given by

$$\mu_i(h^1\otimes\dots\otimes h^{n+1}) = (h^1\otimes\dots\otimes h^ih^{i+1}\otimes\dots\otimes h^{n+1}).$$

With these auxiliary operators, the coboundary operator can be rewritten as

$$\begin{aligned}\delta_n : C^n_{par}(H,A) &\to C^{n+1}_{par}(H,A)\\ f &\mapsto \delta_n(f) := E^n(f)*\left(\prod_{i=1}^{n} f^{(-1)^i}\circ\mu_i\right)* i_{n,n+1}(f^{(-1)^{n+1}}),\end{aligned}$$

and the properties of δ_n are based upon the properties of these operators.

Lemma 3.9 (i) For $f,g\in C^n_{par}(H,A)$, we have $E^n(f*g)=E^n(f)*E^n(g)$.

(ii) $E^n(e_n)=e_{n+1}$.

(iii) For $n<m$, we have $i_{n,m}(f*g) = i_{n,m}(f)*i_{n,m}(g)$, for all $f,\ g\in C^n_{par}(H,A)$.

(iv) $i_{n,m}(e_n)*e_m = e_m$.

(v) For $f,\ g\in C^n_{par}(H,A)$, we have $(f*g)\circ\mu_i = (f\circ\mu_i)*(g\circ\mu_i)$, $\forall\ i\in\{1,\dots,n\}$.

(vi) $(e_n\circ\mu_n)*i_{n,n+1}(e_n) = e_{n+1}$.

(vii) $(e_n\circ\mu_i)*e_{n+1} = e_{n+1}$, $\forall\ i\in\{1,\cdots,n-1\}$.

Proof: Take $(h^1\otimes\cdots\otimes h^{n+1})\in H^{\otimes n+1}$, then,

(i) For $f,\ g \in C^n_{par}(H,A)$,

$$\begin{aligned} & E^n(f*g)(h^1,\dots,\ h^{n+1}) = h^1\cdot(f*g(h^2,\dots,\ h^{n+1}))\\ = \ & h^1\cdot(f(h^2_{(1)},\dots,h^{n+1}_{(1)})g(h^2_{(2)},\dots,h^{n+1}_{(2)}))\\ = \ & (h^1_{(1)}\cdot(f(h^2_{(1)},\dots,h^{n+1}_{(1)}))(h^1_{(2)}\cdot(g(h^2_{(2)},\dots,h^{n+1}_{(2)})))\\ = \ & E^n(f)(h^1_{(1)},\dots,\ h^{n+1}_{(1)})E^n(g)(h^1_{(2)},\dots,\ h^{n+1}_{(2)}). \end{aligned}$$

(ii)

$$\begin{aligned} & E^n(e_n)(h^1,\dots,h^{n+1}) = h^1\cdot e_n(h^2,\dots,h^{n+1})\\ = \ & h^1\cdot(h^2\cdot(\dots\ \cdot(h^{n+1}\cdot 1_A)\dots))\\ = \ & e_{n+1}(h^1,\dots,h^{n+1}). \end{aligned}$$

(iii) For $n<m$ and $f,\ g\in C^n_{par}(H,A)$.

$$\begin{aligned} & i_{n,n+1}(f*g)(h^1,\dots,h^{n+1}) = f*g(h^1,\dots,h^n)\varepsilon(h^{n+1})\\ = \ & f(h^1_{(1)},\dots,h^n_{(1)})g(h^1_{(2)},\dots,h^n_{(2)})\varepsilon(\varepsilon(h^{n+1}_{(1)})h^{n+1}_{(2)})\\ = \ & f(h^1_{(1)},\dots,h^n_{(1)})\varepsilon(h^{n+1}_{(1)})g(h^1_{(2)},\dots,h^n_{(2)})\varepsilon(h^{n+1}_{(2)})\\ = \ & i_{n,n+1}(f)(h^1_{(1)},\dots,h^{n+1}_{(1)})i_{n,n+1}(g)(h^1_{(2)},\dots,h^{n+1}_{(2)})\\ = \ & i_{n,n+1}(f)*i_{n,n+1}(g)(h^1,\dots,h^{n+1}). \end{aligned}$$

(iv) Take $h^1\otimes\ldots\otimes h^m\in H^{\otimes m}$, then,

$$\begin{aligned} & i_{n,m}(e_n)*e_m(h^1,\dots,h^m) =\\ = \ & i_{n,m}(e_n)(h^1_{(1)},\dots,h^m_{(1)})e_m(h^1_{(2)},\dots,h^m_{(2)})\\ = \ & e_n(h^1_{(1)},\dots,h^n_{(1)})\varepsilon(h^{n+1}_{(1)})\dots\varepsilon(h^m_{(1)})(h^1_{(2)}\cdot(\dots\cdot(h^n_{(2)}\cdot(h^{n+1}_{(2)}\cdot(\dots\\ & \dots\cdot(h^m_{(2)}\cdot 1_A)\dots)))\dots))\\ = \ & (h^1_{(1)}\cdot(\dots\cdot(h^n_{(1)}\cdot 1_A)\dots))(h^1_{(2)}\cdot(\dots\cdot(h^n_{(2)}\cdot(h^{n+1}\cdot(\dots\\ & \dots\cdot(h^m\cdot 1_A)\dots)))\dots))\\ \overset{(PA2)}{=} \ & h^1\cdot[(h^2_{(1)}\cdot(\dots\cdot(h^n_{(1)}\cdot 1_A)\dots))(h^2_{(2)}\cdot(\dots\cdot(h^n_{(2)}\cdot(h^{n+1}\cdot(\dots\\ & \dots\cdot(h^m\cdot 1_A)\dots)))\dots))]\\ \overset{(PA2)}{=} \ & h^1\cdot(h^2\cdot(\dots\cdot(h^n\cdot(1_A(h^{n+1}\cdot(\dots\cdot(h^m\cdot 1_A)\dots))))\dots)))\\ = \ & h^1\cdot(h^2\cdot(\dots\ \cdot(h^n\cdot(h^{n+1}\cdot(\dots\ \cdot(h^m\cdot 1_A)\dots)))\dots))\\ = \ & e_m(h^1,\dots,h^m). \end{aligned}$$

(v) For all $i \in \{1, \dots, n\}$ and for $f,\ g \in C^n_{par}(H, A)$

$$
\begin{aligned}
&(f * g) \circ \mu_i(h^1, \dots, h^i, h^{i+1}, \dots, h^{n+1}) = \\
=&(f * g)(h^1, \dots, h^i h^{i+1}, \dots, h^{n+1}) \\
=&f(h^1_{(1)}, \dots, (h^i h^{i+1})_{(1)}, \dots, h^{n+1}_{(1)}) g(h^1_{(2)}, \dots, (h^i h^{i+1})_{(2)}, \dots, h^{n+1}_{(2)}) \\
=&f(h^1_{(1)}, \dots, h^i_{(1)} h^{i+1}_{(1)}, \dots, h^{n+1}_{(1)}) g(h^1_{(2)}, \dots, h^i_{(2)} h^{i+1}_{(2)}, \dots, h^{n+1}_{(2)}) \\
=&f \circ \mu_i(h^1_{(1)}, \dots, h^i_{(1)}, h^{i+1}_{(1)}, \dots, h^{n+1}_{(1)}) g \circ \mu_i(h^1_{(2)}, \dots, h^i_{(2)}, h^{i+1}_{(2)}, \dots, h^{n+1}_{(2)}) \\
=&(f \circ \mu_i) * (g \circ \mu_i)(h^1, \dots, h^i, h^{i+1}, \dots, h^{n+1}).
\end{aligned}
$$

(vi)

$$
\begin{aligned}
& (e_n \circ \mu_n) * i_{n,n+1}(e_n)(h^1, \dots, h^{n+1}) = \\
= \ & e_n \circ \mu_n(h^1_{(1)}, \dots, h^n_{(1)}, h^{n+1}_{(1)}) i_{n,n+1}(e_n)(h^1_{(2)}, \dots, h^n_{(2)}, h^{n+1}_{(2)}) \\
= \ & e_n(h^1_{(1)}, \dots, h^n_{(1)} h^{n+1}_{(1)}) e_n(h^1_{(2)}, \dots, h^n_{(2)}) \varepsilon(h^{n+1}_{(2)}) \\
= \ & (h^1_{(1)} \cdot (h^2_{(1)} \cdot (\dots \cdot (h^n_{(1)} h^{n+1} \cdot 1_A) \dots))) (h^1_{(2)} \cdot (h^2_{(2)} \cdot (\dots \cdot (h^n_{(2)} \cdot 1_A) \dots))) \\
\overset{(PA2)}{=} & h^1 \cdot [(h^2_{(1)} \cdot (\dots \cdot (h^n_{(1)} h^{n+1} \cdot 1_A) \dots)) (h^2_{(2)} \cdot (\dots \cdot (h^n_{(2)} \cdot 1_A) \dots))] \\
= \ & h^1 \cdot (h^2 \cdot (\dots \ \cdot (h^{n-1} \cdot [(h^n_{(1)} h^{n+1} \cdot 1_A)(h^n_{(2)} \cdot 1_A)]) \dots)) \\
\overset{(PA3)}{=} & h^1 \cdot (h^2 \cdot (\dots \ \cdot (h^{n-1} \cdot (h^n \cdot (h^{n+1} \cdot 1_A))) \dots)) \\
= \ & e_{n+1}(h^1, \dots, h^{n+1}).
\end{aligned}
$$

(vii)

$$
\begin{aligned}
& (e_n \circ \mu_i) * e_{n+1}(h^1, \dots, h^i, h^{i+1}, \dots, h^{n+1}) = \\
= \ & e_n \circ \mu_i(h^1_{(1)}, \dots, h^i_{(1)}, h^{i+1}_{(1)}, \dots, h^{n+1}_{(1)}) e_{n+1}(h^1_{(2)}, \dots, h^i_{(2)}, h^{i+1}_{(2)}, \dots, h^{n+1}_{(2)}) \\
= \ & e_n(h^1_{(1)}, \dots, h^i_{(1)} h^{i+1}_{(1)}, \dots, h^{n+1}_{(1)}) (h^1_{(2)} \cdot (\dots \cdot (h^i_{(2)} \cdot (h^{i+1}_{(2)} \cdot (\dots \\
& \dots \ \cdot (h^{n+1}_{(2)} \cdot 1_A) \dots))) \dots)) \\
= \ & (h^1_{(1)} \cdot (\dots \cdot (h^i_{(1)} h^{i+1}_{(1)} \cdot (\dots \cdot (h^{n+1}_{(1)} \cdot 1_A) \dots)) \dots)) (h^1_{(2)} \cdot (\dots \cdot (h^i_{(2)} \cdot (h^{i+1}_{(2)} \cdot (\dots \\
& \dots \ \cdot (h^{n+1}_{(2)} \cdot 1_A) \dots))) \dots)) \\
\overset{(PA2)}{=} & h^1 \cdot [(h^2_{(1)} \cdot (\dots \cdot (h^i_{(1)} h^{i+1}_{(1)} \cdot (\dots \cdot (h^{n+1}_{(1)} \cdot 1_A) \dots)) \dots)) (h^2_{(2)} \cdot (\dots \cdot (h^i_{(2)} \cdot (h^{i+1}_{(2)} \cdot (\dots \\
& \dots \ \cdot (h^{n+1}_{(2)} \cdot 1_A) \dots))) \dots))] \\
= \ & h^1 \cdot (\dots \cdot (h^{i-1} \cdot [(h^i_{(1)} h^{i+1}_{(1)} \cdot (h^{i+2}_{(1)} \cdot (\dots \cdot (h^{n+1}_{(1)} \cdot 1_A) \dots))) (h^i_{(2)} \cdot (h^{i+1}_{(2)} \cdot (\dots \\
& \dots \ \cdot (h^{n+1}_{(2)} \cdot 1_A) \dots)))]) \dots)
\end{aligned}
$$

$$\overset{(PA3)}{=} h^1 \cdot (\cdots (h^{i-1} \cdot [(h^i_{(1)} h^{i+1}_{(1)} \cdot (h^{i+2}_{(1)} \cdot (\cdots (h^{n+1}_{(1)} \cdot 1_A) \dots))) (h^i_{(2)} h^{i+1}_{(2)} \cdot (h^{i+2}_{(2)} \cdot (\cdots (h^{n+1}_{(2)} \cdot 1_A) \dots)))(h^i_{(3)} \cdot 1_A)]) \dots)$$

$$\overset{(PA2)}{=} h^1 \cdot (\dots \cdot (h^{i-1} \cdot [(h^i_{(1)} h^{i+1} \cdot [(h^{i+2}_{(1)} \cdot (\dots \cdot (h^{n+1}_{(1)} \cdot 1_A) \dots))(h^{i+2}_{(2)} \cdot (\dots \cdots (h^{n+1}_{(2)} \cdot 1_A) \dots))])(h^i_{(2)} \cdot 1_A)]) \dots)$$

$$\overset{(PA2)}{=} h^1 \cdot (\dots \cdot (h^{i-1} \cdot [(h^i_{(1)} h^{i+1} \cdot [h^{i+2} \cdot (\dots \cdot (h^{n+1} \cdot 1_A) \dots)])(h^i_{(2)} \cdot 1_A)]) \dots)$$

$$\overset{(PA3)}{=} h^1 \cdot (\cdots (h^{i-1} \cdot (h^i \cdot (h^{i+1} \cdot (h^{i+2} \cdot (\cdots (h^{n+1} \cdot 1_A) \dots))))) \dots)$$

$$= e_{n+1}(h^1, \dots, h^{n+1}).$$

■

Theorem 3.10 *For any $n \geq 0$ and $f \in C^n_{par}(H, A)$, the linear map $\delta_n(f) : H^{\otimes n+1} \to A$ belongs to $C^{n+1}_{par}(H, A)$. Moreover, the map $\delta_n : C^n_{par}(H, A) \to C^{n+1}_{par}(H, A)$ is a morphism of abelian groups.*

Proof: If $f \in C^n_{par}(H, A)$, then, $f = f * e_n$ and it is invertible with respect to the convolution. Consider the expression for $\delta_n(f)$,

$$\delta_n(f) := E^n(f) * \prod_{i=1}^{n} f^{(-1)^i} \circ \mu_i * i_{n,n+1}(f^{(-1)^{n+1}}).$$

Using items (i), (iii) and (v) of Lemma 3.9 and using the fact that the convolution algebra $\mathrm{Hom}_k(H^{\otimes n+1}, A)$ is commutative, we conclude that $\delta_n(f * g) = \delta_n(f) * \delta_n(g)$, in particular $\delta_n(f) = \delta_n(f * e_n) = \delta_n(f) * \delta_n(e_n)$. By item (ii), we know that $E^n(e_n) = e_{n+1}$ and by items (iv), (vi) and (vii) we see that the unit e_{n+1} absorbs the other factors, leading to $\delta_n(e_n) = e_{n+1}$. Then we have $\delta_n(f) = \delta_n(f) * e_{n+1}$. A straightforward calculation leads us to $\delta_n(f^{-1}) = (\delta_n(f))^{-1}$ Therefore, we proved that δ_n is well defined and it is a morphism of abelian groups.

■

In order to prove that $\delta_{n+1} \circ \delta_n(f) = e_{n+2}$, for every $f \in C^n_{par}(H, A)$, we need the following lemma.

Lemma 3.11 *Let $f \in C^{n-1}_{par}(H, A)$, then*

(i) $E^n(i_{n-1,n}(f)) = i_{n,n+1}(E^{n-1}(f))$.

(ii) $(e_{n-1} \circ \mu_i \circ \mu_{i+1}) * e_{n+1} = e_{n+1}$, *for all $i \in \{1, \cdots, n-1\}$.*

(iii) $(e_{n-1} \circ \mu_i \circ \mu_{i+j}) * e_{n+1} = e_{n+1}$, $\forall\ i \in \{1, \cdots, n-1\}$, $j \in \{2, \cdots, n-i\}$.

(iv) $E^n(f \circ \mu_i) = E^{n-1}(f) \circ \mu_{i+1}$, $\forall\ i \in \{1, \ldots, n-1\}$.

(v) $E^n \circ E^{n-1}(f) = i_{1,n+1}(\widetilde{e}_1) * (E^{n-1}(f) \circ \mu_1)$.

(vi) $i_{n,n+1}(f \circ \mu_i) * i_{n-1,n}(f^{-1}) \circ \mu_i = i_{n,n+1}(e_{n-1} \circ \mu_i)$, $\forall i \in \{1, \ldots, n-1\}$.

(vii) $(i_{n-1,n}(f) \circ \mu_n) * i_{n-1,n+1}(f^{-1}) = i_{n-1,n+1}(e_{n-1})$.

(viii) $(f \circ \mu_i \circ \mu_i) * (f^{-1} \circ \mu_i \circ \mu_{i+1}) = e_{n-1} \circ \mu_i \circ \mu_i$, $\forall\ i \in \{1, \ldots, n-1\}$.

(ix) $(f \circ \mu_i \circ \mu_{i+j}) * (f^{-1} \circ \mu_{i+j-1} \circ \mu_i) = e_{n-1} \circ \mu_i \circ \mu_{i+j}$, *for all* $i \in \{1, \ldots, n-2\}$, *for all* $j \in \{2, \ldots, n-i\}$.

Proof: Let $f \in C^{n-1}_{par}(H, A)$ and $h^1 \otimes \ldots \otimes h^{n+1} \in H^{\otimes n+1}$, then

(i)

$$\begin{aligned}
& E^n(i_{n-1,n}(f))(h^1, \ldots, h^{n+1}) = h^1 \cdot (i_{n-1,n}(f)(h^2, \ldots, h^{n+1})) \\
= \ & h^1 \cdot (f(h^2, \ldots, h^n)\varepsilon(h^{n+1})) = (h^1\ h^n))\varepsilon(h^{n+1}) \\
= \ & (E^{n-1}(f))(h^1, \ldots, h^n)\varepsilon(h^{n+1}) = i_{n,n+1}(E^{n-1}(f))(h^1, \ldots, h^{n+1}).
\end{aligned}$$

(ii) For every $i \in \{1, \ldots, n-1\}$,

$$\begin{aligned}
& (e_{n-1} \circ \mu_i \circ \mu_{i+1}) * e_{n+1}(h^1, \ldots, h^{n+1}) = \\
= \ & e_{n-1}(h^1_{(1)}, \ldots, h^i_{(1)} h^{i+1}_{(1)} h^{i+2}_{(1)}, \ldots, h^{n+1}_{(1)}) e_{n+1}(h^1_{(2)}, \ldots, h^{(n+1)}_{(2)}) \\
= \ & h^1_{(1)} \cdot (\ldots \cdot (h^i_{(1)} h^{i+1}_{(1)} h^{i+2}_{(1)} \cdot (\ldots \cdot (h^{n+1}_{(1)} \cdot 1_A) \ldots)) \ldots) \\
& (h^1_{(2)} \cdot (\ldots \cdot (h^i_{(2)} \cdot (h^{i+1}_{(2)} \cdot (h^{i+2}_{(2)} \cdot (\ldots \cdot (h^{n+1}_{(2)} \cdot 1_A) \ldots)))) \ldots) \\
\overset{(PA2)}{=} & h^1 \cdot (h^2 \cdot (\ldots \cdot [(h^i_{(1)} h^{i+1}_{(1)} h^{i+2}_{(1)} \cdot (\ldots \cdot (h^{n+1}_{(1)} \cdot 1_A) \ldots)) \\
& (h^i_{(2)} \cdot (h^{i+1}_{(2)} \cdot (h^{i+2}_{(2)} \cdot (\ldots \cdot (h^{n+1}_{(2)} \cdot 1_A) \ldots))))] \ldots)) \\
\overset{(PA3)}{=} & h^1 \cdot (h^2 \cdot (\ldots \cdot [(h^i_{(1)} h^{i+1}_{(1)} h^{i+2}_{(1)} \cdot (\ldots \cdot (h^{n+1}_{(1)} \cdot 1_A) \ldots)) \\
& (h^i_{(2)} h^{i+1}_{(2)} \cdot (h^{i+2}_{(2)} \cdot (\ldots \cdot (h^{n+1}_{(2)} \cdot 1_A) \ldots)))(h^i_{(3)} \cdot 1_A)] \ldots)) \\
\overset{(PA3)}{=} & h^1 \cdot (h^2 \cdot (\ldots \cdot [(h^i_{(1)} h^{i+1}_{(1)} h^{i+2}_{(1)} \cdot (\ldots \cdot (h^{n+1}_{(1)} \cdot 1_A) \ldots)) \\
& (h^i_{(2)} h^{i+1}_{(2)} h^{i+2}_{(2)} \cdot (\ldots \cdot (h^{n+1}_{(2)} \cdot 1_A) \ldots))(h^i_{(3)} h^{i+1}_{(3)} \cdot 1_A)(h^i_{(4)} \cdot 1_A)] \ldots)) \\
\overset{(PA3)}{=} & h^1 \cdot (h^2 \cdot (\ldots \cdot [(h^i_{(1)} h^{i+1}_{(1)} h^{i+2} \cdot (\ldots \cdot (h^{n+1} \cdot 1_A) \ldots))(h^i_{(2)} h^{i+1}_{(2)} \cdot 1_A) \\
& (h^i_{(3)} \cdot 1_A)] \ldots))
\end{aligned}$$

$$\begin{aligned}
&\overset{(PA3)}{=} h^1\cdot(h^2\cdot(\dots\cdot[(h^i_{(1)}h^{i+1}\cdot(h^{i+2}\cdot(\dots\cdot(h^{n+1}\cdot 1_A)\dots)))(h^i_{(2)}\cdot 1_A)]\dots))\\
&= h^1\cdot(h^2\cdot(\dots\cdot(h^i\cdot(h^{i+1}\cdot(h^{i+2}\cdot(\dots\cdot(h^{n+1}\cdot 1_A)\dots))))\dots))\\
&= e_{n+1}(h^1,\dots,h^{n+1}).
\end{aligned}$$

(iii) For $i\in\{1,\cdots,n-1\},\ j\in\{2,\cdots,n-i\}$, we have

$$\begin{aligned}
&(e_{n-1}\circ\mu_i\circ\mu_{i+j})*e_{n+1}(h^1,\dots,h^{n+1}) =\\
&= e_{n-1}(h^1_{(1)},\dots,h^i_{(1)}h^{i+1}_{(1)},\dots,h^{i+j}_{(1)}h^{i+j+1}_{(1)},\dots,h^{n+1}_{(1)})\\
&\quad e_{n+1}(h^1_{(2)},\dots,h^i_{(2)},h^{i+1}_{(2)},\dots,h^{i+j}_{(2)},h^{i+j+1}_{(2)},\dots,h^{n+1}_{(2)})\\
&= h^1_{(1)}\cdot(\dots\cdot(h^i_{(1)}h^{i+1}_{(1)}\cdot(\dots\cdot(h^{i+j}_{(1)}h^{i+j+1}_{(1)}\cdot(\dots\cdot(h^{n+1}_{(1)}\cdot 1_A)\dots))\dots))\dots)\\
&\quad h^1_{(2)}\cdot(\dots\cdot(h^i_{(2)}\cdot(h^{i+1}_{(2)}\cdot(\dots\cdot(h^{i+j}_{(2)}\cdot(h^{i+j+1}_{(2)}\cdot(\dots\cdot(h^{n+1}_{(2)}\cdot 1_A)\dots)))\dots)))\dots)\\
&\overset{(PA2)}{=} h^1\cdot[(h^2_{(1)}\cdot(\dots\cdot(h^i_{(1)}h^{i+1}_{(1)}\cdot(\dots\cdot(h^{i+j}_{(1)}h^{i+j+1}_{(1)}\cdot(\dots\cdot(h^{n+1}_{(1)}\cdot 1_A)\dots))\dots))\dots))\\
&\quad (h^2_{(2)}\cdot(\dots\cdot(h^i_{(2)}\cdot(h^{i+1}_{(2)}\cdot(\dots\cdot(h^{i+j}_{(2)}\cdot(h^{i+j+1}_{(2)}\cdot(\dots\cdot(h^{n+1}_{(2)}\cdot 1_A)\dots)))\dots)))\dots))]\\
&= h^1\cdot(\dots\cdot(h^{i-1}\cdot[(h^i_{(1)}h^{i+1}_{(1)}\cdot(\dots\cdot(h^{i+j}_{(1)}h^{i+j+1}_{(1)}\cdot(\dots\cdot(h^{n+1}_{(1)}\cdot 1_A)\dots))\dots))\\
&\quad (h^i_{(2)}\cdot(h^{i+1}_{(2)}\cdot(\dots\cdot(h^{i+j}_{(2)}\cdot(h^{i+j+1}_{(2)}\cdot(\dots\cdot(h^{n+1}_{(2)}\cdot 1_A)\dots)))\dots))))])\dots)\\
&\overset{(PA3)}{=} h^1\cdot(\dots\cdot(h^{i-1}\cdot[(h^i_{(1)}h^{i+1}_{(1)}\cdot(h^{i+2}_{(1)}\cdot(\dots\cdot(h^{i+j}_{(1)}h^{i+j+1}_{(1)}\cdot(\dots\cdot(h^{n+1}_{(1)}\cdot 1_A)\dots))\dots)))\\
&\quad (h^i_{(2)}h^{i+1}_{(2)}\cdot(h^{i+2}_{(2)}\cdot(\dots\cdot(h^{i+j}_{(2)}\cdot(h^{i+j+1}_{(2)}\cdot(\dots\cdot(h^{n+1}_{(2)}\cdot 1_A)\dots)))\dots)))(h^i_{(3)}\cdot 1_A)])\dots)\\
&\overset{(PA2)}{=} h^1\cdot(\dots\cdot(h^{i-1}\cdot[(h^i_{(1)}h^{i+1}\cdot[(h^{i+2}_{(1)}\cdot(\dots\cdot(h^{i+j}_{(1)}h^{i+j+1}_{(1)}\cdot(\dots\cdot(h^{n+1}_{(1)}\cdot 1_A)\dots))\dots))\\
&\quad (h^{i+2}_{(2)}\cdot(\dots\cdot(h^{i+j}_{(2)}\cdot(h^{i+j+1}_{(2)}\cdot(\dots\cdot(h^{n+1}_{(2)}\cdot 1_A)\dots)))\dots))])(h^i_{(2)}\cdot 1_A)])\dots)\\
&\overset{(PA2)}{=} h^1\cdot(\dots\cdot(h^{i-1}\cdot[(h^i_{(1)}h^{i+1}\cdot[(h^{i+2}\cdot(\dots\cdot(h^{i+j-1}\cdot[(h^{i+j}_{(1)}h^{i+j+1}_{(1)}\cdot(\dots\\
&\quad \dots\cdot(h^{n+1}_{(1)}\cdot 1_A)\dots))(h^{i+j}_{(2)}\cdot(h^{i+j+1}_{(2)}\cdot(\dots\cdot(h^{n+1}_{(2)}\cdot 1_A)\dots)))])\dots))])(h^i_{(2)}\cdot 1_A)])\dots)\\
&\overset{(PA3)}{=} h^1\cdot(\dots\cdot(h^{i-1}\cdot[(h^i_{(1)}h^{i+1}\cdot[\dots\cdot(h^{i+j-1}\cdot[(h^{i+j}_{(1)}h^{i+j+1}_{(1)}\cdot(h^{i+j+2}_{(1)}\cdot(\dots\\
&\quad \dots\cdot(h^{n+1}_{(1)}\cdot 1_A)\dots)))(h^{i+j}_{(2)}h^{i+j+1}_{(2)}\cdot(h^{i+j+2}_{(2)}\cdot(\dots\cdot(h^{n+1}_{(2)}\cdot 1_A)\dots)))\\
&\quad (h^{i+j}_{(3)}\cdot 1_A)])\dots])(h^i_{(2)}\cdot 1_A)])\dots)\\
&\overset{(PA2)}{=} h^1\cdot(\dots\cdot(h^{i-1}\cdot[(h^i_{(1)}h^{i+1}\cdot[\dots\cdot(h^{i+j-1}\cdot[(h^{i+j}_{(1)}h^{i+j+1}\cdot[(h^{i+j+2}_{(1)}\cdot(\dots\\
&\quad \dots\cdot(h^{n+1}_{(1)}\cdot 1_A)\dots))(h^{i+j+2}_{(2)}\cdot(\dots\cdot(h^{n+1}_{(2)}\cdot 1_A)\dots))])(h^{i+j}_{(2)}\cdot 1_A)])\dots])(h^i_{(2)}\cdot 1_A)])\dots)
\end{aligned}$$

$$\overset{(PA2)}{=} h^1\cdot(\ldots\cdot(h^{i-1}\cdot[(h^i_{(1)}h^{i+1}\cdot[\ldots\cdot(h^{i+j-1}\cdot[(h^{i+j}_{(1)}h^{i+j+1}\cdot[(h^{i+j+2}\cdot(\ldots$$
$$\ldots\cdot(h^{n+1}\cdot 1_A)\ldots))])(h^{i+j}_{(2)}\cdot 1_A)]\ldots])(h^i_{(2)}\cdot 1_A)])\ldots)$$
$$\overset{(PA3)}{=} h^1\cdot(\ldots\cdot(h^{i-1}\cdot[(h^i_{(1)}h^{i+1}\cdot[\ldots\cdot[(h^{i+j}\cdot(h^{i+j+1}\cdot(h^{i+j+2}\cdot(\ldots$$
$$\ldots\cdot(h^{n+1}\cdot 1_A)\ldots))))]\ldots])(h^i_{(2)}\cdot 1_A)])\ldots)$$
$$\overset{(PA3)}{=} h^1\cdot(\ldots\cdot(h^{i-1}\cdot[(h^i\cdot(h^{i+1}\cdot(\ldots\cdot(h^{i+j}\cdot(h^{i+j+1}\cdot(h^{i+j+2}\cdot(\ldots$$
$$\ldots\cdot(h^{n+1}\cdot 1_A)\ldots)))))\ldots)))])\ldots)$$
$$= e_{n+1}(h^1,\ldots,h^{n+1}).$$

(iv) In fact, for every $i\in\{1,\ldots,n-1\}$, we have

$$\begin{aligned} &E^n(f\circ\mu_i)(h^1,\ldots,h^{n+1}) = \\ =\ & h^1\cdot((f\circ\mu_i)(h^2,\ldots,h^i,\underbrace{h^{i+1}}_{i-th\ coord.},h^{i+2},\ldots,h^{n+1})) \\ =\ & h^1\cdot(f(h^2,\ldots,h^i,h^{i+1}h^{i+2},\ldots,h^{n+1})). \end{aligned}$$

On the other hand,

$$\begin{aligned} &E^{n-1}(f)\circ\mu_{i+1}(h^1,\ldots,h^{n+1}) = \\ =\ & E^{n-1}(f)(h^1,\ldots,h^i,h^{i+1}h^{i+2},\ldots,h^{n+1}) \\ =\ & h^1\cdot(f(h^2,\ldots,h^i,h^{i+1}h^{i+2},\ldots,h^{n+1})). \end{aligned}$$

So, we proved the equality.

(v)

$$\begin{aligned} &i_{1,n+1}(\widetilde{e}_1)*(E^{n-1}(f)\circ\mu_1)(h^1,\ldots,h^{n+1}) = \\ =\ & i_{1,n+1}(\widetilde{e}_1)(h^1_{(1)},\ldots,h^{n+1}_{(1)})E^{n-1}(f)\circ\mu_1(h^1_{(2)},\ldots,h^{n+1}_{(2)}) \\ =\ & \widetilde{e}_1(h^1_{(1)})\varepsilon(h^2_{(1)})\ldots\varepsilon(h^{n+1}_{(1)})E^{n-1}(f)(h^1_{(2)}h^2_{(2)},h^3_{(2)},\ldots,h^{n+1}_{(2)}) \\ =\ & (h^1_{(1)}\cdot 1_A)\varepsilon(h^2_{(1)})\ldots\varepsilon(h^{n+1}_{(1)})(h^1_{(2)}h^2_{(2)}\cdot(f(h^3_{(2)},\ldots,h^{n+1}_{(2)}))) \\ =\ & (h^1_{(1)}\cdot 1_A)\varepsilon(h^2_{(1)})\ldots\varepsilon(h^{n+1}_{(1)})(h^1_{(2)}h^2_{(2)}\cdot(f(h^3_{(2)},\ldots,h^{n+1}_{(2)}))) \\ =\ & (h^1_{(1)}\cdot 1_A)(h^1_{(2)}h^2\cdot(f(h^3,\ldots,h^{n+1}))) \\ =\ & h^1\cdot(h^2\cdot(f(h^3,\ldots,h^{n+1}))) \\ =\ & h^1\cdot(E^{n-1}(f)(h^2,h^3,\ldots,h^{n+1})) \\ =\ & E^n(E^{n-1}(f))(h^1,\ldots,h^{n+1}) \\ =\ & E^n\circ E^{n-1}(f)(h^1,\ldots,h^{n+1}). \end{aligned}$$

(vi) For $i \in \{1, \ldots, n-1\}$, we have

$$\begin{aligned}
& i_{n,n+1}(f \circ \mu_i) * i_{n-1,n}(f^{-1}) \circ \mu_i(h^1, \ldots, h^{n+1}) = \\
=& i_{n,n+1}(f \circ \mu_i)(h^1_{(1)}, \ldots, h^{n+1}_{(1)}) i_{n-1,n}(f^{-1}) \circ \mu_i(h^1_{(2)}, \ldots, h^{n+1}_{(2)}) \\
=& f \circ \mu_i(h^1_{(1)}, \ldots, h^n_{(1)}) \varepsilon(h^{n+1}_{(1)}) i_{n-1,n}(f^{-1})(h^1_{(2)}, \ldots, h^i_{(2)} h^{i+1}_{(2)}, \ldots, \underbrace{h^{n+1}_{(2)}}_{n-th\ coord}) \\
=& f(h^1_{(1)}, \ldots, h^i_{(1)} h^{i+1}_{(1)}, \ldots, h^n_{(1)}) \varepsilon(h^{n+1}_{(1)}) f^{-1}(h^1_{(2)}, \ldots, h^i_{(2)} h^{i+1}_{(2)}, \ldots, h^n_{(2)}) \varepsilon(h^{n+1}_{(2)}) \\
=& f \circ \mu_i(h^1_{(1)}, \ldots, h^i_{(1)}, h^{i+1}_{(1)}, \ldots, h^n_{(1)}) f^{-1} \circ \mu_i(h^1_{(2)}, \ldots, h^i_{(2)}, h^{i+1}_{(2)}, \ldots, h^n_{(2)}) \varepsilon(h^{n+1}) \\
=& (f \circ \mu_i) * (f^{-1} \circ \mu_i)(h^1, \ldots, h^n) \varepsilon(h^{n+1}) \\
=& (f * f^{-1}) \circ \mu_i(h^1, \ldots, h^n) \varepsilon(h^{n+1}) \\
=& e_{n-1} \circ \mu_i(h^1, \ldots, h^n) \varepsilon(h^{n+1}) \\
=& i_{n,n+1}(e_n \circ \mu_i)(h^1, \ldots, h^{n+1}).
\end{aligned}$$

(vii)

$$\begin{aligned}
& (i_{n-1,n}(f) \circ \mu_n) * i_{n-1,n+1}(f^{-1})(h^1, \ldots, h^{n+1}) = \\
= \ & (i_{n-1,n}(f) \circ \mu_n)(h^1_{(1)}, \ldots, h^{n+1}_{(1)}) i_{n-1,n+1}(f^{-1})(h^1_{(2)}, \ldots, h^{n+1}_{(2)}) \\
= \ & i_{n-1,n}(f)(h^1_{(1)}, \ldots, h^n_{(1)} h^{n+1}_{(1)}) f^{-1}(h^1_{(2)}, \ldots, h^{n-1}_{(2)}) \varepsilon(h^n_{(2)}) \varepsilon(h^{n+1}_{(2)}) \\
= \ & f(h^1_{(1)}, \ldots, h^{n-1}_{(1)}) \varepsilon(h^n_{(1)} h^{n+1}_{(1)}) f^{-1}(h^1_{(2)}, \ldots, h^{n-1}_{(2)}) \varepsilon(h^n_{(2)}) \varepsilon(h^{n+1}_{(2)}) \\
= \ & f(h^1_{(1)}, \ldots, h^{n-1}_{(1)}) f^{-1}(h^1_{(2)}, \ldots, h^{n-1}_{(2)}) \varepsilon(h^n_{(1)} h^{n+1}_{(1)}) \varepsilon(h^n_{(2)}) \varepsilon(h^{n+1}_{(2)}) \\
= \ & (f * f^{-1})(h^1, \ldots, h^{n-1}) \varepsilon(h^n) \varepsilon(h^{n+1}) \\
= \ & (e_{n-1})(h^1, \ldots, h^{n-1}) \varepsilon(h^n) \varepsilon(h^{n+1}) \\
= \ & i_{n-1,n+1}(e_{n-1})(h^1, \ldots, h^{n+1}).
\end{aligned}$$

(viii) For $i \in \{1, \ldots, n-1\}$, we have

$$\begin{aligned}
& (f \circ \mu_i \circ \mu_i) * (f^{-1} \circ \mu_i \circ \mu_{i+1})(h^1, \ldots, h^{n+1}) = \\
=& f \circ \mu_i \circ \mu_i(h^1_{(1)}, \ldots, h^{n+1}_{(1)}) f^{-1} \circ \mu_i \circ \mu_{i+1}(h^1_{(2)}, \ldots, h^{n+1}_{(2)}) \\
=& f \circ \mu_i(h^1_{(1)}, \ldots, h^i_{(1)} h^{i+1}_{(1)}, h^{i+2}_{(1)}, \ldots, h^{n+1}_{(1)}) f^{-1} \circ \mu_i(h^1_{(2)}, \ldots, h^i_{(2)}, h^{i+1}_{(2)} h^{i+2}_{(2)}, \ldots, h^{n+1}_{(2)}) \\
=& f(h^1_{(1)}, \ldots, h^i_{(1)} h^{i+1}_{(1)} h^{i+2}_{(1)}, \ldots, h^{n+1}_{(1)}) f^{-1}(h^1_{(2)}, \ldots, h^i_{(2)} h^{i+1}_{(2)} h^{i+2}_{(2)}, \ldots, h^{n+1}_{(2)}) \\
=& (f * f^{-1})(h^1, \ldots, h^i h^{i+1} h^{i+2}, \ldots, h^{n+1}) \\
=& e_{n-1}(h^1, \ldots, h^i h^{i+1} h^{i+2}, \ldots, h^{n+1}) \\
=& e_{n-1} \circ \mu_i \circ \mu_i(h^1, \ldots, h^{n+1}).
\end{aligned}$$

(ix) For $i \in \{1, \ldots, n-2\}$ e $j \in \{2, \ldots, n-i\}$, we have

$$
\begin{aligned}
&(f\circ\mu_i\circ\mu_{i+j})*(f^{-1}\circ\mu_{i+j-1}\circ\mu_i)(h^1,\dots,h^{n+1}) =\\
=&f\circ\mu_i\circ\mu_{i+j}(h^1_{(1)},\dots,h^{n+1}_{(1)})f^{-1}\circ\mu_{i+j-1}\circ\mu_i(h^1_{(2)},\dots,h^{n+1}_{(2)})\\
=&f\circ\mu_i(h^1_{(1)},\dots,h^{i+j}_{(1)}h^{i+j+1}_{(1)},\dots,h^{n+1}_{(1)})f^{-1}\circ\mu_{i+j-1}(h^1_{(2)},\dots,h^i_{(2)}h^{i+1}_{(2)},\dots,h^{n+1}_{(2)})\\
=&f(h^1_{(1)},\dots,h^i_{(1)}h^{i+1}_{(1)},\dots,h^{i+j}_{(1)}h^{i+j+1}_{(1)},\dots,h^{n+1}_{(1)})\\
&f^{-1}\circ\mu_{i+j-1}(h^1_{(2)},\dots,\underbrace{h^i_{(2)}h^{i+1}_{(2)}}_{i-th\ coord},h^{i+2}_{(2)},\dots,\underbrace{h^{i+j}_{(2)}}_{(i+j-1)-th\ coord},\dots,h^{n+1}_{(2)})\\
=&f(h^1_{(1)},\dots,h^i_{(1)}h^{i+1}_{(1)},\dots,h^{i+j}_{(1)}h^{i+j+1}_{(1)},\dots,h^{n+1}_{(1)})\\
&f^{-1}(h^1_{(2)},\dots,h^i_{(2)}h^{i+1}_{(2)},\dots,h^{i+j}_{(2)}h^{i+j+1}_{(2)},\dots,h^{n+1}_{(2)})\\
=&(f*f^{-1})(h^1,\dots,h^ih^{i+1},\dots,h^{i+j}h^{i+j+1},\dots,h^{n+1})\\
=&e_{n-1}(h^1,\dots,h^ih^{i+1},\dots,h^{i+j}h^{i+j+1},\dots,h^{n+1})\\
=&e_{n-1}\circ\mu_i\circ\mu_{i+j}(h^1,\dots,h^{n+1}).
\end{aligned}
$$

■

With this lemma, one can prove the following result.

Theorem 3.12 *For any $f\in C^n_{par}(H,A)$, we have that $\delta_{n+1}\circ\delta_n(f)=e_{n+2}$.*

Proof: Indeed, take any $f\in C^n_{par}(H,A)$, then

$$
\begin{aligned}
&\delta_{n+1}(\delta_n(f)) =\\
=&E^{n+1}(\delta_n(f))*\prod_{i=1}^{n+1}(\delta_n(f))^{(-1)^i}\circ\mu_i*i_{n+1,n+2}((\delta_n(f))^{(-1)^{n+2}})\\
=&E^{n+1}(\delta_n(f))*\prod_{i=1}^{n+1}\delta_n(f^{(-1)^i})\circ\mu_i*i_{n+1,n+2}(\delta_n(f^{(-1)^{n+2}}))\\
=&E^{n+1}(E^n(f)*\prod_{j=1}^{n}f^{(-1)^j}\circ\mu_j*i_{n,n+1}(f^{(-1)^{n+1}}))\\
&*\prod_{i=1}^{n+1}(E^n(f^{(-1)^i})*\prod_{j=1}^{n}f^{(-1)^{i+j}}\circ\mu_j*i_{n,n+1}(f^{(-1)^{n+i+1}}))\circ\mu_i\\
&*i_{n+1,n+2}(E^n(f^{(-1)^{n+2}})*\prod_{j=1}^{n}f^{(-1)^{n+j+2}}\circ\mu_j*i_{n,n+1}(f^{(-1)^{2n+3}}))
\end{aligned}
$$

$$= \underbrace{E^{n+1}(E^n(f))}_{Lemma\ 3.11\ (v)} * \prod_{j=1}^{n} \underbrace{E^{n+1}(f^{(-1)^j} \circ \mu_j)}_{Lemma\ 3.11\ (iv)} * \underbrace{E^{n+1}(i_{n,n+1}(f^{(-1)^{n+1}}))}_{Lemma\ 3.11\ (i)}$$

$$* \prod_{i=1}^{n+1} E^n(f^{(-1)^i}) \circ \mu_i * \prod_{i=1}^{n+1}\prod_{j=1}^{n} f^{(-1)^{i+j}} \circ \mu_j \circ \mu_i$$

$$* \prod_{i=1}^{n+1} i_{n,n+1}(f^{(-1)^{n+i+1}}) \circ \mu_i * i_{n+1,n+2}(E^n(f^{(-1)^{n+2}}))$$

$$* i_{n+1,n+2}(\prod_{j=1}^{n} f^{(-1)^{n+j+2}} \circ \mu_j) * i_{n+1,n+2}(i_{n,n+1}(f^{(-1)^{2n+3}}))$$

$$= i_{1,n+2}(\widetilde{e}_1) * E^n(f) \circ \mu_1 * \prod_{j=1}^{n} E^n(f^{(-1)^j}) \circ \mu_{j+1} * i_{n+1,n+2}(E^n(f^{(-1)^{n+1}}))$$

$$* \prod_{i=1}^{n+1} E^n(f^{(-1)^i}) \circ \mu_i * \prod_{i=1}^{n+1}\prod_{j=1}^{n} f^{(-1)^{i+j}} \circ \mu_j \circ \mu_i$$

$$* \prod_{i=1}^{n+1} i_{n,n+1}(f^{(-1)^{n+i+1}}) \circ \mu_i * i_{n+1,n+2}(E^n(f^{(-1)^{n+2}})$$

$$* \prod_{j=1}^{n} i_{n+1,n+2}(f^{(-1)^{n+j+2}} \circ \mu_j) * i_{n,n+2}(f^{(-1)^{2n+3}}))$$

$$= i_{1,n+2}(\widetilde{e}_1) * \prod_{j=1}^{n+1} E^n(f^{(-1)^{j+1}}) \circ \mu_j * \prod_{i=1}^{n+1} E^n(f^{(-1)^i}) \circ \mu_i *$$

$$* i_{n+1,n+2}(E^n(f^{(-1)^{n+1}})) * i_{n+1,n+2}(E^n(f^{(-1)^{n+2}})$$

$$* \prod_{i=1}^{n+1}\prod_{j=1}^{n} f^{(-1)^{i+j}} \circ \mu_j \circ \mu_i * \prod_{j=1}^{n} i_{n+1,n+2}(f^{(-1)^{n+j+2}} \circ \mu_j)$$

$$* \prod_{i=1}^{n} i_{n,n+1}(f^{(-1)^{n+i+1}}) \circ \mu_i * i_{n,n+1}(f^{(-1)^{2n+2}}) \circ \mu_{n+1}$$

$$* i_{n,n+2}(f^{(-1)^{2n+3}}))$$

$$= i_{1,n+2}(\widetilde{e}_1) * \prod_{j=1}^{n+1} E^n(f^{(-1)^{j+1}} * f^{(-1)^j}) \circ \mu_j$$

$$* i_{n+1,n+2}(E^n(f^{(-1)^{n+1}} * f^{(-1)^{n+2}})) * \prod_{i=1}^{n+1}\prod_{j=1}^{n} f^{(-1)^{i+j}} \circ \mu_j \circ \mu_i$$

$$*\underbrace{\prod_{j=1}^{n} i_{n+1,n+2}(f^{(-1)^{n+j+2}} \circ \mu_j) * \prod_{i=1}^{n} i_{n,n+1}(f^{(-1)^{n+i+1}}) \circ \mu_i}_{Lemma\ 3.11\ (vi)}$$

$$*\underbrace{i_{n,n+1}(f^{(-1)^{2n+2}}) \circ \mu_{n+1} * i_{n,n+2}(f^{(-1)^{2n+3}}))}_{Lemma\ 3.11\ (vii)}$$

$$= i_{1,n+2}(\widetilde{e}_1) * \prod_{j=1}^{n+1} \underbrace{E^n(e_n)}_{Lemma\ 3.9\ (i)} \circ \mu_j * i_{n+1,n+2}(\underbrace{E^n(e_n)}_{Lemma\ 3.9\ (i)})$$

$$* \prod_{i=1}^{n+1}\prod_{j=1}^{n} f^{(-1)^{i+j}} \circ \mu_j \circ \mu_i * \prod_{j=1}^{n} i_{n+1,n+2}(e_n \circ \mu_j) * i_{n,n+2}(e_n)$$

$$= i_{1,n+2}(\widetilde{e}_1) * \prod_{j=1}^{n+1} e_{n+1} \circ \mu_j * i_{n+1,n+2}(e_{n+1}) * \prod_{i=1}^{n+1}\prod_{j=1}^{n} f^{(-1)^{i+j}} \circ \mu_j \circ \mu_i$$

$$* \prod_{j=1}^{n} i_{n+1,n+2}(e_n \circ \mu_j) * i_{n+1,n+2}(i_{n,n+1}(e_n))$$

$$= i_{1,n+2}(\widetilde{e}_1) * \prod_{j=1}^{n+1} e_{n+1} \circ \mu_j * i_{n+1,n+2}(e_{n+1}) * \prod_{i=1}^{n+1}\prod_{j=1}^{n} f^{(-1)^{i+j}} \circ \mu_j \circ \mu_i$$

$$* \prod_{j=1}^{n-1} i_{n+1,n+2}(e_n \circ \mu_j) * \underbrace{i_{n+1,n+2}(e_n \circ \mu_n) * i_{n+1,n+2}(i_{n,n+1}(e_n))}_{Lemma\ 3.9\ (vi)}$$

$$= i_{1,n+2}(\widetilde{e}_1) * \prod_{j=1}^{n+1} e_{n+1} \circ \mu_j * i_{n+1,n+2}(e_{n+1}) * \prod_{i=1}^{n+1}\prod_{j=1}^{n} f^{(-1)^{i+j}} \circ \mu_j \circ \mu_i$$

$$\underbrace{* \prod_{j=1}^{n-1} i_{n+1,n+2}(e_n \circ \mu_j) * i_{n+1,n+2}(e_{n+1})}_{Lemma\ 3.9\ (vii)}$$

$$= i_{1,n+2}(\widetilde{e}_1) * \prod_{j=1}^{n+1} e_{n+1} \circ \mu_j * i_{n+1,n+2}(e_{n+1}) * \prod_{i=1}^{n+1}\prod_{j=1}^{n} f^{(-1)^{i+j}} \circ \mu_j \circ \mu_i$$

$$* i_{n+1,n+2}(e_{n+1})$$

$$= i_{1,n+2}(\widetilde{e}_1) * \prod_{j=1}^{n} e_{n+1} \circ \mu_j * \underbrace{e_{n+1} \circ \mu_{n+1} * i_{n+1,n+2}(e_{n+1})}_{Lemma\ 3.9\ (vi)}$$

$$
\begin{aligned}
&* \prod_{i=1}^{n+1}\prod_{j=1}^{n} f^{(-1)^{i+j}} \circ \mu_j \circ \mu_i \\
&= i_{1,n+2}(\widetilde{e}_1) * \underbrace{\prod_{j=1}^{n} e_{n+1} \circ \mu_j * e_{n+2}}_{Lemma\ 3.9\ (vii)} * \prod_{i=1}^{n+1}\prod_{j=1}^{n} f^{(-1)^{i+j}} \circ \mu_j \circ \mu_i \\
&= i_{1,n+2}(\widetilde{e}_1) * e_{n+2} * \prod_{i=1}^{n+1}\prod_{j=1}^{n} f^{(-1)^{i+j}} \circ \mu_j \circ \mu_i \\
&= e_{n+2} * \prod_{i=1}^{n+1}\prod_{j=1}^{n} f^{(-1)^{i+j}} \circ \mu_j \circ \mu_i \\
&= e_{n+2} * \underbrace{\prod_{i=1}^{n} f^{(-1)^{2i}} \circ \mu_i \circ \mu_i * \prod_{i=1}^{n} f^{(-1)^{2i+1}} \circ \mu_i \circ \mu_{i+1}}_{Lemma\ 3.11\ (viii)} * \\
&\quad \underbrace{* \prod_{i=1}^{n-1}\prod_{j=2}^{n-i+1} f^{(-1)^{2i+j}} \circ \mu_i \circ \mu_{i+j} * \prod_{i=1}^{n-1}\prod_{j=2}^{n-i+1} f^{(-1)^{2i+j-1}} \circ \mu_{i+j-1} \circ \mu_i}_{Lemma\ 3.11\ (ix)} \\
&= e_{n+2} * \prod_{i=1}^{n} e_n \circ \mu_i \circ \mu_i * \prod_{i=1}^{n-1}\prod_{j=2}^{n-i+1} e_n \circ \mu_i \circ \mu_{i+j} \\
&= e_{n+2}.
\end{aligned}
$$

Since e_{n+2} absorbs $e_n \circ \mu_i \circ \mu_i$ for all $i \in \{1, \dots, n\}$ and $e_n \circ \mu_i \circ \mu_{i+j}$, for all $i \in \{1, \dots, n-1\}$, $j \in \{2, \dots, n-i-1\}$, by Lemma 3.11 (ii) and (iii) respectively, we conclude that $\delta_{n+1} \circ \delta_n(f) = e_{n+2}$ as desired.

■

Therefore, we ended up with a cochain complex $(C^n_{par}(H, A), \delta_n)_{n\in\mathbb{N}}$, which allows us to define a cohomology theory.

3.3 Cohomologies

Definition 3.13 *Let H be a cocommutative Hopf algebra acting partially over a commutative algebra A and consider the cochain complex $(C^n_{par}(H, A), \delta_n)_{n\in\mathbb{N}}$ as defined in the previous section. For $n > 0$, define the groups of partial n-cocycles, partial n-coboundaries and partial*

n-cohomologies of H taking values in A , respectively, as the abelian groups $Z^n(H,A) = \ker\delta_n$, $B^n(H,A) = Im\ \delta_{n-1}$ *and* $H^n(H,A) = \ker\delta_n/\ Im\ \delta_{n-1}$. $n \geq 1$. *For* $n=0$, *define* $H^0(H,A) = Z^0(H,A) = \ker\delta_0$.

Let us characterize the partial cocycles and the partial coboundaries for $n = 0$, 1 and 2.

For $n = 0$, we have by definition

$$H^0(H,A) = Z^0(H,A) = \{a \in A^\times | h \cdot a = (h \cdot 1_A)a,\ \forall h \in H\}.$$

Thus the partial 0-cocycles are the elements of A invariant under the partial action as defined in [3].

For $n = 1$, the partial 1-coboundaries are

$$B^1(H,A) = \mathrm{Im}\ \delta_0 = \{f \in C^1_{par}(H,A)|\ \exists a \in A^\times, \quad f(h) = \delta_0(a)(h)\},$$

this means

$$B^1(H,A) = \{f \in C^1_{par}(H,A)\ |\ \exists a \in A^\times, \quad f(h) = (h \cdot a)a^{-1}\}.$$

Also, for $f \in C^1_{par}(H,A)$, we have

$$\begin{aligned}\delta_1(f)(h,l) &= E^2(f)(h_{(1)},l_{(1)})f^{-1}(h_{(2)}l_{(2)})f(h_{(3)})\varepsilon(l_{(3)})\\ &= (h_{(1)} \cdot f(l_{(1)}))f^{-1}(h_{(2)}l_{(2)})f(h_{(3)})\varepsilon(l_{(3)}).\end{aligned}$$

Then, for all $h,\ l \in H$, the partial 1-cocycles are

$$\begin{aligned}&Z^1(H,A) = \{f \in C^1_{par}(H,A)|\ \delta_1(f)(h,l) = e_2(h,l)\}\\ &= \{f \in C^1_{par}(H,A)|(h_{(1)} \cdot f(l_{(1)}))f^{-1}(h_{(2)}l_{(2)})f(h_{(3)}) = h \cdot (l \cdot 1_A)\}\\ &= \{f \in C^1_{par}(H,A)|(h_{(1)} \cdot f(l_{(1)}))f(h_{(3)}) = (h_{(1)} \cdot (l_{(1)} \cdot 1_A))f(h_{(2)}l_{(2)})\}.\end{aligned}$$

Due to the fact that for a 1-cocycle f we have $f = e_1 * f$, then the condition of 1-cocycle can also be rewritten as

$$(h_{(1)} \cdot f(l_{(1)}))f(h_{(3)}) = (h_{(1)} \cdot 1_A)f(h_{(2)}l_{(2)}),$$

For $n = 2$, we have the partial 2-coboundaries

$$\begin{aligned}&B^2(H,A) = \{g \in C^2_{par}(H,A)|\exists f \in C^1_{par}(H,A), \delta_1(f)(h,l) = g(h,l)\}\\ =\ &\{g \in C^2_{par}(H,A)|\ g(h,l) = (h_{(1)} \cdot f(l_{(1)}))f^{-1}(h_{(2)},l_{(2)})f(h_{(3)})\}.\end{aligned}$$

Also, for $f \in C^2_{par}(H,A)$, we have

$$\delta_2(f)(h,l,m) = E^2(f) * \prod_{i=1}^{2} f^{(-1)^i} \circ \mu_i * i_{2,3}(f^{(-1)^3})(h,l,m)$$
$$= (h_{(1)} \cdot f(l_{(1)}, m_{(1)})) f^{-1}(h_{(2)} l_{(2)}, m_{(2)}) f(h_{(3)}, l_{(3)} m_{(3)}) f^{-1}(h_{(4)}, l_{(4)}) \varepsilon(m_{(4)}).$$

Then, the partial 2 cocycles are

$$\begin{aligned} Z^2(H,A) &= \{f \in C^2_{par}(H,A) | \delta_2(f)(h,l,m) = e_3(h,l,m), \forall h,l,m \in H\} \\ &= \{f \in C^2_{par}(H,A) | (h_{(1)} \cdot f(l_{(1)}, m_{(1)})) f^{-1}(h_{(2)} l_{(2)}, m_{(2)}) f(h_{(3)}, l_{(3)} m_{(3)}) \\ &\quad f^{-1}(h_{(4)}, l_{(4)}) = h \cdot (l \cdot (m \cdot 1_A)),\ \forall\ h,l,m \in H\} \\ &= \{f \in C^2_{par}(H,A) | (h_{(1)} \cdot f(l_{(1)}, m_{(1)})) f(h_{(2)}, l_{(2)} m_{(2)}) = \\ &\quad (h_{(1)} \cdot (l_{(1)} \cdot (m_{(1)} \cdot 1_A))) f(h_{(2)} l_{(2)}, m_{(2)}) f(h_{(3)}, l_{(3)}),\ \forall\ h,l,m \in H\}. \end{aligned}$$

Again by absorption of units, one can rewrite the condition of 2-cocycle as

$$(h_{(1)} \cdot f(l_{(1)}, m_{(1)})) f(h_{(2)}, l_{(2)} m_{(2)}) = (h_{(1)} \cdot 1_A) f(h_{(2)} l_{(2)}, m_{(2)}) f(h_{(3)}, l_{(3)}),$$

which is the form presented in [5].

Example 3.14 *In the case of a global action of H over A, which is equivalent to say that $h \cdot 1_A = \varepsilon(h) 1_A$, $\forall h \in H$, the cochain complexes are simply given by $C^n(H,A) = Hom_k(H^{\otimes n}, A)^\times$. Then we recover exactly the cohomology theory obtained by Sweedler in [30].*

Example 3.15 *Let G be a group and $H = kG$, the group algebra of G. Using the canonical basis $\{\delta_g \in kG \mid g \in G\}$, the axioms (PA1), (PA2) and (PA3) of partial action read*

(PA1) $\delta_e \cdot a = a$, for every $a \in A$;

(PA2) $\delta_g \cdot (ab) = (\delta_g \cdot a)(\delta_g \cdot b)$, for every $g \in G$ and $a, b \in A$;

(PA3) $\delta_g \cdot (\delta_h \cdot a) = (\delta_g \cdot 1_A)(\delta_{gh} \cdot a)$, for every $g, h \in G$ and $a \in A$.

In order to calculate the partial n-cocycles, partial n-coboundaries and partial n-cohomologies, we denote the coboundary operator by ∂_n instead of δ_n to avoid confusion with the elements $\delta_g \in kG$.

For $n = 0$, we have

$$H^0(kG,A) = Z^0(kG,A) = \{a \in A^\times | (\delta_g \cdot a) a^{-1} = (\delta_g \cdot 1_A),\ \forall \delta_g \in kG\},$$

For $n=1$, the 1-coboundaries are

$$\begin{aligned} B^1(kG,A) &= \{f \in C^1_{par}(kG,A) | \ \exists\ a \in A^{\times} : f(\delta_g) = \partial_0(\delta_g)(a)\} \\ &= \{f \in C^1_{par}(kG,A) | \ f(\delta_g) = (\delta_g \cdot a)a^{-1}\}. \end{aligned}$$

Also, we have, for every $f \in C^1_{par}(kG,A)$,

$$\partial_1(f)(\delta_g,\delta_h) = (\delta_g \cdot f(\delta_h))f^{-1}(\delta_{gh})f(\delta_g).$$

Then, for all $\delta_g,\delta_h \in kG$, we obtain the 1-cocycles

$$\begin{aligned} Z^1(kG,A) &= \{f \in C^1_{par}(kG,A) | \partial_1(f)(\delta_g,\delta_h) = \delta_g \cdot (\delta_h \cdot 1_A)\} \\ &= \{f \in C^1_{par}(kG,A) | (\delta_g \cdot f(\delta_h))f(\delta_g) = (\delta_g \cdot 1_A)f(\delta_{gh})\}. \end{aligned}$$

For $n=2$, the 2-coboundaries are

$$\begin{aligned} B^2(kG,A) &= \{i \in C^2_{par}(kG,A) | \exists f \in C^1_{par}(kG,A) : \partial_1(f)(\delta_g,\delta_h) = i(\delta_g,\delta_h)\} \\ &= \{i \in C^2_{par}(kG,A) | i(\delta_g,\delta_h) = (\delta_g \cdot f(\delta_h))f^{-1}(\delta_{gh})f(\delta_g)\}. \end{aligned}$$

Moreover, for $f \in C^2_{par}(kG,A)$

$$\partial_2(f)(\delta_g,\delta_h,\delta_l) = (\delta_g \cdot f(\delta_h,\delta_l))f^{-1}(\delta_{gh},\delta_l)f(\delta_g,\delta_{hl})f^{-1}(\delta_g,\delta_h).$$

Then, for all $\delta_g,\delta_h,\delta_l \in kG$, the partial 2-cocycles are

$$\begin{aligned} Z^2(kG,A) &= \{f \in C^2_{par}(kG,A) | \ \partial_2(f)(\delta_g,\delta_h,\delta_l) = e_3(\delta_g,\delta_h,\delta_l)\} \\ &= \{f \in C^2_{par}(kG,A) | (\delta_g \cdot f(\delta_h,\delta_l))f(\delta_g,\delta_{hl}) = (\delta_g \cdot 1_A)f(\delta_{gh},\delta_l)f(\delta_g,\delta_h)\}. \end{aligned}$$

This cohomology for partial actions of the group algebra kG corresponds to the cohomology for partial group actions described in [19]. Recall from Example 2.20 that there is a one-to-one correspondence between partial actions of kG and unital partial actions of the group G, given by $A_g = 1_gA$ in which $1_g = \delta_g \cdot 1_A$, and $\alpha_g = (\delta_g \cdot \_\_)|_{A_{g^{-1}}}$. For elements $x_1, \dots x_n \in G$ we define the ideals

$$A_{(x_1,\dots,x_n)} := A_{x_1}A_{x_1x_2}\dots A_{x_1\dots x_n},$$

where $A_{x_i} = 1_{x_i}A$. This expression for the ideals is natural, considering the units

$$e_n(\delta_{x_1},\dots,\delta_{x_n}) = \delta_{x_1} \cdot (\delta_{x_2} \cdot (\cdots(\delta_{x_n} \cdot 1_A))) = 1_{x_1}1_{x_1x_2}\dots 1_{x_1\dots x_n}.$$

The set of these ideals forms a semilattice, because the product of two ideals of this type is also an ideal of this type, this product is commutative and each ideal is idempotent, that is $A_{(x_1,\ldots,x_n)}=A_{(x_1,\ldots,x_n)}A_{(x_1,\ldots,x_n)}$. *This can be viewed easily by the properties of the system of idempotents presented before.*

The correspondence between the cochain complexes presented here and those presented in [19] can be viewed more exactly by the identification of the convolution algebra $Hom_k(kG^{\otimes n}, A)$ *with the algebra of functions* $Fun(G^n, A)$*, moreover, the functions* $f : G^n \to A$ *can also be viewed as collections of elements of* A *indexed by* n*-tuples in* G*, that is* $f(g_1,\ldots,g_n) = f_{g_1,\ldots g_n} \in 1_{(g_1\cdots g_n)}A$. *As the canonical basis elements* δ_g*, for* $g \in G$ *are group-like, the convolution product is in fact the pointwise product, that is, for* $f^1, f^2 \in Fun(G^n, A)$ *and* $g_1,\ldots,g_n \in G$*, we have*

$$f^1 * f^2(g_1,\ldots,g_n) = f^1(g_1,\ldots,g_n)f^2(g_1,\ldots,g_n) = f^1_{g_1,\ldots,g_n} f^2_{g_1,\ldots,g_n}.$$

Therefore, the n*-cochains* $C^n_{par}(kG, A)$ *coincide with the* n*-cochains* $C^n_{par}(G, A)$.

The partial n*-cocycles, partial* n*-coboundaries and partial* n*-cohomologies in the group setting are written as.*

$$H^0(G,A) = Z^0(G,A) = \{a \in A^\times | (\alpha_g(1_{g^{-1}}a))a^{-1} = 1_g,\ \forall g \in G\},$$

For $n = 1$ *the partial* 1*-coboudaries are*

$$\begin{aligned} B^1(G,A) &= \{f \in C^1_{par}(G,A)|\ \exists a \in A^\times, \quad f(g) = \partial_0(g)(a)\} \\ &= \{f \in C^1_{par}(G,A)|\ \exists a \in A^\times, \quad f(g) = (\alpha_g(1_{g^{-1}}a))a^{-1}\}. \end{aligned}$$

Moreover, for $f \in C^1_{par}(G, A)$ *we have*

$$\partial_1(f)(g,h) = (g \cdot (1_{g^{-1}}f(h)))f^{-1}(gh)f(g).$$

Then, the partial 1*-cocycles are*

$$\begin{aligned} Z^1(G,A) &= \{f \in C^1_{par}(G,A)|\ \partial_1(f)(g,h) = e_2(g,h),\ \forall\ g,h \in G\} \\ &= \{f \in C^1_{par}(G,A)|\ (\alpha_g(1_{g^{-1}}f(h)))f(g) = 1_g f(gh), \forall g,h \in G\}. \end{aligned}$$

Note that $\delta_g \cdot (\delta_h \cdot 1_A) = (\delta_g \cdot 1_A)(\delta_{gh} \cdot 1_A) = 1_g 1_{gh}$*, and* 1_{gh} *is absorbed by* $f(gh)$.

For $n = 2$*, the partial* 2*-coboundaries are*

$$\begin{aligned} B^2(G,A) &= \{i \in C^2_{par}(G,A)| \exists f \in C^1_{par}(G,A), \partial_1(f)(g,h) = i(g,h)\} \\ &= \{i \in C^2_{par}(G,A)|\ i(g,h) = (\alpha_g(1_{g^{-1}}f(h)))f^{-1}(gh)f(g)\}. \end{aligned}$$

For $f \in C^2_{par}(G, A)$, we have,

$$\partial_2(f)(g,h,l) = (g \cdot (1_{g^{-1}} f(h,l)))f^{-1}(gh,l)f(g,hl)f^{-1}(g,h),$$

then,

$$Z^2(G,A) = \{f \in C^2_{par}(G,A) |\ \partial_2(f)(g,h,l) = e_3(g,h,l), \forall\ g,h,l \in G\}$$
$$= \{f \in C^2_{par}(G,A) | (\alpha_g(1_{g^{-1}} f(h,l)))f(g,hl) = 1_g f(gh,l)f(g,h), \forall g,h,l \in G\}.$$

Again, the appearance only of 1_g in the right hand side of the 2-cocycle condition is due to absorption of units.

Therefore, the cohomology obtained here is the same as in [19].

In the next subsection we will give more specific examples of cohomologies for partial actions in which the algebra A is the base field k.

3.4 Cohomology for partial actions on the base field

Example 3.16 (Partial group actions over the base field) *Let G be a group. We have already seen in Example 2.22 that partial actions kG over k are in correspondence with subgroups $L \leq G$ by the linear functional*

$$\begin{array}{rccl} \lambda: & kG & \longrightarrow & k \\ & \delta_g & \mapsto & \lambda_g = \lambda(\delta_g) = \left\{ \begin{array}{ll} 1 & , g \in L \\ 0 & , \textit{otherwise} \end{array} \right. \end{array}.$$

Fix the subgroup L of G which defines the partial action. Let us now calculate the cohomologies $H^n_{par}(kG, k)$ (we use the symbol ∂_n for the coboundary map to avoid confusion with the basis elements $\delta_g \in kG$):

- *For $n = 0$, $C^0_{par}(kG, k) = k^\times = k \backslash \{0\}$. Let $a \in C^0$, then,*

$$(\partial_0 a)(\delta_g) = (\delta_g \cdot a)a^{-1} = \lambda_g a a^{-1} = \lambda_g = \left\{ \begin{array}{ll} 1, & g \in L \\ 0, & g \notin L \end{array} \right. .$$

  Therefore, $H^0_{par} = Z^0_{par} = C^0_{par} = k^\times$.

- *For $n = 1$, let $f \in Z^1_{par}(kG, k)$. Then,*

$$(\partial_1 f)(\delta_g, \delta_h) = \lambda_g f(\delta_h) f^{-1}(\delta_{gh}) f(\delta_g) = \lambda_g \lambda_h.$$

Denote $f(\delta_g)$ simply by $f(g)$ using the identification between the convolution algebra and the algebras of functions $f : G \to k$. If $g, h \in L$, then the 1-cocycle condition can be rewritten as $f(gh) = f(g)f(h)$, which means that $f_{|_L} : kG \to k^\times$ is a character of the subgroup L. If $g \notin L$ then it is easy to see that for a 1-cocycle f, we have $f(g) = 0$. As the partial 1-coboundaries are given by $\lambda : G \to k$ such that $\lambda(g) = 1$ for any $g \in L$, we have that $H^1 = Z^1/B^1$ are given by the nontrivial 1-dimensional representations of the subgroup L which determine the partial action.

- *For $n = 2$, let $\omega \in Z^2_{par}(kG, k)$. Then, denoting $\omega(\delta_g, \delta_h)$ simply by $\omega_{g,h}$, we have*

$$(\partial\omega)(\delta_g, \delta_h, \delta_l) = \lambda_g \omega_{h,l} \omega^{-1}_{gh,l} \omega_{g,hl} \omega^{-1}_{g,h} = \lambda_g \lambda_h \lambda_l.$$

It is easy to see from the identity above that, if $(g, h) \notin L \times L$ then $\omega_{g,h} = 0$. Then, defining $\omega : G \times G \to k$ by

$$\omega(g,h) = \begin{cases} 0, & (g,h) \notin L \times L \\ \omega(\delta_g, \delta_h), & (g,h) \in L \times L \end{cases},$$

we have that the partial 2-cocycles relative to G are in fact usual 2-cocycles of the subgroup L [1, 31], in other words

$$Z^2_{par}(kG, k) = Z^2(L, k).$$

Example 3.17 (Partial group gradings over the base field) *Let G be a finite abelian group. In Example 2.23 we saw that the partial actions of the Hopf algebra $H = (kG)^* = \langle p_g \mid g \in G \rangle$ over the base field k are in one-to-one correspondence with subgroups $L \leq G$, namely*

$$\lambda_{p_g} = \begin{cases} \frac{1}{|L|} & , \ g \in L \\ 0 & , \ otherwise. \end{cases}.$$

Let us now calculate explicitly the partial cohomologies for $(kG)^*$.

For $n = 0$, recalling that $\delta_a(h) = (h \cdot a)a^{-1}$, for every $a \in k^\times$, we have $\delta_a(p_g) = \lambda_{p_g}$, and this leads to $Z^0((kG)^*, k) = C^0((kG)^*, k) = H^0((kG)^*, k) = k^\times$. Moreover, the 1-coboundaries are basically given by the functional λ.

For $n = 1$, let $\omega : (kG)^* \to k$ be a partial 1-cocycle, then

$$\begin{aligned}\lambda_{p_g}\lambda_{p_h}=\delta(\omega)(p_g,p_h) &= \sum_{l,m,i\in G}(p_{ml^{-1}}\cdot\omega(p_{hi^{-1}}))\overline{\omega}(p_lp_i)\omega(p_{m^{-1}g})\\ &= \sum_{l,m\in G}\lambda_{ml^{-1}}\omega(p_{hl^{-1}}))\overline{\omega}(p_l)\omega(p_{m^{-1}g}).\end{aligned}$$

Recalling that $e_1*\omega=\omega=\omega*e_1$ *and* $e_1(p_g)=\lambda_{p_g}$, *we have*

$$(e_1*\omega)(p_g)=\sum_{h\in G}\lambda_{p_h}\omega(p_{h^{-1}g})\Rightarrow\frac{1}{|L|}\sum_{h\in L}\omega(p_{gh^{-1}})=\omega(p_g)=\frac{1}{|L|}\sum_{h\in L}\omega(p_{h^{-1}g}).$$

This means that

$$\omega(p_g)=\frac{1}{|L|}\sum_{h\in L}\omega(p_{hg})=\frac{1}{|L|}\sum_{h\in L}\omega(p_{gh}).$$

For any $g\in L$, *we have* $\omega(p_g)=\frac{1}{|L|}\sum_{h\in L}\omega(p_h)$, *which is an invariance by translations in the subgroup. Furthermore, using the normalization condition,* $\omega(1)=1$, *we have that*

$$|L|\omega(p_e)=\sum_{g\in L}\omega(p_g)=1\Rightarrow\omega(p_g)=\frac{1}{|L|},\forall\ g\in L.$$

We don't have, a priori any further constraint for the values of $\omega(p_g)$, *for* $g\notin L$. *If we impose that,* $\omega(p_g)=0$, *for* $g\notin L$, *then the only possible choice is the linear functional* $\lambda:(kG)^*\to k$, *which defines the partial action. Therefore,*

$$\begin{aligned}Z^1_{par}((kG)^*,k) &=\{\omega:kg\to k,\quad \omega(p_g)=\frac{1}{|L|},\quad g\in L\},\\ B^1_{par}((kG)^*,k) &=\{\lambda\},\\ H^1_{par}((kG)^*,k) &=\{\omega:kg\to k,\ \omega(p_g)=\frac{1}{|L|},\ g\in L,\omega(p_g)\neq 0,\ g\notin L\}.\end{aligned}$$

For $n=2$, *first recall that* $e_2=\widetilde{e}_{2,1}*\widetilde{e}_{2,2}$, *in which* $e_2(h,l)=h\cdot(l\cdot 1_A)$, $\widetilde{e}_{2,1}(h,l)=(h\cdot 1_A)\varepsilon(l)$ *and* $\widetilde{e}_{2,2}(h,l)=hl\cdot 1_A$. *Then, for* $g,h\in G$, *we have*

$$\omega(p_g,p_h)=\widetilde{e}_{2,1}*\omega(p_g,p_h)=\sum_{l\in G}\lambda_{p_l}\omega(p_{l^{-1}g},p_h)=\frac{1}{|L|}\sum_{l\in L}\omega(p_{l^{-1}g},p_h).$$

This leads to an invariance by translation on the left slot, that is $\omega(p_{lg}, p_h) = \omega(p_g, p_h)$, *for any* $g, h \in G$ *and* $l \in L$. *On the other hand,*

$$\begin{aligned}\omega(p_g, p_h) &= \widetilde{e}_{2,2} * \omega(p_g, p_h) = \sum_{l,m\in L} \lambda_{p_l p_m} \omega(p_{l^{-1}g}, p_{m^{-1}h}) \\ &= \frac{1}{|L|}\sum_{l\in L}\omega(p_{l^{-1}g}, p_{l^{-1}h}) = \frac{1}{|L|}\sum_{l\in L}\omega(p_g, p_{l^{-1}h}).\end{aligned}$$

This gives an invariance by translation on the right slot, that is, $\omega(p_g, p_{lh}) = \omega(p_g, p_h)$, *for any* $g, h \in G$ *and* $l \in L$. *As these invariances are independent, we have finally*

$$\omega(p_{lg}, p_{mh}) = \omega(p_g, p_h),$$

for any $g, h \in G$ *and* $l, m \in L$ *[6]. This translation invariance is a useful tool for searching solutions of partial 2-cocycles in specific cases. Besides the translation invariance, we have the normalization constraint, given by*

$$\omega(1, p_g) = \omega(p_g, 1) = \lambda_{p_g} \Rightarrow \frac{1}{|L|} = \sum_{h\in L}\omega(p_g, p_h) = \sum_{h\in L}\omega(p_h, p_g),\ \forall\, g \in L.$$

Finally, we have the cocycle condition. For $g, h, i \in L$, *we have*

$$\begin{aligned}&\frac{1}{|L|^3} = \lambda_{p_g}\lambda_{p_h}\lambda_{p_i} = \delta\omega(p_g, p_h, p_i) \\ &= \sum_{\substack{l,m,n,r,s,\\ t,x,y\in L}} \lambda_{p_l}\omega(p_r, p_x)\overline{\omega}(\underbrace{p_{l^{-1}m}p_{r^{-1}s}}_{\substack{\Rightarrow\ l^{-1}m = r^{-1}s\\ \Rightarrow\ l = rms^{-1}}}, p_{x^{-1}y})\omega(p_{m^{-1}n}, \underbrace{p_{s^{-1}t}p_{y^{-1}i}}_{\substack{\Rightarrow s^{-1}t = y^{-1}i\\ \Rightarrow\ t^{-1} = i^{-1}ys^{-1}}})\overline{\omega}(p_{n^{-1}g}, p_{t^{-1}h}) \\ &= \frac{1}{|L|}\sum_{\substack{m,n,r,s,\\ x,y\in L}}\omega(p_r, p_x)\overline{\omega}(p_{r^{-1}s}, p_{x^{-1}y})\omega(p_{m^{-1}n}, p_{y^{-1}i})\overline{\omega}(p_{n^{-1}g}, p_{i^{-1}ys^{-1}h}) \\ &= \frac{1}{|L|}\sum_{m,n,s,y\in L}\lambda_{p_s}\lambda_{p_y}\omega(p_{m^{-1}n}, p_{y^{-1}i})\overline{\omega}(p_{n^{-1}g}, p_{i^{-1}ys^{-1}h}) \\ &= \frac{1}{|L|^2}\sum_{n,s,y\in L}\left(\sum_{m\in L}\frac{1}{|L|}\omega(p_{m^{-1}n}, p_{y^{-1}i})\right)\overline{\omega}(p_{n^{-1}g}, p_{i^{-1}ys^{-1}h}) \\ &= \frac{1}{|L|}\sum_{n,s,y\in L}\omega(p_n, p_{y^{-1}i})\overline{\omega}(p_{n^{-1}g}, p_{s^{-1}yi^{-1}h})\frac{1}{|L|}\end{aligned}$$

$$= \frac{1}{|L|}\sum_{n,y\in L}\omega(p_n,p_{y^{-1}i})\overline{\omega}(p_{n^{-1}g},p_{yi^{-1}h})$$

$$\overset{(*)}{=} \frac{1}{|L|}\sum_{n,x\in L}\omega(p_n,p_x)\overline{\omega}(p_{n^{-1}g},p_{x^{-1}h}),$$

in which $(*)$ *is taken putting* $y^{-1}i = x$ *and* $i^{-1}y = yi^{-1} = x^{-1}$. *Therefore,*

$$\frac{1}{|L|^2} = \sum_{n,x\in L}\omega(p_n,p_x)\overline{\omega}(p_{n^{-1}g},p_{x^{-1}h}).$$

The next example is a specific case of a partial grading of the base field for a fixed group G and a fixed subgroup $L \le G$ defining the partial action.

Example 3.18 *Fix* $G=\langle a,b \mid a^2=b^2=e\rangle=\{e,a,b,ab\}$ *and* $L=\langle a\rangle$, *then* $|L|=2$. *Let us calculate the partial* 1*-cocycles in this case. The invariance by translations gives us* $\omega(p_e)=\omega(p_a)=x$ *and* $\omega(p_b)=\omega(p_{ab})=y$, $\overline{\omega}(p_e)=\overline{\omega}(p_a)=\overline{x}$ *and* $\overline{\omega}(p_b)=\overline{\omega}(p_{ab})=\overline{y}$.

By the normalization constraint $\sum_{g\in G}\omega(p_g) = 1 = \sum_{g\in G}\overline{\omega}(p_g)$, *we have*

$$x + y = \frac{1}{2}. \tag{3.1}$$

and

$$\overline{x} + \overline{y} = \frac{1}{2}. \tag{3.2}$$

Moreover, the condition $\omega * \overline{\omega} = e$, *which can be written as*

$$\sum_{h\in G}\omega(p_h)\overline{\omega}(p_{h^{-1}g}) = \begin{cases} \frac{1}{|L|} & g\in L \\ 0 & g\notin L \end{cases},$$

gives us two equations,

$$x\overline{x} + y\overline{y} = \frac{1}{4}, \tag{3.3}$$

$$x\overline{y} + y\overline{x} = 0. \tag{3.4}$$

Finally, the cocycle condition,

$$\lambda_{p_g}\lambda_{p_h} = \sum_{m\in G}\omega(p_m)\overline{\omega}(p_{m^{-1}h})\omega(p_{mh^{-1}g}),$$

gives us:

- *For $g = h = e$,*

$$\omega(p_e)\overline{\omega}(p_e)\omega(p_e) + \omega(p_a)\overline{\omega}(p_a)\omega(p_a) + \\ +\omega(p_b)\overline{\omega}(p_b)\omega(p_b) + \omega(p_{ab})\overline{\omega}(p_{ab})\omega(p_{ab}) = \frac{1}{4}.$$

$$\begin{aligned} &\Rightarrow \quad 2x^2\overline{x} + 2y^2\overline{y} = \frac{1}{4} \Rightarrow x^2\overline{x} + y^2\overline{y} = \frac{1}{8} \\ &\Rightarrow \quad x^2\overline{x} + xy\overline{y} - xy\overline{y} + y^2\overline{y} = \frac{1}{8} \\ &\overset{x\overline{y}=-y\overline{x}}{\Rightarrow} \quad x^2\overline{x} + xy\overline{y} + yy\overline{x} + y^2\overline{y} = \frac{1}{8} \\ &\Rightarrow \quad \frac{1}{8} = x^2\overline{x} + xy\overline{y} + yy\overline{x} + y^2\overline{y} = (x+y)(x\overline{x} + y\overline{y}). \end{aligned}$$

This is the product of equations (3.1) and (3.3), therefore, no new information is added. The same occurs for $g = e$ and $h = a$, $g = a$ and $h = e$, and $g = h = a$.

- *For $g = e$ e $h = b$,*

$$\omega(p_e)\overline{\omega}(p_b)\omega(p_b) + \omega(p_a)\overline{\omega}(p_{ab})\omega(p_{ab}) + \\ +\omega(p_b)\overline{\omega}(p_e)\omega(p_e) + \omega(p_{ab})\overline{\omega}(p_a)\omega(p_a) = 0.$$

$$\Rightarrow xy\overline{y} + xy\overline{y} + xy\overline{x} + xy\overline{x} = 0 \Rightarrow xy\overline{y} + yx\overline{x} = 0.$$

As $\overline{x} + \overline{y} = \frac{1}{2}$, then we have $xy = 0$. The same condition is obtained for the cases $g = e$ and $h = ab$, $g = a$ and $h = b$, $g = a$ and $h = ab$, $g = b$ and $h = e$, $g = b$ and $h = a$, $g = ab$ and $h = e$, and $g = ab$ and $h = a$.

- *For $g = b$ e $h = b$,*

$$\omega(p_e)\overline{\omega}(p_b)\omega(p_e) + \omega(p_a)\overline{\omega}(p_{ab})\omega(p_a) + \\ +\omega(p_b)\overline{\omega}(p_e)\omega(p_b) + \omega(p_{ab})\overline{\omega}(p_a)\omega(p_{ab}) = 0.$$

$$\begin{aligned} &\Rightarrow \quad x^2\overline{y} + x^2\overline{y} + y^2\overline{x} + y^2\overline{x} = 0 \Rightarrow x^2\overline{y} + y^2\overline{x} = 0 \\ &\Rightarrow \quad x^2\overline{y} + xy\overline{y} - xy\overline{y} + y^2\overline{x} = 0 \\ &\overset{xy\overline{x}=-xy\overline{y}}{\Rightarrow} \quad x^2\overline{y} + xy\overline{y} + yx\overline{x} + y^2\overline{x} = 0 \\ &\Rightarrow \quad 0 = x^2\overline{y} + xy\overline{y} + yx\overline{x} + y^2\overline{x} = (x+y)(x\overline{y} + y\overline{x}). \end{aligned}$$

This equation is the product of (3.1) and (3.4), therefore, no new information is added. The same occurs if we take $g = b$ *and* $h = ab$, $g = ab$ *and* $h = b$, *and* $g = h = ab$.

Resuming, we have the following equations:

$$x + y = \frac{1}{2}, \quad \overline{x} + \overline{y} = \frac{1}{2}, \quad x\overline{x} + y\overline{y} = \frac{1}{4}, \quad x\overline{y} + y\overline{x} = 0, \quad xy = 0,$$

whose unique possible solution is $\omega(p_g) = \lambda_{p_g} = \begin{cases} \frac{1}{|L|} & , g \in L \\ 0 & , g \notin L \end{cases}$.

For $n = 2$, *first note that* $\omega(p_g, p_h) = \omega(p_{ag}, p_h) = \omega(p_g, p_{ah}) = \omega(p_{ag}, p_{ah})$, *for every* $g, h \in G$, *then,*

- $\omega(p_e, p_e) = \omega(p_a, p_e) = \omega(p_e, p_a) = \omega(p_a, p_a)$;
- $\omega(p_b, p_e) = \omega(p_{ab}, p_e) = \omega(p_b, p_a) = \omega(p_{ab}, p_a)$;
- $\omega(p_e, p_b) = \omega(p_a, p_b) = \omega(p_e, p_{ab}) = \omega(p_a, p_{ab})$;
- $\omega(p_b, p_b) = \omega(p_{ab}, p_b) = \omega(p_b, p_{ab}) = \omega(p_{ab}, p_{ab})$.

The normalization constraint gives us

$$\sum_{h \in G} \omega(p_g, p_h) = \lambda_{p_g} = \sum_{h \in G} \omega(p_h, p_g).$$

Applying the above normalization constraint respectively for $g = e, a, b, ab$, *we have*

- *For* $g = e$ $(\lambda_{p_e} = 1/2)$,

$$\begin{aligned} \omega(p_e, p_e) + \omega(p_e, p_a) + \omega(p_e, p_b) + \omega(p_e, p_{ab}) &= \frac{1}{2} \\ \omega(p_e, p_e) + \omega(p_a, p_e) + \omega(p_b, p_e) + \omega(p_{ab}, p_e) &= \frac{1}{2}, \end{aligned}$$

$$\Rightarrow \omega(p_e, p_e) + \omega(p_e, p_b) = \frac{1}{4} \quad \text{and} \quad \omega(p_e, p_b) = \omega(p_b, p_e).$$

The same is obtained for $g = a$ $(\lambda_{p_a} = \frac{1}{2})$.

- *For* $g = b$ $(\lambda_{p_b} = 0)$,

$$\begin{aligned} \omega(p_b, p_e) + \omega(p_b, p_a) + \omega(p_b, p_b) + \omega(p_b, p_{ab}) &= 0 \\ \omega(p_e, p_b) + \omega(p_a, p_b) + \omega(p_b, p_b) + \omega(p_{ab}, p_b) &= 0, \end{aligned}$$

$$\Rightarrow \omega(p_b, p_b) = -\omega(p_e, p_b) = -\omega(p_b, p_e).$$

The same is obtained for $g = ab$ $(\lambda_{p_{ab}} = 0)$.

Therefore, the only remaining independent components are $\omega(p_e, p_e)$ *and* $\omega(p_e, p_b)$. *Moreover*

$$\omega(p_e, p_e) + \omega(p_e, p_b) = \frac{1}{4}.$$

The cocycle condition

$$h_{(1)} \cdot \omega(l_{(1)}, m_{(1)})\omega(h_{(2)}, l_{(2)}m_{(2)}) = (h_{(1)} \cdot 1)\omega(h_{(2)}, l_{(1)})\omega(h_{(3)}l_{(2)}, m),$$

can be written, in our case, as

$$\sum_{s\in G}\omega(p_g, p_s)\omega(p_{hs^{-1}}, p_{ls^{-1}}) = \sum_{s\in G}\omega(p_{gs^{-1}}, p_{hs^{-1}})\omega(p_s, p_l).$$

This condition gives us 64 equations which are, in fact redundant, that is, every 2-cochain in this case is a 2-cocycle

Finally, from $\omega * \overline{\omega} = e_2$, *taking* $x = \omega(p_e, p_e)$ *and* $y = \overline{\omega}(p_e, p_e)$, *we obtain the equation [6]*

$$16xy - 3(x + y) + \frac{1}{2} = 0.$$

Chapter 4

The associated Hopf algebra of a partial action

In [19], for a partial action θ of a group G on a commutative algebra A, the authors introduced the inverse semigroup $\widetilde{A}$, given by the invertible elements of all ideals of the form $1_{x_1}\dots 1_{x_n}A$, for $x_1,\dots,x_n \in G$ and $n \in \mathbb{N}$. Once showed that $\theta_x(1_{x^{-1}}\widetilde{A}) = 1_x\widetilde{A}$, in other words, θ restricted to $\widetilde{A}$ defines a partial action $\widetilde{\theta}$ of G on $\widetilde{A}$ such that their cohomologies are the same, that is, $H^n_{par}(G, A) \cong H^n_{par}(G, \widetilde{A})$.

This construction brings advantages because $\widetilde{A}$ possesses a richer structure than A and then one can study, for example, extension theory by partial group actions from a wider perspective, namely, the theory of extensions of inverse semigroups [20].

In our context, we can also have similar constructions, allowing us to trade partial actions of a cocommutative Hopf algebra H on a commutative algebra A by a partial action of H on a commutative and cocommutative Hopf algebra $\widetilde{A}$ generating the same cohomology.

In order to proceed with the construction of this new Hopf algebra, one has a technical obstruction concerning the invertible elements of the algebra A. Indeed, the multiplicative abelian group $A^{\times}$ embeds into the abelian group $C^n_{par}(H, A)$ for each $n \in \mathbb{N}$ by the group monomorphisms $\phi_n : A^{\times} \to C^n_{par}(H, A)$, given by $\phi_n(a) = ae_n$. These morphisms ϕ_n are coherent with the coboundary morphisms, that is, for each $n \in \mathbb{N}$, we have $\delta_n \circ \phi_n(a) = \delta_n(ae_n) = ae_{n+1} = \phi_{n+1}(a)$. Therefore, one can construct a new cochain complex which gets rid of these invertible elements and yet defining the same cohomology.

Definition 4.1 *Let H be a cocommutative Hopf algebra and A be a commutative partial H-module algebra. We define, for $n \in \mathbb{N}$ the n-th reduced partial cochain group $\widetilde{C}^n_{par}(H,A)$ as the quotient abelian group $C^n_{par}(H,A)/A^\times$.*

Proposition 4.2 *The reduced partial cochain complex $\widetilde{C}^\bullet_{par}(H,A)$ generates cohomology groups isomorphic to those relative to the cochain complex $C^\bullet_{par}(H,A)$.*

Proof: Indeed, denote, for each $n \in \mathbb{N}$, the n-th reduced cohomology group by $\widetilde{H}^n_{par}(H,A)$ and define the map

$$\psi^n : H^n_{par}(H,A) \to \widetilde{H}^n_{par}(H,A) \text{ by } \psi^n([f]) \mapsto [fA^\times].$$

One can easily see that this map is well defined, surjective and it is a morphism of abelian groups. The injectivity comes from the fact that given a partial n-cochain $f \in C^n_{par}(H,A)$ and $a \in A^\times$, we have $af * f^{-1} = ae_n = \delta_{n-1}(ae_{n-1})$, then f and af are cohomologous. Therefore the cohomology groups $H^n_{par}(H,A)$ and $\widetilde{H}^n_{par}(H,A)$ are isomorphic.

■

Remark 4.3 *We will denote the reduced n-cochains again by f instead of $fA^\times$ in order to make the notation cleaner. It is clear also that at level zero we have $\widetilde{C}^0_{par}(H,A) = \{1_A\}$.*

Now, define the algebra $\widetilde{A}$ as the quotient $\widetilde{A} = \dfrac{\widehat{A}}{\mathcal{J}}$, in which $\widehat{A}$ is the free commutative unital algebra

$$\widehat{A} = k[X_{1_A}, X_{f(h^1,\ldots,h^n)} \mid n \geq 1,\ h^1,\ldots,h^n \in H,\ f \in \widetilde{C}^n_{par}(H,A)].$$

The set of variables runs over the distinct $f \in \widetilde{C}^n_{par}(H,A)$, that is, if f and g are two partial n-cochains such that $f = g$, then, for every $h^1 \otimes \cdots \otimes h^n \in H^{\otimes n}$ we have $X_{f(h^1,\ldots h^n)} = X_{g(h^1,\ldots h^n)}$. The ideal $\mathcal{J}$ is taken exactly to recover certain properties from the original algebra A and from the partial action of H. This ideal is generated by elements of the type

$$X_{1_A} - 1; \tag{4.1}$$

$$X_{f(h^1,\ldots,\sum_i \lambda_i h^j_i,\ldots,h^n)} - \sum_i \lambda_i X_{f(h^1,\ldots,h^j_i,\ldots h^n)}, \tag{4.2}$$

for each $f \in \widetilde{C}^n_{par}(H,A),\ \forall n > 0$;

$$\sum_i \lambda_i X_{e_{n_1}(h^{1,1},\ldots,h^{1,n_1})} \cdots X_{e_{n_{k_i}}(h^{k_i,1},\ldots,h^{k_i,n_{k_i}})}\ , \tag{4.3}$$

for each zero combination

$$\sum_i \lambda_i e_{n_1}(h^{1,1},\ldots,h^{1,n_1})\ldots e_{n_{k_i}}(h^{k_i,1},\ldots,h^{k_i,n_{k_i}}) = 0 \in A;$$

$$X_{f(h^1_{(1)},\ldots,h^n_{(1)})} X_{g(h^1_{(2)},\ldots,h^n_{(2)})} - X_{(f*g)(h^1,\ldots h^n)}; \tag{4.4}$$

$$X_{(h\cdot(f_1(h^{1,1},\ldots,h^{1,n_1})+\ldots+f_m(l^{m,1},\ldots,l^{m,n_m})))} - \\ - \left(X_{(h\cdot f_1(h^{1,1},\ldots,h^{1,n_1}))} + \ldots + X_{(h\cdot f_m(l^{m,1},\ldots,l^{m,n_m}))}\right); \tag{4.5}$$

$$X_{(1_H\cdot f(h^1,\ldots,h^n))} - X_{f(h^1,\ldots,h^n)}; \tag{4.6}$$

$$X_{(h\cdot(f_1(h^{1,1},\ldots,h^{1,n_1})\ldots f_m(l^{m,1},\ldots,l^{m,n_m})))} - \\ - X_{(h_{(1)}\cdot f_1(h^{1,1},\ldots,h^{1,n_1}))} \cdots X_{(h_{(m)}\cdot f_m(l^{m,1},\ldots,l^{m,n_m}))}; \tag{4.7}$$

and

$$X_{(h\cdot(k\cdot f(h^1,\ldots,h^n)))} - X_{(h_{(1)}\cdot 1_A)} X_{(h_{(2)}k\cdot f(h^1,\ldots,h^n))}. \tag{4.8}$$

Remark 4.4 *1. Note that $\mathcal{I}$ is indeed an ideal of the algebra $\widehat{A}$, for example, an element $h\cdot(f_1(h^{1,1},\ldots,h^{1,n_1})\ldots f_m(l^{m,1},\ldots,l^{m,n_m}))$ can be written as*

$$E^{n_1+\cdots n_m}\left(i_{n_1,n_1+\cdots+n_m}(f_1) * (\varepsilon^{\otimes n_1} \otimes i_{n_2,n_2+\cdots+n_m}(f_2)) * \cdots \right.\\ \left.\cdots *(\varepsilon^{\otimes(n_1+\cdots+n_{m-1})} \otimes f_m)\right)(h, h^{1,1},\ldots,h^{1,n_1},\ldots,l^{m,1},\ldots,l^{m,n_m}),$$

according to Definition 3.8.

2. *The condition (4.1) means that the unit of the algebra A will play the role of the unit of the algebra $\widetilde{A}$.*

3. *The condition (4.3) refers to every linear combination of monomials involving the units of the cochain groups $\widetilde{C}^\bullet_{par}(H,A)$ which vanish in the algebra A. Of course, some of these relations are in fact present among elements of the form (4.2), but there are other vanishing linear combinations in A involving partial actions of elements of H upon the unit 1_A which needed to be ruled out in order to remember the structure of A.*

4. *Casting out elements of the form (4.2) is needed to remember that the generators of* $\widetilde{A}$ *are linear maps between* $H^{\otimes n}$ *and* A*. In particular, in the quotient we have identities of the type* $X_{f(h^1,..,h^i,..,h^n)} = X_{f(h^1,..,h^i_{(1)},..,h^n)}\varepsilon_H(h^i_{(2)}) = \varepsilon_H(h^i_{(1)})X_{f(h^1,..,h^i_{(2)},..,h^n)}$*, for each* $i \in \{1, ..., n\}$*, for each* $f \in \widetilde{C}^n_{par}(H, A)$ *for all* $n > 0$*.*

5. *Casting out elements of the form (4.4) is needed in order to make the relations coming from the convolution product between cochains be still valid in* $\widetilde{A}$*.*

6. *Finally, we need to mod out elements of the form (4.5), (4.6), (4.7) and (4.8) in order to recover the linearity of the partial action of* H *on* A *and the identities coming from axioms (PA1), (PA2) and (PA3).*

After taking the quotient, as far as it doesn't lead to a misunderstanding, we are going to denote the classes $X_{f(h^1,...,h^n)} + \mathcal{I} \in \widetilde{A}$ simply by $X_{f(h^1,...,h^n)}$

Define also the subalgebra of A, $E(A) = \langle h \cdot 1_A \mid h \in H\rangle$ and the unit map $\eta : E(A) \to \widetilde{A}$ given by

$$\eta((h^1 \cdot 1_A) \dots (h^n \cdot 1_A)) = X_{e_1(h^1)} \dots X_{e_1(h^n)}.$$

This map is well defined, because among the generators of the ideal $\mathcal{I}$ which defines the algebra $\widetilde{A}$ there are all linear combinations representing null combinations in A involving the units of the cochain complex. Also, by construction it is an algebra morphism (note that $\eta(1_A) = X_{1_A} = 1_{\widetilde{A}} \in \widetilde{A}$, consequently, $\widetilde{A}$ is a $E(A)$ algebra. Moreover, the unit map is injective. This can be easily seen considering the evaluation map $\widehat{\text{ev}} : \widehat{A} \to A$ which simply associate to each element of $\widehat{a} \in \widehat{A}$ its value at $\widehat{\text{ev}}(\widehat{a}) \in A$. It is easy to see that $\widehat{\text{ev}}(\mathcal{I}) = 0$, then one can define a linear map $\text{ev} : \widetilde{A} \to A$ with the same content. Therefore, if $\eta(a) = 0$ in $\widetilde{A}$, then $a = \text{ev}(\eta(a)) = 0$. By the injectivity of the unit map, one can identify $E(A)$ with its image in $\eta(E(A)) \subseteq \widetilde{A}$.

Every element in the image of $E(A)$ different from $1_{\widetilde{A}}$ can be written as a combination of variables X_e, in which is the image of an idempotent in $\widetilde{C}^n_{par}(H, A)$ for some $n > 0$. In fact, what we are going to prove is that one can rewrite an element of the form $(h^1 \cdot 1_A) \dots (h^n \cdot 1_A)$ as a linear combination of images of the idempotent $e_n \in \widetilde{C}^n_{par}(H, A)$. Let us make induction on the number n of factors $h \cdot 1_A$ involved. For $n = 1$, we have $h \cdot 1_A = e_1(h)$.

Now, suppose that the result is valid for $r \in \mathbb{N}$, $1 \leq r < n$, that is,

$$(h^1 \cdot 1_A)(h^2 \cdot 1_A) \dots (h^r \cdot 1_A) = \sum_{i=1}^{s} e_r(l_i^1, \dots, l_i^r)$$

for some elements $l_i^j \in H$, for $i \in \{1, \dots, s\}$ and $j \in \{1, \dots r\}$. Take $h^1, \dots, h^n \in H$, then

$$\begin{aligned}
& (h^1 \cdot 1_A)(h^2 \cdot 1_A) \dots (h^n \cdot 1_A) = (h^1 \cdot 1_A)[(h^2 \cdot 1_A) \dots (h^n \cdot 1_A)] \\
= & \sum_i (h^1 \cdot 1_A) e_{n-1}(l_i^1, \dots, l_i^{n-1}) \\
= & \sum_i (h^1 \cdot 1_A)(l_i^1 \cdot (l_i^2 \cdot (\dots \cdot (l_i^{n-1} \cdot 1_A) \dots))) \\
= & \sum_i (h^1_{(1)} \cdot 1_A)(h^1_{(2)} S(h^1_{(3)}) l_i^1 \cdot (l_i^2 \cdot (\dots \cdot (l_i^{n-1} \cdot 1_A) \dots))) \\
= & \sum_i h^1_{(1)} \cdot (S(h^1_{(2)}) l_i^1 \cdot (l_i^2 \cdot (\dots \cdot (l_i^{n-1} \cdot 1_A) \dots))) \\
= & \sum_i e_n(h^1_{(1)}, S(h_{(2)}) l_i^1, l_i^2, \dots, l_i^{n-1}).
\end{aligned}$$

This proves our claim. Moreover, for each $n > 0$, we have $e_n(h^1, \dots, h^n) \in E(A)$. Indeed,

$$\begin{aligned}
& e_n(h^1, \dots, h^n) = h^1 \cdot (h^2 \cdot (\dots \cdot (h^n \cdot 1_A) \dots)) \\
= & (h^1_{(1)} \cdot 1_A)(h^1_{(2)} h^2 \cdot (\dots \cdot (h^n \cdot 1_A) \dots)) \\
= & (h^1_{(1)} \cdot 1_A)(h^1_{(2)} h^2_{(1)} \cdot 1_A)(h^1_{(3)} h^2_{(2)} h^3 \cdot (\dots \cdot (h^n \cdot 1_A) \dots)) \\
= & (h^1_{(1)} \cdot 1_A)(h^1_{(2)} h^2_{(1)} \cdot 1_A) \dots (h^1_{(n)} h^2_{(n-1)} \dots h^{n-1}_{(2)} h^n \cdot 1_A) \in E(A).
\end{aligned}$$

Our construction will enable us to see a richer structure on the algebra $\widetilde{A}$ with the advantage of getting the same cohomology theory as the original algebra A.

Theorem 4.5 *Let H be a cocommutative Hopf algebra H and A be commutative partial H-module algebra A. Then the algebra $\widetilde{A}$ is a commutative and cocommutative Hopf algebra which is also a partial H-module algebra such that for only $n \in \mathbb{N}$, the n-cohomology group $H^n_{par}(H, \widetilde{A})$ is isomorphic to the n-cohomology $H^n_{par}(H, A)$.*

Proof: We have already shown that $\widetilde{A}$ is a commutative algebra over $E(A)$. For the coalgebra structure, define the map $\widehat{\Delta} : \widehat{A} \to \widehat{A} \otimes_{E(A)} \widehat{A}$, given by,

$$\widehat{\Delta}(X_{f_1(h^{1,1},\dots,h^{1,n_1})}\dots X_{f_m(l^{m,1},\dots,l^{m,n_m})}) =$$
$$= X_{f_1(h^{1,1}_{(1)},\dots,h^{1,n_1}_{(1)})}\dots X_{f_m(l^{m,1}_{(1)},\dots,l^{m,n_m}_{(1)})} \otimes X_{f_1(h^{1,1}_{(2)},\dots,h^{1,n_1}_{(2)})}\dots X_{f_m(l^{m,1}_{(2)},\dots,l^{m,n_m}_{(2)})},$$

for $f_1 \in \widetilde{C}^{n_1}(H,A),\dots,f^m \in \widetilde{C}^{n_m}(H,A)$. And the map $\widehat{\varepsilon} : \widehat{A} \to E(A)$, given by

$$\widehat{\varepsilon}(X_{f_1(h^{1,1},\dots,h^{1,n_1})}\dots X_{f_m(l^{m,1},\dots,l^{m,n_m})}) =$$
$$e_{n_1}(h^{1,1},\dots,h^{1,n_1})\dots e_{n_m}(l^{m,1},\dots,l^{m,n_m}).$$

Finally, we define the antipode $\widehat{S} : \widehat{A} \to \widehat{A}$ as

$$\widehat{S}(X_{f_1(h^{1,1},\dots,h^{1,n_1})}\dots X_{f_m(l^{m,1},\dots,l^{m,n_m})}) =$$
$$= X_{f_1^{-1}(h^{1,1},\dots,h^{1,n_1})}\dots X_{f_m^{-1}(l^{m,1},\dots,l^{m,n_m})},$$

for $f_1 \in \widetilde{C}^{n_1}(H,A),\dots,f^m \in \widetilde{C}^{n_m}(H,A)$.

One needs first to show that these maps can be well defined in $\widetilde{A}$ that is, we must verify that $\mathcal{I}$ is a Hopf ideal. Most of the verifications are long, but straightforward. Basically, for $\widehat{\varepsilon}$, as its image lies in $E(A) \subseteq A$, where the relations are valid, then $\widehat{\varepsilon}(\mathcal{I}) = 0$. For $\widehat{S}$, it is also easy to see that $\widehat{S}(\mathcal{I}) \subseteq \mathcal{I}$. Therefore, one can define algebra maps $\varepsilon : \widetilde{A} \to E(A)$ and $S : \widetilde{A} \to \widetilde{A}$, ($S$ is an algebra map because $\widetilde{A}$ is commutative) in the same way.

The most involved ones are the verifications for $\widehat{\Delta}$. For this task, it is convenient to divide the process into two steps. First, we consider the ideal $\mathcal{J} \trianglelefteq \widehat{A}$ generated only by elements of the form (4.1), (4.2) and (4.3). For elements of the form (4.1) and (4.2), it is quite straightforward, now take an element of the form (4.3) that is, a linear combination

$$x = \sum_i \lambda_i X_{e_{n_1}(h^{1,1},\dots,h^{1,n_1})}\dots X_{e_{n_{k_i}}(h^{k_i,1},\dots,h^{k_i,n_{k_i}})} \in \mathcal{J}.$$

such that $\widehat{\mathrm{ev}}(x)=0$. Then, we have

$$\begin{aligned}
&\widehat{\Delta}(x)= \\
&=\sum_i \lambda_i X_{e_{n_1}(h^{1,1}_{(1)},\dots,h^{1,n_1}_{(1)})}\dots X_{e_{n_{k_i}}(h^{k_i,1}_{(1)},\dots,h^{k_i,n_{k_i}}_{(1)})}\otimes X_{e_{n_1}(h^{1,1}_{(2)},\dots,h^{1,n_1}_{(2)})}\dots \\
&\qquad\dots X_{e_{n_{k_i}}(h^{k_i,1}_{(2)},\dots,h^{k_i,n_{k_i}}_{(2)})} \\
&=\sum_i \lambda_i X_{e_{n_1}(h^{1,1}_{(1)},\dots,h^{1,n_1}_{(1)})}\dots X_{e_{n_{k_i}}(h^{k_i,1}_{(1)},\dots,h^{k_i,n_{k_i}}_{(1)})} X_{e_{n_1}(h^{1,1}_{(2)},\dots,h^{1,n_1}_{(2)})}\dots \\
&\qquad\dots X_{e_{n_{k_i}}(h^{k_i,1}_{(2)},\dots,h^{k_i,n_{k_i}}_{(2)})}\otimes 1 \\
&=\sum_i \lambda_i X_{e_{n_1}(h^{1,1}_{(1)},\dots,h^{1,n_1}_{(1)})} X_{e_{n_1}(h^{1,1}_{(2)},\dots,h^{1,n_1}_{(2)})}\dots X_{e_{n_{k_i}}(h^{k_i,1}_{(1)},\dots,h^{k_i,n_{k_i}}_{(1)})} \\
&\qquad X_{e_{n_{k_i}}(h^{k_i,1}_{(2)},\dots,h^{k_i,n_{k_i}}_{(2)})}\otimes 1 \\
&=\sum_i \lambda_i \Bigg(\bigg(\Big(\Big(X_{e_{n_1}(h^{1,1}_{(1)},\dots,h^{1,n_1}_{(1)})} X_{e_{n_1}(h^{1,1}_{(2)},\dots,h^{1,n_1}_{(2)})} - X_{e_{n_1}(h^{1,1},\dots,h^{1,n_1})}\Big) \\
&\quad X_{e_{n_2}(h^{2,1}_{(1)},\dots,h^{2,n_2}_{(1)})} X_{e_{n_2}(h^{2,1}_{(2)},\dots,h^{2,n_2}_{(2)})}\dots X_{e_{n_{k_i}}(h^{k_i,1}_{(1)},\dots,h^{k_i,n_{k_i}}_{(1)})} X_{e_{n_{k_i}}(h^{k_i,1}_{(2)},\dots,h^{k_i,n_{k_i}}_{(2)})}\Big)\otimes 1 \\
&\quad+\Big(X_{e_{n_1}(h^{1,1},\dots,h^{1,n_1})}\Big(X_{e_{n_2}(h^{2,1}_{(1)},\dots,h^{2,n_2}_{(1)})} X_{e_{n_2}(h^{2,1}_{(2)},\dots,h^{2,n_2}_{(2)})} - X_{e_{n_2}(h^{2,1},\dots,h^{2,n_2})}\Big) \\
&\quad X_{e_{n_3}(h^{3,1},\dots,h^{3,n_3})}\dots X_{e_{n_{k_i}}(h^{k_i,1}_{(1)},\dots,h^{k_i,n_{k_i}}_{(1)})} X_{e_{n_{k_i}}(h^{k_i,1}_{(2)},\dots,h^{k_i,n_{k_i}}_{(2)})}\Big)\otimes 1+\dots \\
&\quad\dots+\Big(X_{e_{n_1}(h^{1,1},\dots,h^{1,n_1})} X_{e_{n_2}(h^{2,1},\dots,h^{2,n_2})}\dots X_{e_{n_{k_i-1}}(h^{k_i-1,1},\dots,h^{k_i-1,n_{k_i-1}})} \\
&\quad\Big(X_{e_{n_{k_i}}(h^{k_i,1}_{(1)},\dots,h^{k_i,n_{k_i}}_{(1)})} X_{e_{n_{k_i}}(h^{k_i,1}_{(2)},\dots,h^{k_i,n_{k_i}}_{(2)})} - X_{e_{n_{k_i}}(h^{k_i,1},\dots,h^{k_i,n_{k_i}})}\Big)\bigg)\otimes 1\Bigg) \\
&\quad+\sum_i \lambda_i X_{e_{n_1}(h^{1,1},\dots,h^{1,n_1})}\dots X_{e_{n_{k_i}}(h^{k_i,1},\dots,h^{k_i,n_{k_i}})}\otimes 1.
\end{aligned}$$

Therefore, $\widehat{\Delta}(x)\in \widehat{A}\otimes \mathcal{J}+\mathcal{J}\otimes \widehat{A}$. Then, one can define a new linear map

$$\overline{\Delta}:\widehat{A}/\mathcal{J}\to \left(\widehat{A}/\mathcal{J}\right)\otimes_{E(A)}\left(\widehat{A}/\mathcal{J}\right)\cong \frac{\left(\widehat{A}\otimes_{E(A)}\widehat{A}\right)}{\left(\widehat{A}\otimes_{E(A)}\mathcal{J}+\mathcal{J}\otimes_{E(A)}\widehat{A}\right)}.$$

with the same form. Recall that in $\widehat{A}/\mathcal{J}$ we have identities of the form

$$X_{f(h^1,\dots,h^n)}=X_{f(h^1_{(1)},\dots,h^n_{(1)})}\varepsilon_H(h^1_{(2)})\dots\varepsilon_H(h^n_{(2)}).$$

Now define the ideal $\mathcal{I}' \trianglelefteq \widehat{A}/\mathcal{J}$ generated by the elements of the form (4.4), (4.5), (4.6), (4.7) and (4.8). Take an element of the form (4.4), $x = X_{f(h^1_{(1)},\dots,h^n_{(1)})} X_{g(h^1_{(2)},\dots,h^n_{(2)})} - X_{(f*g)(h^1,\dots h^n)} \in \mathcal{I}'$, then,

$$\begin{aligned}
&\overline{\Delta}(x) = X_{f(h^1_{(1)},\dots,h^n_{(1)})} X_{g(h^1_{(3)},\dots,h^n_{(3)})} \otimes X_{f(h^1_{(2)},\dots,h^n_{(2)})} X_{g(h^1_{(4)},\dots,h^n_{(4)})} \\
&\quad - X_{(f*g)(h^1_{(1)},\dots h^n_{(1)})} \otimes X_{(f*g)(h^1_{(2)},\dots h^n_{(2)})} \\
&= X_{f(h^1_{(1)},\dots,h^n_{(1)})} X_{g(h^1_{(2)},\dots,h^n_{(2)})} \otimes X_{f(h^1_{(3)},\dots,h^n_{(3)})} X_{g(h^1_{(4)},\dots,h^n_{(4)})} \\
&\quad - X_{(f*g)(h^1_{(1)},\dots h^n_{(1)})} \otimes X_{f(h^1_{(2)},\dots,h^n_{(2)})} X_{g(h^1_{(3)},\dots,h^n_{(3)})} \\
&\quad + X_{(f*g)(h^1_{(1)},\dots h^n_{(1)})} \otimes X_{f(h^1_{(2)},\dots,h^n_{(2)})} X_{g(h^1_{(3)},\dots,h^n_{(3)})} \\
&\quad - X_{(f*g)(h^1_{(1)},\dots h^n_{(1)})} \otimes X_{(f*g)(h^1_{(2)},\dots h^n_{(2)})} \\
&= X_{f(h^1_{(1)},\dots,h^n_{(1)})} X_{g(h^1_{(2)},\dots,h^n_{(2)})} \otimes X_{f(h^1_{(3)},\dots,h^n_{(3)})} X_{g(h^1_{(4)},\dots,h^n_{(4)})} \\
&\quad - X_{(f*g)(h^1_{(1)},\dots h^n_{(1)})} \varepsilon_H(h^1_{(2)}) \dots \varepsilon_H(h^n_{(2)}) \otimes X_{f(h^1_{(3)},\dots,h^n_{(3)})} X_{g(h^1_{(4)},\dots,h^n_{(4)})} \\
&\quad + X_{(f*g)(h^1_{(1)},\dots h^n_{(1)})} \otimes \Big(X_{f(h^1_{(2)},\dots,h^n_{(2)})} X_{g(h^1_{(3)},\dots,h^n_{(3)})} - X_{(f*g)(h^1_{(2)},\dots h^n_{(2)})} \Big) \\
&= \Big(X_{f(h^1_{(1)},\dots,h^n_{(1)})} X_{g(h^1_{(2)},\dots,h^n_{(2)})} - X_{(f*g)(h^1_{(1)},\dots h^n_{(1)})} \varepsilon_H(h^1_{(2)}) \dots \varepsilon_H(h^n_{(2)}) \Big) \\
&\quad \otimes X_{f(h^1_{(3)},\dots,h^n_{(3)})} X_{g(h^1_{(4)},\dots,h^n_{(4)})} \\
&\quad + X_{(f*g)(h^1_{(1)},\dots h^n_{(1)})} \otimes \Big(X_{f(h^1_{(2)},\dots,h^n_{(2)})} X_{g(h^1_{(3)},\dots,h^n_{(3)})} - X_{(f*g)(h^1_{(2)},\dots h^n_{(2)})} \Big).
\end{aligned}$$

Therefore $\overline{\Delta}(x) \in \mathcal{I}' \otimes \Big(\widehat{A}/\mathcal{J}\Big) + \Big(\widehat{A}/\mathcal{J}\Big) \otimes \mathcal{I}'$. With similar strategies, one can prove the same for elements of the form (4.5), (4.6), (4.7) and (4.8). Therefore, there exists a well defined algebra map $\Delta : \widetilde{A} \to \widetilde{A} \otimes_{E(A)} \widetilde{A}$ with the same form on generators.

It is easy to see that $(\widetilde{A}, \mu, \eta, \Delta, \varepsilon)$ gives a commutative and cocommutative bialgebra over the base algebra $E(A)$.

Let us verify the antipode axioms, $(I * S) = (S * I) = \eta \circ \varepsilon$. Indeed, for $f_1 \in \widetilde{C}^{n_1}(H, A), \dots f^m \in \widetilde{C}^{n_m}(H, A)$ we have

$$\begin{aligned}
&(S * I)(X_{f_1(h^{1,1},\dots,h^{1,n_1})} \dots X_{f_m(l^{m,1},\dots,l^{m,n_m})}) \\
&= \mu(S \otimes I) \circ \Delta(X_{f_1(h^{1,1},\dots,h^{1,n_1})} \dots X_{f_m(l^{m,1},\dots,l^{m,n_m})}) \\
&= S(X_{f_1(h^{1,1}_{(1)},\dots,h^{1,n_1}_{(1)})} \dots X_{f_m(l^{m,1}_{(1)},\dots,l^{m,n_m}_{(1)})}) X_{f_1(h^{1,1}_{(2)},\dots,h^{1,n_1}_{(2)})} \dots X_{f_m(l^{m,1}_{(2)},\dots,l^{m,n_m}_{(2)})} \\
&= X_{f_1^{-1}(h^{1,1}_{(1)},\dots,h^{1,n_1}_{(1)})} \dots X_{f_m^{-1}(l^{m,1}_{(1)},\dots,l^{m,n_m}_{(1)})} X_{f_1(h^{1,1}_{(2)},\dots,h^{1,n_1}_{(2)})} \dots X_{f_m(l^{m,1}_{(2)},\dots,l^{m,n_m}_{(2)})} \\
&= X_{(f_1^{-1} * f_1)(h^{1,1},\dots,h^{1,n_1})} \dots X_{(f_m^{-1} * f_m)(l^{m,1},\dots,l^{m,n_m})} \\
&= X_{e_{n_1}(h^{1,1},\dots,h^{1,n_1})} \dots X_{e_{n_m}(l^{m,1},\dots,l^{m,n_m})} \\
&= \eta \circ \varepsilon(X_{f_1(h^{1,1},\dots,h^{1,n_1})} \dots X_{f_m(l^{m,1},\dots,l^{m,n_m})}).
\end{aligned}$$

Analogously, we have the equality $(I * S) = \eta \circ \varepsilon$. Therefore, $\widetilde{A}$ is a commutative and cocommutative Hopf algebra over $E(A)$.

One can define a partial action of H on $\widetilde{A}$, $\bullet : H \otimes \widetilde{A} \to \widetilde{A}$. First, define a linear map $\blacktriangleright: H \otimes \widehat{A} \to \widehat{A}$ given by

$$h \blacktriangleright (X_{f_1(h^{1,1},\ldots,h^{1,n_1})} \ldots X_{f_m(l^{m,1},\ldots,l^{m,n_m})}) = \\ = X_{(h_{(1)} \cdot f_1(h^{1,1},\ldots,h^{1,n_1}))} \ldots X_{(h_{(m)} \cdot f_m(l^{m,1},\ldots,l^{m,n_m})}$$

For each $h \in H$, one can prove that $h \blacktriangleright \mathcal{I} \subseteq \mathcal{I}$. For example, taking an element

$$x = X_{f(h^1_{(1)},\ldots,h^n_{(1)})} X_{g(h^1_{(2)},\ldots,h^n_{(2)})} - X_{(f*g)(h^1,\ldots h^n)},$$

we have

$$h \blacktriangleright x = X_{h_{(1)} \cdot f(h^1_{(1)},\ldots,h^n_{(1)})} X_{h_{(2)} \cdot g(h^1_{(2)},\ldots,h^n_{(2)})} - X_{h \cdot (f*g)(h^1,\ldots h^n)} \\ = X_{h_{(1)} \cdot f(h^1_{(1)},\ldots,h^n_{(1)})} X_{h_{(2)} \cdot g(h^1_{(2)},\ldots,h^n_{(2)})} - X_{h \cdot (f(h^1_{(1)},\ldots,h^n_{(1)}) g(h^1_{(2)},\ldots,h^n_{(2)}))} \in \mathcal{I}.$$

Then, there is a well defined map $\bullet : H \otimes \widetilde{A} \to \widetilde{A}$, again, given by.

$$h \bullet (X_{f_1(h^{1,1},\ldots,h^{1,n_1})} \ldots X_{f_m(l^{m,1},\ldots,l^{m,n_m})}) = \\ = X_{(h_{(1)} \cdot f_1(h^{1,1},\ldots,h^{1,n_1}))} \ldots X_{(h_{(m)} \cdot f_m(l^{m,1},\ldots,l^{m,n_m}))}.$$

It is straightforward to show that $\bullet$ is a partial action of H on $\widetilde{A}$. This follows directly from the fact that $\cdot$ is a partial action of H on A.

Finally, it remains to verify that A and $\widetilde{A}$ generate the same cohomology groups, that is, for any $n \in \mathbb{N}$ we have $H^n_{par}(H, \widetilde{A}) \cong H^n_{par}(H, A)$. In fact, what we are going to prove is that $H^n_{par}(H, \widetilde{A}) \cong \widetilde{H}^n_{par}(H, A)$, which implies our result.

First note that, for any $n \in \mathbb{N}$ and $h^1 \otimes \cdots \otimes h^n \in H^{\otimes n}$ we have

$$X_{e_n(h^1,\ldots,h^n)} = X_{h^1 \cdot (\ldots (h^n \cdot 1_A) \ldots)} = h^1 \bullet (\ldots (h^n \bullet X_{1_A}) \ldots) = h^1 \bullet (\ldots (h^n \bullet 1_{\widetilde{A}}) \ldots).$$

For each $n \in \mathbb{N}$, a reduced partial n-cochain $f \in \widetilde{C}^n_{par}(H, A)$ generates a partial n-cochain $\widetilde{f} \in C^n_{par}(H, \widetilde{A})$ given by $\widetilde{f}(h^1, \ldots, h^n) = X_{f(h^1,\ldots,h^n)}$. On the other hand, each n-cochain $g \in C^n_{par}(H, \widetilde{A})$, in order to be convolution invertible, must be of the form $g(h^1, \ldots h^n) = X_{\overline{g}(h^1,\ldots,h^n)}$, for some $\overline{g} \in \widetilde{C}^n_{par}(H, A)$, for each $h^1 \otimes \cdots \otimes h^n \in H^{\otimes n}$. Therefore, one can define, for each $n \in \mathbb{N}$, two mutually inverse well defined morphisms of abelian groups $\Phi : \widetilde{H}^n_{par}(H, A) \to H^n_{par}(H, \widetilde{A})$ given by $\Phi([f]) \mapsto [\widetilde{f}]$ and $\Psi : H^n_{par}(H, \widetilde{A}) \to \widetilde{H}^n_{par}(H, A)$ given by $\Psi([g]) \mapsto [\overline{g}]$.

These maps produce the isomorphism between the cohomology groups $H^n_{par}(H, \widetilde{A})$ and $\widetilde{H}^n_{par}(H, A)$, and consequently between $H^n_{par}(H, \widetilde{A})$ and $H^n_{par}(H, A)$. ■

Remark 4.6 *For the classical case of a global action of a cocommutative Hopf algebra H on a commutative algebra A, [30], one can still construct this Hopf algebra $\widetilde{A}$, and in this case, as $h \cdot 1_A = \varepsilon_H(h)1_A$, the base subalgebra $E(A)$ coincides with the base field. Therefore, the Hopf algebra $\widetilde{A}$ is a commutative and cocommutative Hopf algebra over k which gives the same classical cohomological theory as A. The properties of this Hopf algebra and its role in the classical cohomology theory is still an interesting topic to be explored.*

Chapter 5

Twisted partial actions and crossed products

In reference [5], the authors introduced the notion of a twisted partial action of a Hopf algebra H over an algebra A and described the construction of the crossed product by a 2-cocycle. They introduced also the notion of symmetric twisted partial Hopf actions and in this context, they were able to decide whether two twisted partial actions give rise to the same crossed product. Recall that, in the classical case, two crossed products are isomorphic if, and only if, the associated twisted (global) actions can be transformed one into another by some kind of coboundary (see [28], Theorem 7.3.4). So, it's analogue the partial case takes the form: two crossed products are isomorphic if their associated cocycles are related by a linear map which has properties similar to a convolution invertible 2-coboundary. Nevertheless, the authors of [5] still did not have a cohomology theory underlying those crossed products. In what follows, we shall see that in the case of cocommutative Hopf algebras acting partially over commutative algebras the crossed products are indeed classified by the second cohomology group,as defined before.

Definition 5.1 *[5] Let H be a Hopf algebra and A be a unital algebra (with unit 1_A). Let $\cdot : H \otimes A \to A$ and $\omega : H \otimes H \to A$ be two linear maps. The pair $(\cdot, \omega)$ is called a twisted partial action of H over A if,*

(TPA1) $1_H \cdot a = a$, for every $a \in A$.

(TPA2) $h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b)$, for every $h \in H$ and $a, b \in A$.

(TPA3) $(h_{(1)} \cdot (l_{(1)} \cdot a))\omega(h_{(2)}, l_{(2)}) = \omega(h_{(1)}, l_{(1)})(h_{(2)}l_{(2)} \cdot a)$, for every $h, l \in H$ and $a \in A$.

(TPA4) $\omega(h, l) = \omega(h_{(1)}, l_{(1)})(h_{(2)}l_{(2)} \cdot 1_A)$, for every $h, l \in H$.

In this case, we say that $(A, \cdot, \omega)$ is a twisted partial H-module algebra.

Definition 5.2 *[5] Let H be a Hopf algebra and $(A, \cdot, \omega)$ be a twisted partial H-module algebra as above. Define over $A \otimes H$ a multiplication given by*

$$(a \otimes h)(b \otimes l) = \sum a(h_{(1)} \cdot b)\omega(h_{(2)}, l_{(1)}) \otimes h_{(3)}l_{(2)}$$

for every $a, b \in A$ e $h, l \in H$. We define the partial crossed product as $\underline{A\#_\omega H} = (A \otimes H)(1_A \otimes 1_H)$.

Proposition 5.3 *[5] Given a Hopf algebra H and a twisted partial H-module algebra $(A, \cdot, \omega)$, the partial crossed product $\underline{A\#_\omega H}$ is unital if, and only if,*

$$\omega(h, 1_H) = \omega(1_H, h) = h \cdot 1_A, \qquad \forall h \in H. \tag{5.1}$$

Moreover, the crossed product is associative if, and only if

$$(h_{(1)} \cdot \omega(l_{(1)}, m_{(1)}))\omega(h_{(2)}, l_{(2)}m_{(2)}) = \omega(h_{(1)}, l_{(1)})\omega(h_{(2)}l_{(2)}, m), \forall h, l, m \in H. \tag{5.2}$$

□

A linear map $\omega : H^{\otimes 2} \to A$ satisfying (5.1) and (5.2) of the above Proposition is called a normalized cocycle.

We denote by $a\#h$ the element

$$(a \otimes h)(1_A \otimes 1_H) = a(h_{(1)} \cdot 1_A) \otimes h_{(2)} \in \underline{A\#_\omega H}.$$

One can easily deduce that

$$a\#h = a(h_{(1)} \cdot 1_A)\#h_{(2)}.$$

There is an injective algebra morphism $i : A \to \underline{A\#_\omega H}$, given by $i(a) = a\#1$, this endows the crossed product $\underline{A\#_\omega H}$ with a left A-module structure. Also one can show that the linear map

$$\begin{array}{rccc} \rho : & \underline{A\#_\omega H} & \to & \underline{A\#_\omega H} \otimes H \\ & a\#h & \mapsto & a\#h_{(1)} \otimes h_{(2)} \end{array}$$

defines a right H-comodule algebra structure on $\underline{A\#_\omega H}$. This left A-module and right H-comodule structures on the partial crossed product will be important in order to relate crossed products with extensions of A by H.

Definition 5.4 *[5] Let $A = (A, \cdot, \omega)$ be a twisted partial H-module algebra. We say that the twisted partial action is symmetric if*

(i) The linear maps $\tilde{e}_{1,2}, \tilde{e}_2 : H \otimes H \to A$, given by $\tilde{e}_{1,2}(h,l) = (h \cdot 1_A)\varepsilon(l)$ and $\tilde{e}_2(h,l) = hl \cdot 1_A$ are central idempotents in the convolution algebra $Hom_k(H \otimes H, A)$;

(ii) The map ω satisfies the cocycle condition (5.2) and it is an invertible element in the ideal $\langle \tilde{e}_{1,2} * \tilde{e}_2 \rangle \subset Hom(H \otimes H, A)$.

(iii) For any $h, l \in H$, we have

$$e_2(h,l) = (h \cdot (l \cdot 1_A)) = \sum (h_{(1)} \cdot 1_A)(h_{(2)} l \cdot 1_A) = (\tilde{e}_{1,2} * \tilde{e}_2)(h,l)$$

The algebra A is called, in this case, a symmetric twisted partial H-module algebra.

For the case of a cocommutative Hopf algebra H and a commutative algebra A, every symmetric twisted partial action of H over A is in fact a partial action.

Proposition 5.5 *Let H be a cocommutative Hopf algebra and A be a commutative symmetric twisted partial H-module algebra, then A is a partial H-module algebra.*

Proof: Indeed, by axiom (TPA3) from Definition 5.1,

$$\sum (h_{(1)} \cdot (l_{(1)} \cdot a))\omega(h_{(2)}, l_{(2)}) = \sum \omega(h_{(1)}, l_{(1)})(h_{(2)} l_{(2)} \cdot a),$$

we conclude that

$$\begin{aligned}
\sum (h \cdot (l \cdot a)) &= \sum \omega(h_{(1)}, l_{(1)})(h_{(2)} l_{(2)} \cdot a)\omega^{-1}(h_{(3)}, l_{(3)}) \\
&= \sum \omega(h_{(1)}, l_{(1)})\omega^{-1}(h_{(2)}, l_{(2)})(h_{(3)} l_{(3)} \cdot a) \\
&= \sum (h_{(1)} \cdot (l_{(1)} \cdot 1_A))(h_{(2)} l_{(2)} \cdot a) \\
&= \sum (h_{(1)} \cdot 1_A)(h_{(2)} l_{(1)} \cdot 1_A)(h_{(2)} l_{(2)} \cdot a) \\
&= \sum (h_{(1)} \cdot 1_A)(h_{(2)} l \cdot a).
\end{aligned}$$

By the commutativity of the convolution algebra $Hom(H \otimes H, A)$, we also conclude that $h \cdot (l \cdot a) = (h_{(1)} l \cdot a)(h_{(2)} \cdot 1_A)$. Therefore, A is a partial H-module algebra.

■

In the case of H being a cocommutative Hopf algebra and A being a partial H-module algebra, we can still define a partial crossed product for each 2-cocycle $\omega \in Z^2_{par}(H, A)$. In fact, all possible partial crossed products which can be constructed in this case are classified by the second cohomology group $H^2_{par}(H, A)$.

5.1 Partial crossed products and $H^2_{par}(H, A)$

Theorem 4.1 from reference [5] gives a necessary and sufficient condition on two different symmetric partial actions of a Hopf algebra H over an algebra A to have isomorphic crossed products.

Theorem 5.6 *[5] Let A be a unital algebra and H a Hopf algebra with two symmetric twisted partial actions, $h \otimes a \mapsto h \cdot a$ and $h \otimes a \mapsto h \bullet a$, with cocycles ω and σ, respectively. Suppose that there is an isomorphism*

$$\Phi : \underline{A \#_\omega H} \to \underline{A \#_\sigma H}$$

which is also a left A-module and a right H-comodule map. Then there exist linear maps $u, v \in Hom_k(H, A)$ such that for all $h, k \in H$, $a \in A$

(i) $u * v(h) = h \cdot 1_A$;

(ii) $u(h) = u(h_{(1)})(h_{(2)} \cdot 1_A) = (h_{(1)} \cdot 1_A) u(h_{(2)})$;

(iii) $h \bullet a = v(h_{(1)})(h_{(2)} \cdot a) u(h_{(3)})$

(iv) $\sigma(h, k) = v(h_{(1)})(h_{(2)} \cdot v(h_{(2)}))\omega(h_{(3)}, k_{(2)}) u(h_{(4)} k_{(3)})$;

(v) $\Phi(a \#_\omega h) = a u(h_{(1)}) \#_\sigma h_{(2)}$.

Conversely, given maps $u, v \in Hom_k(H, A)$, satisfying (i),(ii), (iii) and (iv) and, in addition $u(1_H) = v(1_H) = 1_A$, then the map Φ, as presented in (v), is an isomorphism of algebras.

□

For the case of a cocommutative Hopf algebra H and a commutative algebra A, items (i) and (ii) imply that u is the convolution inverse of v in the ideal $e_1 * \mathrm{Hom}_k(H, A)$. Item (iii), in its turn, implies that the two partial actions $\bullet$ and $\cdot$ are equal. Finally, item (iv) can be rewritten as

$$\sigma * \omega^{-1}(h,k) = (h_{(1)} \cdot v(k_{(1)}))u(h_{(2)}k_{(2)})v(h_{(3)}) = \delta_1(v)(h,k).$$

Therefore, one can rewrite Theorem 5.6 as:

Theorem 5.7 *Let H be a cocommutative Hopf algebra and A be a partial H-module algebra. Then, given two partial 2-cocycles $\omega, \sigma \in Z^2_{par}(H,A)$, the associated partial crossed products $\underline{A\#_\omega H}$ and $\underline{A\#_\sigma H}$ are isomorphic if, and only if, ω and σ are cohomologous, that is, they belong to the same class in the cohomology group $H^2_{par}(H,A)$.*

□

In order to conclude that the second partial cohomology fully classifies all the isomorphism classes of partial crossed products, it remains to check that every class in $H^2_{par}(H,A)$ contains a normalized 2-cocycle.

Proposition 5.8 *Given a partial 2-cocycle $\omega \in Z^2_{par}(H,A)$, there exists a normalized 2-cocycle $\widetilde{\omega} \in Z^2(H,A)$, which is cohomologous to ω.*

Proof: Indeed, take a 2-cocycle ω, then ω satisfies

$$(h_{(1)} \cdot \omega(k_{(1)}, l_{(1)}))\omega(h_{(2)}, k_{(2)}l_{(2)}) = \omega(h_{(1)}, k_{(1)})\omega(h_{(2)}k_{(2)}, l).$$

Putting $h = 1_H$ in the expression above , we have

$$\begin{aligned}
&(1_H \cdot \omega(k_{(1)}, l_{(1)}))\omega(1_H, k_{(2)}l_{(2)}) = \omega(1_H, k_{(1)})\omega(1_H k_{(2)}, l)\\
\Rightarrow\ &\omega(k_{(1)}, l_{(1)})\omega(1_H, k_{(2)}l_{(2)}) = \omega(1_H, k_{(1)})\omega(k_{(2)}, l)\\
\Rightarrow\ &\overline{\omega}(k_{(1)},l_{(1)})\omega(k_{(2)},l_{(2)})\omega(1_H,k_{(3)}l_{(3)}) = \omega(1_H,k_{(1)})\omega(k_{(2)},l_{(1)})\overline{\omega}(k_{(3)},l_{(2)})\\
\Rightarrow\ &k_{(1)} \cdot (l_{(1)} \cdot 1_A)\omega(1_H, k_{(2)}l_{(2)}) = \omega(1_H, k_{(1)})(k_{(2)} \cdot (l_{(1)} \cdot 1_A))\\
\overset{k=1_H}{\Rightarrow}&(l_{(1)} \cdot 1_A)\omega(1_H, l_{(2)}) = \omega(1_H, 1_H) \cdot (l \cdot 1_A).
\end{aligned}$$

As $\omega^{-1}(1_H,1_H)\omega(1_H,1_H) = 1_H \cdot 1_A = 1_A$, we conclude that $\omega(1_H,1_H) \in A^\times$. Then, one can define $\widetilde{\omega}(h,k) = \omega(h,k)(\omega(1_H,1_H))^{-1}$. It is easy to see that $\widetilde{\omega}(1_H, l) = (l \cdot 1_A)$.

On the other hand, putting $l = 1_H$ in the 2-cocycle condition, we have

$$
\begin{aligned}
& (h_{(1)} \cdot \omega(k_{(1)}, 1_H))\omega(h_{(2)}, k_{(2)}) = \omega(h_{(1)}, k_{(1)})\omega(h_{(2)}k_{(2)}, 1_H) \\
\Rightarrow\ & h_{(1)} \cdot \omega(k_{(1)}, 1_H))\omega(h_{(2)}, k_{(2)})\overline{\omega}(h_{(3)}, k_{(3)}) = \\
& \qquad = \overline{\omega}(h_{(1)}, k_{(1)})\omega(h_{(2)}, k_{(2)})\omega(h_{(3)}k_{(3)}, 1_H) \\
\Rightarrow\ & (h_{(1)} \cdot \omega(k_{(1)}, 1_H))(h_{(2)} \cdot (k_{(2)} \cdot 1_A)) = (h_{(1)} \cdot (k_{(1)} \cdot 1_A))\omega(h_{(2)}k_{(2)}, 1_H) \\
\Rightarrow\ & (h_{(1)} \cdot \omega(k_{(1)}, 1_H)(k_{(2)} \cdot 1_A)) = (h_{(1)} \cdot 1_A)(h_{(2)}k_{(1)} \cdot 1_A)\omega(h_{(3)}k_{(2)}, 1_H) \\
\Rightarrow\ & (h \cdot \omega(k, 1_H) = (h_{(1)} \cdot 1_A)\omega(h_{(2)}k, 1_H) \\
\overset{k=1_H}{\Rightarrow}\ & h \cdot \omega(1_H, 1_H) = (h_{(1)} \cdot 1_A)\omega(h_{(2)}, 1_H) \\
\Rightarrow\ & h \cdot \omega(1_H, 1_H) = \omega(h, 1_H).
\end{aligned}
$$

Therefore, $h \cdot \widetilde{\omega}(1_H, 1_H) = h \cdot (1_H \cdot 1_A) = h \cdot 1_A = \widetilde{\omega}(h, 1_H)$.

Finally, let us verify that $\widetilde{\omega}$ is cohomologous to ω, that is, there exists $\phi \in C^1_{par}(H, A)$ such that $\widetilde{\omega} * \omega^{-1} = \delta_1\phi$. Indeed, on the one hand, note that

$$\widetilde{\omega}(h_{(1)}, k_{(1)})\omega^{-1}(h_{(2)}, k_{(2)}) = (h \cdot (k \cdot 1_A))(\omega(1_H, 1_H))^{-1}.$$

On the other hand,

$$\delta\phi(h, k) = (h_{(1)} \cdot \phi(k_{(1)}))\phi^{-1}(h_{(2)}k_{(2)})\phi(h_{(3)}).$$

Then, if we define $\phi(k) = (k \cdot 1_A)(\omega(1_H, 1_H))^{-1}$, we have

$$
\begin{aligned}
\delta\phi(h, k) &= h_{(1)} \cdot ((k_{(1)} \cdot 1_A)(\omega(1_H, 1_H))^{-1})(h_{(2)}k_{(2)} \cdot 1_A)(\omega(1_H, 1_H)) \\
& \qquad (h_{(3)} \cdot 1_A)(\omega(1_H, 1_H))^{-1} \\
&= (h_{(1)} \cdot (k_{(1)} \cdot 1_A))(\omega(1_H, 1_H))^{-1}(h_{(2)}k_{(2)} \cdot 1_A)(h_{(3)} \cdot 1_A) \\
&= (h_{(1)} \cdot (k_{(1)} \cdot 1_A))(\omega(1_H, 1_H))^{-1}.
\end{aligned}
$$

This concludes our proof.

■

5.2 The Hopf algebroid structure of the partial crossed product

We saw that the cohomology theory for a cocommutative Hopf algebra H acting partially over a commutative algebra A is equivalent to the cohomology theory of the same Hopf algebra H acting on a commutative and cocommutative Hopf algebra $\widetilde{A}$ whose base ring is the commutative algebra $E(A)$. This replacement gives us a deeper understanding about the structure of crossed products. In fact, we shall see that the crossed product has a structure of a Hopf algebroid over the base algebra $E(A)$. Let us recall briefly the definition of a Hopf algebroid, for a detailed presentation, see the reference [11].

Definition 5.9 *[11] Given a k-algebra A, a left (resp. right) bialgebroid over A is given by the data* $(\mathcal{H}, A, s_l, t_l, \Delta_l, \varepsilon_l)$ *(resp.* $(\mathcal{H}, A, s_r, t_r, \Delta_r, \varepsilon_r)$*) such that:*

1. $\mathcal{H}$ *is a k-algebra.*

2. *The map* s_l *(resp.* s_r*) is a morphism of algebras between A and* $\mathcal{H}$*, and the map* t_l *(resp.* t_r*) is an anti-morphism of algebras between A and* $\mathcal{H}$*. Their images commute, that is, for every* $a, b \in A$ *we have* $s_l(a)t_l(b) = t_l(b)s_l(a)$ *(resp.* $s_r(a)t_r(b) = t_r(b)s_r(a)$*). By the maps* s_l, t_l *(resp.* s_r, t_r*) the algebra* $\mathcal{H}$ *inherits a structure of A-bimodule given by* $a \triangleright h \triangleleft b = s_l(a)t_l(b)h$ *(resp.* $a \blacktriangleright h \blacktriangleleft b = hs_r(b)t_r(a)$*).*

3. *The triple* $(\mathcal{H}, \Delta_l, \varepsilon_l)$ *(resp.* $(\mathcal{H}, \Delta_r, \varepsilon_r)$*) is an A-coring relative to the structure of A-bimodule defined by* s_l *and* t_l *(resp.* s_r*, and* t_r*).*

4. *The image of* Δ_l *(resp.* Δ_r*) lies in the Takeuchi subalgebra*

$$\mathcal{H}_A \times \mathcal{H} = \left\{ \sum_i h_i \otimes k_i \in \mathcal{H} \otimes_{A,\triangleright\triangleleft} \mathcal{H} \mid \sum_i h_i t_l(a) \otimes k_i = \sum_i h_i \otimes k_i s_l(a) \;\; \forall a \in A \right\},$$

respectively,

$$\mathcal{H} \times_A \mathcal{H} = \left\{ \sum_i h_i \otimes k_i \in \mathcal{H} \otimes_{A,\blacktriangleright\blacktriangleleft} \mathcal{H} \mid \sum_i s_r(a) h_i \otimes k_i = \sum_i h_i \otimes t_r(a) k_i \;\; \forall a \in A \right\},$$

and it is an algebra morphism.

5. *For every* $h, k \in \mathcal{H}$*, we have* $\varepsilon_l(hk) = \varepsilon_l(hs_l(\varepsilon_l(k))) = \varepsilon_l(ht_l(\varepsilon_l(k)))$*, respectively,* $\varepsilon_r(hk) = \varepsilon_r(s_r(\varepsilon_r(h))k) = \varepsilon_r(t_r(\varepsilon_r(h))k)$*.*

Given two anti-isomorphic algebras A_l and A_r (i.e., $A_l \cong A_r^{op}$), a left A_l-bialgebroid $(\mathcal{H}, A_l, s_l, t_l, \Delta_l, \varepsilon_l)$ and a right A_r-bialgebroid $(\mathcal{H}, A_r, s_r, t_r, \Delta_r, \varepsilon_r)$, a Hopf algebroid structure on $\mathcal{H}$ is given by an antipode, that is, an algebra anti-homomorphism $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ such that

(i) $s_l \circ \varepsilon_l \circ t_r = t_r$, $t_l \circ \varepsilon_l \circ s_r = s_r$, $s_r \circ \varepsilon_r \circ t_l = t_l$ *and* $t_r \circ \varepsilon_r \circ s_l = s_l$*;*

(ii) $(\Delta_l \otimes_{A_r} I) \circ \Delta_r = (I \otimes_{A_l} \Delta_r) \circ \Delta_l$ *and* $(I \otimes_{A_r} \Delta_l) \circ \Delta_r = (\Delta_r \otimes_{A_l} I) \circ \Delta_l$*;*

(iii) $\mathcal{S}(t_l(a) h t_r(b')) = s_r(b') \mathcal{S}(h) s_l(a)$, *for all* $a \in A_l$, $b' \in A_r$ *and* $h \in \mathcal{H}$*;*

(iv) $\mu_{\mathcal{H}} \circ (\mathcal{S} \otimes_{A_l} I) \circ \Delta_l = s_r \circ \varepsilon_r$ *and* $\mu_{\mathcal{H}} \circ (I \otimes_{A_r} S) \circ \Delta_r = s \circ \varepsilon_l$*.*

In our case, both algebras, A_l and A_r, coincide with the commutative algebra $E(A)$ and the crossed product $\underline{\widetilde{A} \#_\omega H}$ will play the role of the Hopf algebroid $\mathcal{H}$ of the previous definition.

Theorem 5.10 *Let H be a cocommutative Hopf algebra and A be a commutative partial H-module algebra. Consider the commutative and cocommutative Hopf algebra $\widetilde{A}$, constructed in Theorem 4.5, over the commutative algebra $E(A)$, which is also a partial H-module algebra. Then, the crossed product $\underline{\widetilde{A} \#_\omega H}$, in which ω is a partial 2-cocycle from $H^2_{par}(H, A) = H^2_{par}(H, \widetilde{A})$, is a Hopf algebroid over the base algebra $E(A)$.*

We note that in this demonstration we used an abuse of notations by writing $\omega(h, k)$ to understand $X_{\omega(h,k)}$ in the crossed product.

Proof: The source and target maps, both left and right, are defined by the restriction to $E(A)$ of the canonical inclusion of $\widetilde{A}$ into the crossed product

$$s_l, t_l, s_r, t_r : \begin{array}{ccc} E(A) & \longrightarrow & \underline{\widetilde{A} \#_\omega H} \\ a & \mapsto & \underline{a \# 1_H} \end{array}.$$

We have already seen that this inclusion is an algebra map and by the commutativity of $\widetilde{A}$, the images of source and target maps commute among themselves. Note that, even though the left and right sources and targets are equal, their associated bimodule structures are different nonetheless. Indeed, for $a, a' \in E(A)$ and $b \# h \in \underline{\widetilde{A} \#_\omega H}$ we have

$$a \triangleright (b \# h) \triangleleft a' = (a \# 1_H)(a' \# 1_H)(b \# h) = a a' b \# h,$$

and

$$a \blacktriangleright (b \# h) \blacktriangleleft a' = (b \# h)(a \# 1_H)(a' \# 1_H) = b(h_{(1)} \cdot a a') \# h_{(2)}.$$

The left and right comultiplication maps are defined, respectively, as

$$\begin{array}{rccc} \Delta_l : & \widetilde{A}\#_\omega H & \longrightarrow & \widetilde{A}\#_\omega H \otimes \widetilde{A}\#_\omega H \\ & a\#h & \mapsto & a_{(1)}\#h_{(1)} \otimes_{E(A),\triangleright\triangleleft} a_{(2)}\#h_{(2)} \end{array}$$

and

$$\begin{array}{rccc} \Delta_r : & \widetilde{A}\#H & \longrightarrow & \widetilde{A}\#H \otimes \widetilde{A}\#H \\ & a\#h & \mapsto & a_{(1)}\#h_{(1)} \otimes_{E(A),\blacktriangleright\blacktriangleleft} a_{(2)}\#h_{(2)} \end{array},$$

in which the tensor product $\otimes_{E(A),\triangleright\triangleleft}$ (resp. $\otimes_{E(A),\blacktriangleright\blacktriangleleft}$) is balanced with respect to the $E(A)$-bimodule structure implemented by s_l, t_l (resp. s_r, t_r). It is easy to see that

$$\begin{aligned} (\Delta_l \otimes_{E(A),\blacktriangleright\blacktriangleleft} I) \circ \Delta_r &= (I \otimes_{E(A),\triangleright\triangleleft} \Delta_r) \circ \Delta_l \\ (I \otimes_{E(A),\blacktriangleright\blacktriangleleft} \Delta_l) \circ \Delta_r &= (\Delta_r \otimes_{E(A),\triangleright\triangleleft} I) \circ \Delta_l. \end{aligned}$$

One can see also that, for any $a \in E(A)$,

$$\Delta_l(s_l(a)) = s_l(a) \otimes (1_A\#1_H), \quad \text{and} \quad \Delta_l(t_l(a)) = (1_A\#1_H) \otimes t_l(a).$$

This is because any element of $E(A)$ is a linear combination of monomials of the form $(h^1 \cdot 1_A)\ldots(h^n \cdot 1_A)$, for $h^1, \ldots, h^n \in H$, and then

$$\begin{aligned} &\Delta_l(s_l((h^1 \cdot 1_A)\ldots(h^n \cdot 1_A))) = \\ &= (h^1_{(1)} \cdot 1_A)\ldots(h^n_{(1)} \cdot 1_A)\#1_H \otimes (h^1_{(2)} \cdot 1_A)\ldots(h^n_{(2)} \cdot 1_A)\#1_H \\ &= (h^1_{(1)} \cdot 1_A)\ldots(h^n_{(1)} \cdot 1_A)\#1_H \otimes ((h^1_{(2)} \cdot 1_A)\ldots(h^n_{(2)} \cdot 1_A)\#1_H)(1_A\#1_H) \\ &= (h^1_{(1)} \cdot 1_A)\ldots(h^n_{(1)} \cdot 1_A)\#1_H \otimes s_l((h^1_{(2)} \cdot 1_A)\ldots(h^n_{(2)} \cdot 1_A))(1_A\#1_H) \\ &= (h^1_{(1)} \cdot 1_A)\ldots(h^n_{(1)} \cdot 1_A)\#1_H \otimes ((h^1_{(2)} \cdot 1_A)\ldots(h^n_{(2)} \cdot 1_A)) \triangleright (1_A\#1_H) \\ &= ((h^1_{(1)} \cdot 1_A)\ldots(h^n_{(1)} \cdot 1_A)\#1_H) \triangleleft ((h^1_{(2)} \cdot 1_A)\ldots(h^n_{(2)} \cdot 1_A)) \otimes 1_A\#1_H \\ &= t_l((h^1_{(2)} \cdot 1_A)\ldots(h^n_{(2)} \cdot 1_A))((h^1_{(1)} \cdot 1_A)\ldots(h^n_{(1)} \cdot 1_A)\#1_H) \otimes 1_A\#1_H \\ &= ((h^1_{(2)} \cdot 1_A)\ldots(h^n_{(2)} \cdot 1_A)\#1_H)((h^1_{(1)} \cdot 1_A)\ldots(h^n_{(1)} \cdot 1_A)\#1_H) \otimes 1_A\#1_H \\ &= ((h^1_{(1)} \cdot 1_A)\ldots(h^n_{(1)} \cdot 1_A)(h^1_{(2)} \cdot 1_A)\ldots(h^n_{(2)} \cdot 1_A)\#1_H) \otimes 1_A\#1_H \\ &= (h^1 \cdot 1_A)\ldots(h^n \cdot 1_A)\#1_H \otimes 1_A\#1_H \\ &= s_l((h^1 \cdot 1_A)\ldots(h^n \cdot 1_A)) \otimes 1_A\#1_H. \end{aligned}$$

From this we deduce that Δ_l is a morphism of $E(A)$-bimodules with the bimodule structure given by $\triangleright$ and $\triangleleft$.

The same occurs for Δ_r, that is, for any $a \in E(A)$, we have

$$\Delta_r(t_r(a)) = t_r(a) \otimes (1_A\#1_H), \quad \text{and} \quad \Delta_r(s_r(a)) = (1_A\#1_H) \otimes s_r(a),$$

and then Δ_r is a morphism of $E(A)$-bimodules, with the bimodule structure given by $\blacktriangleright$ and $\blacktriangleleft$.

The image of the left comultiplication Δ_l lies in the left Takeuchi product $\underline{\widetilde{A}\#_\omega H}_{E(A)} \times \underline{\widetilde{A}\#_\omega H}$. Indeed, for $a\#h \in \underline{\widetilde{A}\#_\omega H}$ and $b \in E(A)$, we have

$$\begin{aligned}
&(a_{(1)}\#h_{(1)})t_l(b) \otimes a_{(2)}\#h_{(2)} = (a_{(1)}\#h_{(1)})(b\#1_H) \otimes a_{(2)}\#h_{(2)} \\
&= a_{(1)}(h_{(1)} \cdot b)\omega(h_{(2)}, 1_H)\#h_{(3)} \otimes a_{(2)}\#h_{(4)} \\
&= a_{(1)}(h_{(1)} \cdot b)(h_{(2)} \cdot 1_A)\#h_{(3)} \otimes a_{(2)}\#h_{(4)} \\
&= (h_{(1)} \cdot b)a_{(1)}\#h_{(2)} \otimes a_{(2)}\#h_{(3)} \\
&= ((h_{(1)} \cdot b)\#1_H)(a_{(1)}\#h_{(2)}) \otimes a_{(2)}\#h_{(3)} \\
&= t_l(h_{(1)} \cdot b)(a_{(1)}\#h_{(2)}) \otimes a_{(2)}\#h_{(3)} \\
&= a_{(1)}\#h_{(2)} \otimes s_l(h_{(1)} \cdot b)(a_{(2)}\#h_{(3)}) \\
&= a_{(1)}\#h_{(2)} \otimes ((h_{(1)} \cdot b)\#1_H)(a_{(2)}\#h_{(3)}) \\
&= a_{(1)}\#h_{(1)} \otimes a_{(2)}(h_{(2)} \cdot b)\#h_{(3)} \\
&= a_{(1)}\#h_{(1)} \otimes (a_{(2)}\#h_{(2)})(b\#1_H) \\
&= a_{(1)}\#h_{(1)} \otimes (a_{(2)}\#h_{(2)})s_l(b).
\end{aligned}$$

Moreover, Δ_l is a morphism of algebras. Take any $a\#h, b\#l \in \underline{\widetilde{A}\#_\omega H}$, then

$$\begin{aligned}
&\Delta_l((a\#h)(b\#l)) = \Delta_l(a(h_{(1)} \cdot b)\omega(h_{(2)}, l_{(1)})\#h_{(3)}l_{(2)}) \\
&=(a(h_{(1)}\cdot b)\omega(h_{(2)},l_{(1)}))_{(1)}\#(h_{(3)}l_{(2)})_{(1)}\otimes(a(h_{(1)}\cdot b)\omega(h_{(2)},l_{(1)}))_{(2)}\#(h_{(3)}l_{(2)})_{(2)} \\
&=a_{(1)}(h_{(1)}\cdot b_{(1)})\omega(h_{(3)}, l_{(1)})\#h_{(5)}l_{(3)}\otimes a_{(2)}(h_{(2)}\cdot b_{(2)})\omega(h_{(4)}, l_{(2)})\#h_{(6)}l_{(4)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\Delta_l(a\#h)\Delta_l(b\#l) = (a_{(1)}\#h_{(1)} \otimes a_{(2)}\#h_{(2)})(b_{(1)}\#l_{(1)} \otimes b_{(2)}\#l_{(2)}) \\
&=(a_{(1)}\#h_{(1)})(b_{(1)}\#l_{(1)}) \otimes (a_{(2)}\#h_{(2)})(b_{(2)}\#l_{(2)}) \\
&=a_{(1)}(h_{(1)}\cdot b_{(1)})\omega(h_{(2)}, l_{(1)})\#h_{(3)}l_{(2)}\otimes a_{(2)}(h_{(4)}\cdot b_{(2)})\omega(h_{(5)}, l_{(3)})\#h_{(6)}l_{(4)}.
\end{aligned}$$

The equality follows from the co-commutativity of H.

Analogously, one can prove that the image of the right comultiplication lies in the right Takeuchi product $\underline{\widetilde{A}\#_\omega H} \times_{E(A)} \underline{\widetilde{A}\#_\omega H}$, and it is an algebra morphism.

The left and right counits are defined, respectively, as

$$\begin{array}{rccl}
\varepsilon_l : & \widetilde{A}\#_\omega H & \longrightarrow & E(A) \\
& a\#h & \mapsto & \varepsilon_l(a\#h) := \varepsilon_{\widetilde{A}}(a)(h \cdot 1_A)
\end{array}$$

and

$$\begin{array}{rccl} \varepsilon_r : & \underline{\widetilde{A}\#_\omega H} & \longrightarrow & E(A) \\ & a\#h & \mapsto & \varepsilon_r(a\#h) := S(h)\cdot\varepsilon_{\widetilde{A}}(a) \end{array} .$$

First, both are morphisms of $E(A)$-bimodules, with their respective structures. Take $a, a' \in E(A)$ and $b\#h \in \underline{\widetilde{A}\#_\omega H}$, then

$$\begin{aligned} \varepsilon_l(a \triangleright (b\#h) \triangleleft a') &= \varepsilon_l((aa'b\#h)) = \varepsilon_{\widetilde{A}}(aa')\varepsilon_{\widetilde{A}}(b)(h\cdot 1_A) \\ &= a\varepsilon_{\widetilde{A}}(b)(h\cdot 1_A)a' = a\varepsilon_l(b\#h)a', \end{aligned}$$

and

$$\begin{aligned} &\varepsilon_l(a \blacktriangleright (b\#h) \blacktriangleleft a') = \varepsilon_l(b(h_{(1)}\cdot aa')\#h_{(2)}) \\ =&S(h_{(2)})\cdot(\varepsilon_{\widetilde{A}}(b)\varepsilon_{\widetilde{A}}(h_{(1)}\cdot aa')) = (S(h_{(3)})\cdot\varepsilon_{\widetilde{A}}(b))(S(h_{(2)})\cdot(h_{(1)}\cdot aa')) \\ =&(S(h_{(3)})\cdot\varepsilon_{\widetilde{A}}(b))(S(h_{(1)})h_{(2)}\cdot aa') = (S(h)\cdot b)aa' = a\varepsilon_r(b\#h)a'. \end{aligned}$$

One can easily verify the compatibility relations with the left and right counits and the left and right source and targets, that is, $s_l\circ\varepsilon_l\circ t_r = t_r$, $t_l\circ\varepsilon_l\circ s_r = s_r$, $s_r\circ\varepsilon_r\circ t_l = t_l$ and $t_r\circ\varepsilon_r\circ s_l = s_l$.

Let us verify that $(\underline{\widetilde{A}\#_\omega H}, \Delta_l, \varepsilon_l)$ and $(\underline{\widetilde{A}\#_\omega H}, \Delta_r, \varepsilon_r)$ are corings over $E(A)$ with their respective bimodule structures. We have already seen that Δ_l, ε_l, Δ_r and ε_r are morphisms of $E(A)$ bimodules. It is easy to see that the left and right comultiplications are coassociative. It remains to verify the counit axiom for both structures. Take $a\#h \in \underline{\widetilde{A}\#_\omega H}$, then

$$\begin{aligned} & (\varepsilon_l \otimes_{E(A),\triangleright\triangleleft} I)\circ\Delta_l(a\#h) = \varepsilon_l(a_{(1)}\#h_{(1)}) \triangleright (a_{(2)}\#h_{(2)}) \\ =\; & (\varepsilon_{\widetilde{A}}(a_{(1)})(h_{(1)}\cdot 1_A)\#1_H)(a_{(2)}\#h_{(2)}) \\ =\; & \varepsilon_{\widetilde{A}}(a_{(1)})(a_{(2)}(h_{(1)}\cdot 1_A)\#h_{(2)}) \\ =\; & a(h_{(1)}\cdot 1_A)\#h_{(2)}) \\ =\; & a\#h. \end{aligned}$$

Using the cocommutativity of H and the commutativity of A it is also easy to see that

$$(I \otimes_{E(A),\triangleright\triangleleft} \varepsilon_l)\circ\Delta_l(a\#h) = a\#h.$$

Therefore, $(\underline{\widetilde{A}\#_\omega H}, \Delta_l, \varepsilon_l)$ is a coring over $E(A)$. For the right structure, for $a\#h \in \underline{\widetilde{A}\#_\omega H}$, we have

$$
\begin{aligned}
&(\varepsilon_r \otimes_{E(A),\blacktriangleright\blacktriangleleft} I) \circ \Delta_r(a\#h) = \varepsilon_r(a_{(1)}\#h_{(1)}) \blacktriangleright (a_{(2)}\#h_{(2)})\\
&= (a_{(2)}\#h_{(2)})((S(h_{(1)}) \cdot \varepsilon_{\widetilde{A}}(a_{(1)}))\#1_H)\\
&= (a_{(2)}\#h_{(1)})((S(h_{(2)}) \cdot \varepsilon_{\widetilde{A}}(a_{(1)}))\#1_H)\\
&= a_{(2)}(h_{(1)} \cdot (S(h_{(4)}) \cdot \varepsilon_{\widetilde{A}}(a_{(1)})))\omega(h_{(2)}, 1_H)\#h_{(3)}\\
&= a_{(2)}(h_{(1)} \cdot (S(h_{(4)}) \cdot \varepsilon_{\widetilde{A}}(a_{(1)})))(h_{(2)} \cdot 1_A)\#h_{(3)}\\
&= a_{(2)}(h_{(1)}S(h_{(4)}) \cdot \varepsilon_{\widetilde{A}}(a_{(1)}))(h_{(2)} \cdot 1_A)\#h_{(3)}\\
&= a_{(2)}(h_{(1)}S(h_{(2)}) \cdot \varepsilon_{\widetilde{A}}(a_{(1)}))(h_{(3)} \cdot 1_A)\#h_{(4)}\\
&= a_{(2)}\varepsilon_{\widetilde{A}}(a_{(1)})(h_{(1)} \cdot 1_A)\#h_{(2)}\\
&= a(h_{(1)} \cdot 1_A)\#h_{(2)})\\
&= a\#h.
\end{aligned}
$$

Analogously, one can prove that

$$(I \otimes_{E(A),\blacktriangleright\blacktriangleleft} \varepsilon_r) \circ \Delta_r(a\#h) = a\#h.$$

Therefore, $(\underline{\widetilde{A}\#_\omega H}, \Delta_r, \varepsilon_r)$ is a coring over $E(A)$.

Let us verify now that

$$\varepsilon_l((a\#h)(b\#k)) = \varepsilon_l((a\#h)s_l(\varepsilon_l(b\#k))) = \varepsilon_l((a\#h)t_l(\varepsilon_l(b\#k))),$$

and

$$\varepsilon_r((a\#h)(b\#k)) = \varepsilon_r(s_r(\varepsilon_r(a\#h))(b\#k)) = \varepsilon_r(t_r(\varepsilon_r(a\#h))(b\#k)),$$

for any $a\#h, b\#k \in \underline{\widetilde{A}\#_\omega H}$. For the left counit, on the one hand,

$$
\begin{aligned}
&\varepsilon_l((a\#h)(b\#k)) = \varepsilon_l(a(h_{(1)} \cdot b)\omega(h_{(2)}, k_{(1)})\#h_{(3)}k_{(2)})\\
= \;&\varepsilon_{\widetilde{A}}(a)\varepsilon_{\widetilde{A}}(h_{(1)} \cdot b)\varepsilon_{\widetilde{A}}(\omega(h_{(2)}, k_{(1)}))(h_{(3)}k_{(2)} \cdot 1_A)\\
= \;&\varepsilon_{\widetilde{A}}(a)(h_{(1)} \cdot \varepsilon_{\widetilde{A}}(b))(h_{(2)} \cdot (k_{(1)} \cdot 1_A))(h_{(3)}k_{(2)} \cdot 1_A)\\
= \;&\varepsilon_{\widetilde{A}}(a)(h_{(1)} \cdot \varepsilon_{\widetilde{A}}(b))(h_{(2)} \cdot (k \cdot 1_A))\\
= \;&\varepsilon_{\widetilde{A}}(a)(h \cdot (\varepsilon_{\widetilde{A}}(b)(k \cdot 1_A))).
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&\varepsilon_l((a\#h)s_l(\varepsilon_l(b\#k))) = \varepsilon_l((a\#h)s_l(\varepsilon_{\widetilde{A}}(b)(k \cdot 1_A)))\\
= \;&\varepsilon_l((a\#h)(\varepsilon_{\widetilde{A}}(b)(k \cdot 1_A)\#1_H))\\
= \;&\varepsilon_l(a(h_{(1)} \cdot (\varepsilon_{\widetilde{A}}(b)(k \cdot 1_A)))\omega(h_{(2)}, 1_H)\#h_{(3)})\\
= \;&\varepsilon_l(a(h_{(1)} \cdot (\varepsilon_{\widetilde{A}}(b)(k \cdot 1_A)))(h_{(2)} \cdot 1_H)\#h_{(3)})\\
= \;&\varepsilon_{\widetilde{A}}(a(h_{(1)} \cdot (\varepsilon_{\widetilde{A}}(b)(k \cdot 1_A))))(h_{(2)} \cdot 1_A)\\
= \;&\varepsilon_{\widetilde{A}}(a)(h_{(1)} \cdot (\varepsilon_{\widetilde{A}}(b)(k \cdot 1_A)))(h_{(2)} \cdot 1_A)\\
= \;&\varepsilon_{\widetilde{A}}(a)(h \cdot (\varepsilon_{\widetilde{A}}(b)(k \cdot 1_A))).
\end{aligned}
$$

Therefore, $\varepsilon_l((a\#h)(b\#k))=\varepsilon_l((a\#h)s_l(\varepsilon_l(b\#k)))=\varepsilon_l((a\#h)t_l(\varepsilon_l(b\#k)))$.
For the right counit, on the one hand

$$\begin{aligned}
&\varepsilon_r((a\#h)(b\#k)) = \varepsilon_r(a(h_{(1)}\cdot b)\omega(h_{(2)},k_{(1)})\#h_{(3)}k_{(2)})\\
&= S(h_{(3)}k_{(2)})\cdot \varepsilon_{\widetilde{A}}(a(h_{(1)}\cdot b)\omega(h_{(2)},k_{(1)}))\\
&= S(h_{(3)}k_{(2)})\cdot (\varepsilon_{\widetilde{A}}(a)(h_{(1)}\cdot \varepsilon_{\widetilde{A}}(b))(h_{(2)}\cdot (k_{(1)}\cdot 1_A))\\
&= (S(h_{(5)}k_{(4)})\cdot \varepsilon_{\widetilde{A}}(a))(S(h_{(4)}k_{(3)})\cdot (h_{(1)}\cdot \varepsilon_{\widetilde{A}}(b)))(S(h_{(3)}k_{(2)})\cdot (h_{(2)}\cdot (k_{(1)}\cdot 1_A)))\\
&= (S(h_{(5)}k_{(4)})\cdot \varepsilon_{\widetilde{A}}(a))(S(h_{(4)}k_{(3)})h_{(1)}\cdot \varepsilon_{\widetilde{A}}(b))(S(h_{(3)}k_{(2)})h_{(2)}\cdot (k_{(1)}\cdot 1_A))\\
&= (S(k_{(4)})S(h_{(5)})\cdot \varepsilon_{\widetilde{A}}(a))(S(k_{(3)})S(h_{(4)})h_{(1)}\cdot \varepsilon_{\widetilde{A}}(b))\\
&\qquad\qquad (S(k_{(2)})S(h_{(3)})h_{(2)}\cdot (k_{(1)}\cdot 1_A))\\
&= (S(k_{(4)})S(h_{(5)})\cdot \varepsilon_{\widetilde{A}}(a))(S(k_{(3)})S(h_{(3)})h_{(4)}\cdot \varepsilon_{\widetilde{A}}(b))\\
&\qquad\qquad (S(k_{(2)})S(h_{(1)})h_{(2)}\cdot (k_{(1)}\cdot 1_A))\\
&= (S(k_{(4)})S(h)\cdot \varepsilon_{\widetilde{A}}(a))(S(k_{(3)})\cdot \varepsilon_{\widetilde{A}}(b))(S(k_{(2)})\cdot (k_{(1)}\cdot 1_A))\\
&= (S(k_{(4)})S(h)\cdot \varepsilon_{\widetilde{A}}(a))(S(k_{(3)})\cdot \varepsilon_{\widetilde{A}}(b))(S(k_{(2)})k_{(1)}\cdot 1_A)\\
&= (S(k_{(4)})S(h)\cdot \varepsilon_{\widetilde{A}}(a))(S(k_{(3)})\cdot \varepsilon_{\widetilde{A}}(b))(S(k_{(1)})k_{(2)}\cdot 1_A)\\
&= (S(k_{(2)})S(h)\cdot \varepsilon_{\widetilde{A}}(a))(S(k_{(1)})\cdot \varepsilon_{\widetilde{A}}(b))\\
&= (S(k_{(2)})\cdot (S(h)\cdot \varepsilon_{\widetilde{A}}(a)))(S(k_{(1)})\cdot \varepsilon_{\widetilde{A}}(b))\\
&= S(k)\cdot ((S(h)\cdot \varepsilon_{\widetilde{A}}(a))\varepsilon_{\widetilde{A}}(b)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\varepsilon_r(s_r(\varepsilon_r(a\#h))(b\#k)) = \varepsilon_r((S(h)\cdot \varepsilon_{\widetilde{A}}(a))\#1_H)(b\#k))\\
= \;&\varepsilon_r((S(h)\cdot \varepsilon_{\widetilde{A}}(a))b\#k)\\
= \;&S(k)\cdot ((S(h)\cdot \varepsilon_{\widetilde{A}}(a))\varepsilon_{\widetilde{A}}(b)).
\end{aligned}$$

Therefore,

$$\varepsilon_r((a\#h)(b\#k)) = \varepsilon_r(s_r(\varepsilon_r(a\#h))(b\#k)) = \varepsilon_r(t_r(\varepsilon_r(a\#h))(b\#k)).$$

Finally, we define the antipode as

$$\begin{array}{rccc}
\mathcal{S}: & \underline{\widetilde{A}\#_\omega H} & \longrightarrow & \underline{\widetilde{A}\#_\omega H} \\
& a\#h & \mapsto & (S_H(h_{(3)})\cdot S_{\widetilde{A}}(a))\omega^{-1}(S_H(h_{(2)}),h_{(4)})\#S_H(h_{(1)})
\end{array}.$$

Take $b,c\in E(A)$ and $a\#h\in \underline{\widetilde{A}\#_\omega H}$, then, one can prove that

$$\mathcal{S}(t(b)(a\#h)t(c)) = s(c)\mathcal{S}(a\#h)s(b).$$

Indeed,

$$\begin{aligned}
&\mathcal{S}(t_l(b)(a\#h)t_r(c)) = \mathcal{S}((b\#1_H)(a\#h)(c\#1_H))\\
= \;& \mathcal{S}((ba\#h)(c\#1_H))\\
= \;& \mathcal{S}(ba(h_{(1)}\cdot c)\omega(h_{(2)},1_H)\#h_{(3)})\\
= \;& \mathcal{S}(ba(h_{(1)}\cdot c)\#h_{(2)})\\
= \;& (S(h_{(4)})\cdot S_{\widetilde{A}}(ba(h_{(1)}\cdot c)))\omega^{-1}(S(h_{(3)}),h_{(5)})\#S(h_{(2)})\\
\overset{(*)}{=} \;& (S(h_{(4)})\cdot((h_{(1)}\cdot c)S_{\widetilde{A}}(a)b))\omega^{-1}(S(h_{(3)}),h_{(5)})\#S(h_{(2)})
\end{aligned}$$

in which the equality $(*)$ is valid because $S_{\widetilde{A}}$ restricted to the base algebra $E(A)$ is equal to the identity. On the other hand.

$$\begin{aligned}
&s_r(c)\mathcal{S}(a\#h)s_l(b) =\\
&= (c\#1_H)((S(h_{(3)})\cdot S_{\widetilde{A}}(a))\omega^{-1}(S(h_{(2)}),h_{(4)})\#S(h_{(1)}))(b\#1_H)\\
&= (c(S(h_{(4)})\cdot S_{\widetilde{A}}(a))\omega^{-1}(S(h_{(3)}),h_{(5)})\omega(1_H,S(h_{(2)}))\#S(h_{(1)}))(b\#1_H)\\
&= (c(S(h_{(3)})\cdot S_{\widetilde{A}}(a))\omega^{-1}(S(h_{(2)}),h_{(4)})\#S(h_{(1)}))(b\#1_H)\\
&= c(S(h_{(4)})\cdot S_{\widetilde{A}}(a))\omega^{-1}(S(h_{(3)}),h_{(5)})(S(h_{(2)})\cdot b)\#S(h_{(1)})\\
&= (S(h_{(6)})\cdot b)(S(h_{(5)})\cdot S_{\widetilde{A}}(a))(S(h_{(3)})h_{(4)}\cdot c)\omega^{-1}(S(h_{(2)}),h_{(7)})\#S(h_{(1)})\\
&= (S(h_{(6)})\cdot b)(S(h_{(5)})\cdot S_{\widetilde{A}}(a))(S(h_{(3)})\cdot(h_{(4)}\cdot c))\omega^{-1}(S(h_{(2)}),h_{(7)})\#S(h_{(1)})\\
&= (S(h_{(6)})\cdot b)(S(h_{(5)})\cdot S_{\widetilde{A}}(a))(S(h_{(4)})\cdot(h_{(1)}\cdot c))\omega^{-1}(S(h_{(3)}),h_{(7)})\#S(h_{(2)})\\
&= (S(h_{(4)})\cdot((h_{(1)}\cdot c)S_{\widetilde{A}}(a)b))\omega^{-1}(S(h_{(3)}),h_{(5)})\#S(h_{(2)}).
\end{aligned}$$

It remains to check that

$$\mu(\mathcal{S}\otimes_{E(A),\triangleright\triangleleft} Id)\circ\Delta_l = s_r\circ\varepsilon_r$$

and

$$\mu(Id\otimes_{E(A),\blacktriangleright\blacktriangleleft}\mathcal{S})\circ\Delta_r = s_l\circ\varepsilon_l.$$

Take $a\#h\in\underline{\widetilde{A}\#_\omega H}$, then

$$\begin{aligned}
&\mu(\mathcal{S}\otimes_{E(A),\triangleright\triangleleft} Id)\circ\Delta_l(a\#h) = \mathcal{S}(a_{(1)}\#h_{(1)})(a_{(2)}\#h_{(2)})\\
&= (S(h_{(3)})\cdot S_{\widetilde{A}}(a_{(1)})\omega^{-1}(S(h_{(2)}),h_{(4)})\#S(h_{(1)}))(a_{(2)}\#h_{(5)})\\
&= (S(h_{(5)})\cdot S_{\widetilde{A}}(a_{(1)}))\omega^{-1}(S(h_{(4)}),h_{(6)})(S(h_{(3)})\cdot a_{(2)})\omega(S(h_{(2)}),h_{(7)})\#S(h_{(1)})h_{(8)}\\
&= (S(h_{(4)})\cdot S_{\widetilde{A}}(a_{(1)}))(S(h_{(3)})\cdot a_{(2)})\omega^{-1}(S(h_{(2)}),h_{(5)})\omega(S(h_{(1)}),h_{(6)})\#S(h_{(7)})h_{(8)}
\end{aligned}$$

$$
\begin{aligned}
&= (S(h_{(3)}) \cdot S_{\widetilde{A}}(a_{(1)}))(S(h_{(2)}) \cdot a_{(2)})(S(h_{(1)}) \cdot (h_{(4)} \cdot 1_A))\#1_H \\
&= (S(h_{(3)}) \cdot S_{\widetilde{A}}(a_{(1)}))(S(h_{(2)}) \cdot a_{(2)})(S(h_{(1)})h_{(4)} \cdot 1_A)\#1_H \\
&= (S(h_{(2)}) \cdot S_{\widetilde{A}}(a_{(1)}))(S(h_{(1)}) \cdot a_{(2)})\#1_H \\
&= (S(h) \cdot (S_{\widetilde{A}}(a_{(1)})a_{(2)}))\#1_H \\
&= s_r(S(h) \cdot \varepsilon_{\widetilde{A}}(a)) \\
&= s \circ \varepsilon_r(a\#h).
\end{aligned}
$$

and

$$
\begin{aligned}
&\mu(Id \otimes_{E(A),\blacktriangleright\blacktriangleleft} \mathcal{S}) \circ \Delta_r(a\#h) = (a_{(1)}\#h_{(1)})\mathcal{S}(a_{(2)}\#h_{(2)}) \\
=&(a_{(1)}\#h_{(1)})(S(h_{(4)}) \cdot S_{\widetilde{A}}(a_{(2)})\omega^{-1}(S(h_{(3)}), h_{(5)})\#S(h_{(2)})) \\
=&(a_{(1)}(h_{(1)} \cdot (S(h_{(7)}) \cdot S_{\widetilde{A}}(a_{(2)})\omega^{-1}(S(h_{(6)}), h_{(8)})))\omega(h_{(2)}, S(h_{(5)}))\#h_{(3)}S(h_{(4)}) \\
=&(a_{(1)}(h_{(1)} \cdot (S(h_{(6)}) \cdot S_{\widetilde{A}}(a_{(2)})))(h_{(2)} \cdot \omega^{-1}(S(h_{(5)}), h_{(7)}))\omega(h_{(3)}, S(h_{(4)}))\#1_H \\
=&(a_{(1)}(h_{(1)} \cdot (S(h_{(2)}) \cdot S_{\widetilde{A}}(a_{(2)})))(h_{(3)} \cdot \omega^{-1}(S(h_{(6)}), h_{(7)}))\omega(h_{(4)}, S(h_{(5)}))\#1_H \\
=&(a_{(1)}(h_{(1)}S(h_{(2)}) \cdot S_{\widetilde{A}}(a_{(2)})))(h_{(3)} \cdot \omega^{-1}(S(h_{(6)}), h_{(7)}))\omega(h_{(4)}, S(h_{(5)}))\#1_H \\
=&a_{(1)}S_{\widetilde{A}}(a_{(2)})(h_{(1)} \cdot \omega^{-1}(S(h_{(4)}), h_{(5)}))\omega(h_{(2)}, S(h_{(3)}))\#1_H \\
\overset{(*)}{=}&\varepsilon_{\widetilde{A}}(a)\omega^{-1}(h_{(1)}S(h_{(8)}), h_{(9)})\omega(h_{(2)}, S(h_{(7)})h_{(10)})\omega^{-1}(h_{(3)}, S(h_{(6)}))\omega(h_{(4)}, S(h_{(5)}))\#1_H \\
=&\varepsilon_{\widetilde{A}}(a)\omega^{-1}(h_{(1)}S(h_{(6)}), h_{(7)})\omega(h_{(2)}, S(h_{(5)})h_{(8)})(h_{(3)} \cdot (S(h_{(4)}) \cdot 1_A))\#1_H \\
=&\varepsilon_{\widetilde{A}}(a)\omega^{-1}(h_{(1)}S(h_{(5)}), h_{(6)})\omega(h_{(2)}, S(h_{(4)})h_{(7)})(h_{(3)} \cdot 1_A)\#1_H \\
=&\varepsilon_{\widetilde{A}}(a)\omega^{-1}(h_{(1)}S(h_{(2)}), h_{(7)})\omega(h_{(3)}, S(h_{(5)})h_{(6)})(h_{(4)} \cdot 1_A)\#1_H \\
=&\varepsilon_{\widetilde{A}}(a)\omega^{-1}(1_H, h_{(3)})\omega(h_{(1)}, 1_H)(h_{(2)} \cdot 1_A)\#1_H \\
=&\varepsilon_{\widetilde{A}}(a)(h \cdot 1_A)\#1_H = s_l \circ \varepsilon_l(a\#h).
\end{aligned}
$$

The equality $(*)$ we used that, for any $h, k, l \in H$, we have

$$h \cdot \omega^{-1}(k,l) = \omega^{-1}(h_{(1)}k_{(1)}, l_{(1)})\omega(h_{(2)}, k_{(2)}l_{(2)})\omega^{-1}(h_{(3)}, k_{(3)}).$$

This follows easily from

$$h \cdot (\omega(k_{(1)}, l_{(1)})\omega^{-1}(k_{(2)}, l_{(2)}) = h \cdot (k \cdot (l \cdot 1_A)),$$

and

$$h \cdot \omega(k,l) = \omega(h_{(1)}, k_{(1)})\omega^{-1}(h_{(2)}, k_{(2)}l_{(1)})\omega(h_{(3)}k_{(3)}, l_{(2)}),$$

which, in its turn is an immediate consequence of the 2-cocycle identity.

Therefore, $(\underline{\widetilde{A}\#_\omega H}, s_l, t_l, s_r, t_r, \Delta_l, \Delta_r, \varepsilon_l, \varepsilon_r, \mathcal{S})$ is a structure of a Hopf algebroid over $E(A)$.

■

Chapter 6

Partially cleft extensions and cleft extensions by Hopf algebroids

In [5], the authors introduced the notion of a partially cleft extension of an algebra A by a Hopf algebra H, and proved that partially cleft extensions are related with partial crossed product. In our case, which means H cocommutative and A commutative, the results developed in [5] remain valid, so, using the Hopf algebroid structure of the crossed product, one can rethink partially cleft extensions of commutative algebras by cocommutative Hopf algebras in a broader scenario, namely, the theory of cleft extensions of algebras by Hopf algebroids developed by G. Böhm and T. Brzezinski [12].

Definition 6.1 *[5] Let H be a Hopf algebra, and $A \subset B$ be an H-extension, that is B is a right H-comodule algebra and $A = B^{coH}$. The extension $A \subset B$ is partially cleft if there exists a pair of linear maps $\gamma, \overline{\gamma} : H \to B$ such that:*

(i) $\gamma(1_H) = 1_B$;

(ii) The following diagrams are commutative

$$\begin{array}{ccc} H & \xrightarrow{\gamma} & B \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \rho} \\ H \otimes H & \xrightarrow[\gamma \otimes Id_H]{} & B \otimes H \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{\overline{\gamma}} & B \\ \downarrow{\scriptstyle \Delta^{cop}} & & \downarrow{\scriptstyle \rho} \\ H \otimes H & \xrightarrow[\overline{\gamma} \otimes S]{} & B \otimes H \end{array}$$

(iii) $(\gamma*\overline{\gamma})\circ\mu$ *is a central element in the convolution algebra* $Hom_k(H\otimes H, B)$*, in which* $\mu : H\otimes H \to H$ *is the multiplication in* H *and* $(\overline{\gamma}*\gamma)(h)$ *commutes with every element of* A*, for each* $h \in H$*, and, for all* $b\in B$*,* $h, l \in H$*, if we write* $e_h = (\gamma*\overline{\gamma})(h)$ *and* $\widetilde{e}_h = (\overline{\gamma}*\gamma)(h)$*, then:*

(iv) $\sum b_{(0)}\overline{\gamma}(b_{(1)})\gamma(b_{(2)}) = b$;

(v) $\gamma(h)e_l = \sum e_{h_{(1)}l}\gamma(h_{(2)})$;

(vi) $\overline{\gamma}(l)\widetilde{e}_h = \sum \widetilde{e}_{hl_{(1)}}\overline{\gamma}(l_{(2)})$;

(vii) $\sum \gamma(hl_{(1)})\widetilde{e}_{l_{(2)}} = \sum e_{h_{(1)}}\gamma(h_{(2)}l)$.

Partially cleft extensions are related to partial crossed products, as one can see in the next two results.

Proposition 6.2 *[5] If* $(A, \cdot, (\omega, \omega^{-1}))$ *is a symmetric twisted partial* H*-module algebra with a 2-cocycle* ω*, then,* $A \subset \underline{A\#_\omega H}$ *is a partially cleft* H*-extension.*

□

For the crossed product $\underline{A\#_\omega H}$, the cleaving maps $\gamma, \overline{\gamma} : H \to \underline{A\#_\omega H}$ are given by

$$\gamma(h) = 1_A\#h, \quad \text{and} \quad \overline{\gamma}(h) = \omega^{-1}(S(h_{(2)}), h_{(3)})\#S(h_{(1)}). \qquad (6.1)$$

Theorem 6.3 *[5] Let* B *be* H*-comodule algebra and* $A = B^{coH}$*. Then the* H*-extension* $A \subset B$ *is partially cleft if, and only if there is a symmetric twisted partial action* $\cdot : H\otimes A \to A$ *with a 2-cocycle* $\omega : H\otimes H \to A$ *such that* B *is isomorphic to the partial crossed product* $\underline{A\#_\omega H}$*.*

□

In the case of a cocommutative Hopf algebra H acting partially over a commutative algebra A, we have already seen that we can replace A by a commutative and cocommutative Hopf algebra $\widetilde{A}$ with the same cohomology theory. Then, we observe that $Hom(H\otimes H, \widetilde{A})$ is a commutative algebra, so, (iii) in Definition 6.1 becomes trivial and $\widetilde{e}_h = e_h$. By (iv), $b^{(0)}e_{b^{(1)}} = b$ and we conclude that $\overline{\gamma} = \gamma^{-1}$ in the ideal $\langle e\rangle$.

Moreover, the crossed product $\underline{\widetilde{A}\#_\omega H}$ has a structure of Hopf algebroid over the base algebra $E(A)$. Then it is interesting to see whether one can replace the crossed product $\underline{A\#_\omega H}$ by the crossed product $\underline{\widetilde{A}\#_\omega H}$ in the analysis of cleft extensions by the Hopf algebra H. First note that the H-comodule structure on both crossed products is the same, namely $\rho(a\#h) = a\#h_{(1)} \otimes h_{(2)}$. Furthermore $(\underline{A\#_\omega H})^{coH} \cong A$ and $(\underline{\widetilde{A}\#_\omega H})^{coH} \cong \widetilde{A}$, then both crossed products are H-extensions of their respectives algebras of coinvariants. Finally, the cleaving maps $\gamma, \overline{\gamma} : H \to \underline{A\#_\omega H}$, given by (6.1), take their values actually in $\underline{\widetilde{A}\#_\omega H}$.

In what follows, we shall see that there exists a Hopf algebroid $\mathcal{H}$ such that the crossed product $\underline{\widetilde{A}\#_\omega H}$ can be viewed as a cleft extension of $\widetilde{A}$ by $\mathcal{H}$. For this purpose, it is important introduce some results about the theory of cleft extensions for Hopf algebroids developed by G. Böhn and T. Brzezinski in [12].

Definition 6.4 *[12] Let $(\mathcal{H}, L, R, s_L, t_L, s_R, t_R, \Delta_L, \Delta_R, \epsilon_L, \epsilon_R, \mathcal{S})$ be a Hopf algebroid and A be a right $\mathcal{H}$-comodule algebra. Denote by $\eta_R(r) = r \cdot 1_A = 1_A \cdot r$ the unit map of the corresponding R-ring structure of A. Let B be the subalgebra of $\mathcal{H}_R$-coinvariants in A. The extension $B \subset A$ is called $\mathcal{H}$-cleft if*

(a) A is an L-ring (with unit $\eta_L : L \to A$) and B is an L-subring of A;

(b) there exists a convolution invertible left L-linear right $\mathcal{H}$-colinear morphism $\gamma : \mathcal{H} \to A$.

A map γ satisfying condition (b) is called a cleaving map.

Remark 6.5 *Some small remarks have to be made about this definition.*

(1) First is that the structure of right $\mathcal{H}$-comodule algebra on A is related to the base ring R, that is $\rho : A \to A \otimes_R \mathcal{H}$ is a right $R - R$-bilinear map in the sense that, for every $a \in A$ and $r \in R$, $\rho(\eta_R(r) a \eta_R(s)) = a^{(0)} \otimes_R s_R(r) a^{(1)} s_R(s)$.

(2) The map $\gamma : \mathcal{H} \to A$ being $\mathcal{H}$-colinear implies that it is right R-linear in the sense that $\gamma(h s_R(r)) = \gamma(h)\eta_R(r)$ and left R-linear in the sense that $\gamma(s_R(r)h) = \eta_R(r)\gamma(h)$, for any $a \in A$ and $r \in R$.

(3) The notion of a convolution invertible map $\gamma : \mathcal{H} \to A$, in which A is both L-ring and R-ring, means that there is a unique $\overline{\gamma} : \mathcal{H} \to A$ such that [12]

$$\begin{aligned} \mu_A \circ (\gamma \otimes_R \overline{\gamma}) \circ \Delta_R &= \eta_L \circ \epsilon_L \\ \mu_A \circ (\overline{\gamma} \otimes_L \gamma) \circ \Delta_L &= \eta_R \circ \epsilon_R. \end{aligned}$$

(4) The left L-linearity of the map γ in condition (b) of Definition 6.4 means, that the cleaving map satisfies $\gamma(s_L(l)h) = \eta_L(l)\gamma(h)$, for any $a \in A$ and $l \in L$.

The lemmas below show us the behaviour of the convolution inverse of a cleaving map ($\overline{\gamma}$) with respect to (1) and (2) of Remark 6.5. We suggest [12] for details of proofs.

Lemma 6.6 *[12] Let $\mathcal{H}$ be a Hopf algebroid and $B \subseteq A$ an $\mathcal{H}$-cleft extension, with a cleaving map γ. Then,*

$$\overline{\gamma}(t_R(r)h) = \overline{\gamma}(h)\eta_R(r), \qquad \text{for all } r \in R, h \in H. \tag{6.2}$$

□

In the case of a Hopf algebra cleft extensions, the convolution inverse of a cleaving map is a right colinear map, where the right coaction is given by the coproduct followed by the antipode and a flip. So, since we have two coactions in the Hopf algebroid case, one for each constituent bialgebroid, the next Lemma follows:

Lemma 6.7 *[12] Let $\mathcal{H}$ be a Hopf algebroid and $B \subseteq A$ an $\mathcal{H}$-cleft extension with a cleaving map γ. Then, for all $h \in H$,*

$$\rho^A(\overline{\gamma}(h)) = \overline{\gamma}(h_{(2)}) \otimes_R S(h_{(1)}), \tag{6.3}$$

and

$$\lambda^A(\overline{\gamma}(h)) = \overline{\gamma}(h^{(2)}) \otimes_L S(h^{(1)}), \tag{6.4}$$

where ρ^A and γ^A are the right coactions of the constituent right and left bialgebroids, respectivelly.

□

To conclude our preliminaries, we recall that Doi and Takeuchi, in [15], characterized Cleft extensions as Galois extensions with the normal basis property when H is a Hopf algebra. A similar result is obtained in this context of Hopf algebroids. The main difference with the regular case is that a Cleft $\mathcal{H}$-extension is a Galois extension with respect to the right bialgebroid $\mathcal{H}_R$ but it has a normal basis property with respect to the base algebra L of the left bialgebroid $\mathcal{H}_L$. By Galois extension, we understand that,

Definition 6.8 *[12] Let $\mathcal{H}$ be a Hopf algebroid, A be a right $\mathcal{H}$-comodule algebra and B be the subalgebra of $\mathcal{H}_R$-coinvariants in A. The extension $B \subset A$ is called $\mathcal{H}$-Galois if the canonical map*

$$\begin{array}{rccc} can_{\mathcal{H}}: & A \otimes_B A & \longrightarrow & A \otimes_{\mathcal{H}} H \\ & a \otimes_B a' & \mapsto & aa'^{[0]} \otimes_{\mathcal{H}} a'^{[1]} \end{array}$$

The next two lemmas give to us the idea of how to prove the result mentioned above.

Lemma 6.9 *[12] Let $\mathcal{H}$ be a Hopf algebroid and $B \subseteq A$ an $\mathcal{H}$-cleft extension with a cleaving map γ. Then, for all $a \in A$, $a^{(0)}\overline{\gamma}(a^{(1)}) \in B$.*

□

Lemma 6.10 *[12] Let $\mathcal{H}$ be a Hopf algebroid and $B \subseteq A$ an $\mathcal{H}$-cleft extension. Then the inclusion $B \subseteq A$ splits in the category of left B-modules. If, in addition, the antipode of $\mathcal{H}$ is bijective, then the inclusion $B \subseteq A$ splits also in the category of right B-modules.*

□

Theorem 6.11 *[12] Let $\mathcal{H}$ be a Hopf algebroid and $B \subseteq A$ a right $\mathcal{H}$-extension. Then the following statements are equivalent:*

(1) $B \subseteq A$ is an $\mathcal{H}$-cleft extension.

(2) (a) The extension $B \subseteq A$ is $\mathcal{H}_R$-Galois;

(b) $A \simeq B \otimes_L H$ as left B-modules and right $\mathcal{H}$-comodules.

□

In our case, for a cocommutative Hopf algebra H acting partially upon a commutative and cocommutative Hopf algebra $\widetilde{A}$, the base algebras L and R will coincide with the commutative subalgebra $E(A)$ of $\widetilde{A}$ and then many distinctions between the left and right structures will coalesce. The Hopf algebroid is given by the partial smash product $\underline{E(A)\#H}$. For a proof that this partial smash product is in fact a Hopf algebroid over $E(A)$, see reference [9], Theorem 3.5. The extension to be considered is the previously defined partially H-cleft extension $\widetilde{A} \subset \underline{\widetilde{A}\#_\omega H}$. Then we have the following theorem.

Theorem 6.12 *Let H be a cocommutative Hopf algebra acting partially on a commutative and cocommutative Hopf algebra $\widetilde{A}$ and let ω be a partial 2-cocycle in $H^2_{par}(H, \widetilde{A})$. Then the partial crossed product $\underline{\widetilde{A}\#_\omega H}$ is a right $\mathcal{H} = \underline{E(A)\#H}$-module algebra with $\widetilde{A} \cong (\underline{\widetilde{A}\#_\omega H})^{co\mathcal{H}}$. Moreover, the extension $\widetilde{A} \subset \underline{\widetilde{A}\#_\omega H}$ is $\mathcal{H}$-cleft in the sense of Definition 6.4.*

Proof: First, define the linear map

$$\widetilde{\rho}: \begin{array}{ccc} \underline{\widetilde{A}\#_\omega H} & \longrightarrow & \underline{\widetilde{A}\#_\omega H} \otimes_{E(A),\blacktriangleright\blacktriangleleft} \underline{E(A)\#H} \\ a\#h & \mapsto & a\#h_{(1)} \otimes (h_{(2)} \cdot 1_A)\#h_{(3)} \end{array}$$

Note that the expression of $\widetilde{\rho}(a\#h)$ can also be written as

$$\widetilde{\rho}(a\#h) = a\#h_{(1)} \otimes 1_{E(A)}\#h_{(2)} = a\#h_{(1)} \otimes 1_A\#h_{(2)},$$

that is because

$$\begin{aligned} & a\#h_{(1)} \otimes (h_{(2)} \cdot 1_A)\#h_{(3)} = a\#h_{(1)} \otimes (1_A\#h_{(2)})(1_A\#1_H) \\ = \; & a\#h_{(1)} \otimes (1_A\#h_{(2)}) \blacktriangleleft 1_A = a\#h_{(1)} \otimes (1_A\#h_{(2)}). \end{aligned}$$

It is easy to see that $(\widetilde{\rho} \otimes_{E(A),\blacktriangleright\blacktriangleleft} Id) \circ \widetilde{\rho} = (Id \otimes_{E(A),\blacktriangleright\blacktriangleleft} \widetilde{\Delta}_r) \circ \widetilde{\rho}$, in which $\widetilde{\Delta}_r$ is the right comultiplication in $\mathcal{H}$, given by

$$\widetilde{\Delta}_r(r\#h) = r\#h_{(1)} \otimes 1_A\#h_{(2)}, \qquad \forall r \in E(A), \forall h \in H.$$

Also, one can check that the map $\widetilde{\rho}$ is $E(A)$-bilinear. Indeed, for $a\#h \in \underline{\widetilde{A}\#_\omega H}$ and $r, s \in E(A)$ then

$$\begin{aligned} & \widetilde{\rho}((r\#1_H)(a\#h)(s\#1_H)) = \widetilde{\rho}(ra(h_{(1)} \cdot s)\#h_{(2)}) \\ = \; & ra(h_{(1)} \cdot s)\#h_{(2)} \otimes 1_A\#h_{(3)} = (ra\#h_{(1)}) \blacktriangleleft s \otimes 1_A\#h_{(2)} \\ = \; & ra\#h_{(1)} \otimes s \blacktriangleright (1_A\#h_{(2)}) = ra\#h_{(1)} \otimes (1_A\#h_{(2)})(s\#1_H) \\ = \; & a(h_{(1)}S(h_{(2)}) \cdot r)\#h_{(3)} \otimes (1_A\#h_{(4)})(s\#1_H) \\ = \; & a(h_{(1)} \cdot (S(h_{(3)}) \cdot r))\#h_{(2)} \otimes (1_A\#h_{(4)})(s\#1_H) \\ = \; & (a\#h_{(1)}) \blacktriangleleft (S(h_{(2)}) \cdot r) \otimes (1_A\#h_{(3)})(s\#1_H) \\ = \; & a\#h_{(1)} \otimes (S(h_{(2)}) \cdot r) \blacktriangleright (1_A\#h_{(3)})(s\#1_H) \\ = \; & a\#h_{(1)} \otimes (1_A\#h_{(2)})((S(h_{(3)}) \cdot r)\#1_H)(s\#1_H) \\ = \; & a\#h_{(1)} \otimes ((h_{(2)} \cdot (S(h_{(4)}) \cdot r))\#h_{(3)})(s\#1_H) \\ = \; & a\#h_{(1)} \otimes ((h_{(2)}S(h_{(4)}) \cdot r)\#h_{(3)})(s\#1_H) \\ = \; & a\#h_{(1)} \otimes (r\#h_{(2)})(s\#1_H) \\ = \; & a\#h_{(1)} \otimes (r\#1_H)(1_A\#h_{(2)})(s\#1_H). \end{aligned}$$

Denote by $\widetilde{\epsilon}_r : \underline{E(A)\#H} \to E(A)$ the right counit of the partial smash product $\underline{E(A)\#H}$, given by $\widetilde{\epsilon}_r(a\#h) = S(h) \cdot 1_A$. Then, we have

$$\begin{aligned} & (Id \otimes_{E(A),\blacktriangleright\blacktriangleleft} \widetilde{\epsilon}_r) \circ \widetilde{\rho}(a\#h) = (a\#h_{(1)}) \blacktriangleleft \widetilde{\epsilon}_r(1_A\#h_{(2)}) \\ = \; & (a\#h_{(1)})((S(h_{(2)}) \cdot 1_A)\#1_H) = a(h_{(1)} \cdot (S(h_{(3)}) \cdot 1_A))\#h_{(2)} \\ = \; & a(h_{(1)}S(h_{(2)}) \cdot 1_A)\#h_{(3)} = a\#h. \end{aligned}$$

Finally, for $a\#h,\ b\#k \in \underline{\widetilde{A}\#_\omega H}$, we have

$$\begin{aligned}\widetilde{\rho}((a\#h)(b\#k)) &= \widetilde{\rho}(a(h_{(1)}\cdot b)\omega(h_{(2)},k_{(1)})\#h_{(3)}k_{(2)}) \\ &= a(h_{(1)}\cdot b)\omega(h_{(2)},k_{(1)})\#h_{(3)}k_{(2)}\otimes 1_A\#h_{(4)}k_{(3)},\end{aligned}$$

on the other hand,

$$\begin{aligned}&\widetilde{\rho}(a\#h)\widetilde{\rho}(b\#k) = (a\#h_{(1)})(b\#k_{(1)})\otimes(1_A\#h_{(2)})(1_A\#k_{(2)})\\
&= a(h_{(1)}\cdot b)\omega(h_{(2)},k_{(1)})\#h_{(3)}k_{(2)}\otimes(h_{(4)}\cdot 1_A)\#h_{(5)}k_{(3)}\\
&= a(h_{(1)}\cdot b)\omega(h_{(2)},k_{(1)})\#h_{(3)}k_{(2)}\otimes(h_{(4)}k_{(3)}S(k_{(4)})\cdot 1_A)\#h_{(5)}k_{(5)}\\
&= a(h_{(1)}\cdot b)\omega(h_{(2)},k_{(1)})\#h_{(3)}k_{(2)}\otimes(h_{(4)}k_{(3)}\cdot(S(k_{(4)})\cdot 1_A))\#h_{(5)}k_{(4)}\\
&= a(h_{(1)}\cdot b)\omega(h_{(2)},k_{(1)})\#h_{(3)}k_{(2)}\otimes(S(k_{(4)})\cdot 1_A)\blacktriangleright(1_A\#h_{(4)}k_{(3)})\\
&= (a(h_{(1)}\cdot b)\omega(h_{(2)},k_{(1)})\#h_{(3)}k_{(2)})\blacktriangleleft(S(k_{(4)})\cdot 1_A)\otimes 1_A\#h_{(4)}k_{(3)}\\
&= a(h_{(1)}\cdot b)\omega(h_{(2)},k_{(1)})(h_{(3)}k_{(2)}\cdot(S(k_{(5)})\cdot 1_A))\#h_{(4)}k_{(3)}\otimes 1_A\#h_{(5)}k_{(4)}\\
&= (a(h_{(1)}\cdot b)\omega(h_{(2)},k_{(1)})(h_{(3)}k_{(2)}S(k_{(3)})\cdot 1_A)\#h_{(4)}k_{(4)})\otimes 1_A\#h_{(5)}k_{(5)}\\
&= a(h_{(1)}\cdot b)\omega(h_{(2)},k_{(1)})(h_{(3)}\cdot 1_A)\#h_{(4)}k_{(2)}\otimes 1_A\#h_{(5)}k_{(3)}\\
&= a(h_{(1)}\cdot b)\omega(h_{(2)},k_{(1)})\#h_{(3)}k_{(2)}\otimes 1_A\#h_{(4)}k_{(3)}.\end{aligned}$$

Therefore, the crossed product $\underline{\widetilde{A}\#_\omega H}$ is a right $\mathcal{H}$-comodule algebra. It is obvious that $i(\widetilde{A})\subseteq(\underline{\widetilde{A}\#_\omega H})^{co\mathcal{H}}$, now take $\sum_i a_i\#h_i\in(\underline{\widetilde{A}\#_\omega H})^{co\mathcal{H}}$, then

$$\sum_i a_i\#h_{i(1)}\otimes 1_A\#h_{i(2)} = \sum_i a_i\#h_i\otimes 1_A\#1_H\in\widetilde{A}\otimes H\otimes_{E(A)}E(A)\otimes H.$$

Applying $Id\otimes\epsilon_H\otimes Id\otimes Id$ to this identity and identifying $\widetilde{A}\otimes_{E(A)}E(A)\cong\widetilde{A}$, we obtain

$$\sum_i a_i\#h_i = \sum_i a_i\epsilon_H(h_i)\#1_H.$$

Therefore $i(\widetilde{A}) = (\underline{\widetilde{A}\#_\omega H})^{co\mathcal{H}}$.

In order to see that the $\mathcal{H}$-extension is cleft, one needs only to define the cleaving map, as item (a) of Definition 6.4 is automatically satisfied, since $L=R=E(A)$. Define the maps

$$\begin{aligned}\widetilde{\gamma}:\ \underline{E(A)\#H} &\to \underline{\widetilde{A}\#_\omega H}\\ r\#h &\mapsto r\#h\end{aligned}$$

and

$$\begin{aligned}\overline{\widetilde{\gamma}}:\ \underline{E(A)\#H} &\to \underline{\widetilde{A}\#_\omega H}\\ r\#h &\mapsto r\omega^{-1}(S(h_{(2)}),h_{(3)})\#S(h_{(1)})\end{aligned}$$

Note that $a\#h$ in the domain of $\widetilde{\gamma}$ means something quite different from $a\#h$ in the image because the first is in the partial smash product, while the second lies in the partial crossed product. It is easy to see that $\widetilde{\gamma}$ is left $E(A)$-linear. Indeed, for $r \in E(A)$ and $a\#h \in \underline{E(A)\#H}$, we have

$$\begin{aligned} \widetilde{\gamma}(s_l(r)(a\#h)) &= \widetilde{\gamma}((r\#1_H)(a\#h)) = \widetilde{\gamma}(ra\#h) \\ &= ra\#h = (r\#1_H)(a\#h). \end{aligned}$$

Also, the cleaving map is a morphism of right $\mathcal{H}$-comodules. Consider $a\#h \in \underline{E(A)\#H}$, then

$$\begin{aligned} \widetilde{\rho} \circ \widetilde{\gamma}(a\#h) &= \widetilde{\rho}(a\#h) = a\#h_{(1)} \otimes 1_A\#h_{(2)} \\ &= \widetilde{\gamma}(a\#h_{(1)} \otimes 1_A\#h_{(2)} = (\widetilde{\gamma} \otimes_{E(A),\blacktriangleright\blacktriangleleft} Id) \circ \widetilde{\rho}(a\#h). \end{aligned}$$

Finally, let us check that the maps $\widetilde{\gamma}$ and $\overline{\widetilde{\gamma}}$ are mutually inverse by convolution in the sense that

$$\begin{aligned} \mu \circ (\widetilde{\gamma} \otimes_{E(A),\blacktriangleright\blacktriangleleft} \overline{\widetilde{\gamma}}) \circ \widetilde{\Delta}_r &= i \circ \widetilde{\epsilon}_l \\ \mu \circ (\overline{\widetilde{\gamma}} \otimes_{E(A),\triangleright\triangleleft} \widetilde{\gamma}) \circ \widetilde{\Delta}_l &= i \circ \widetilde{\epsilon}_r. \end{aligned}$$

Consider $a\#h \in \underline{E(A)\#H}$, then

$$\begin{aligned} &\mu \circ (\gamma \otimes_{E(A),\blacktriangleright\blacktriangleleft} \overline{\gamma}) \circ \widetilde{\Delta}_r(a\#h) = \widetilde{\gamma}(a\#h_{(1)})\overline{\widetilde{\gamma}}(1_A\#h_{(2)}) \\ &= (a\#h_{(1)})(\omega^{-1}(S(h_{(3)}), h_{(4)})\#S(h_{(2)})) \\ &= a(h_{(1)} \cdot \omega^{-1}(S(h_{(6)}), h_{(7)}))\omega(h_{(2)}, S(h_{(5)}))\#h_{(3)}S(h_{(4)}) \\ &= a(h_{(1)} \cdot \omega^{-1}(S(h_{(4)}), h_{(5)}))\omega(h_{(2)}, S(h_{(3)}))\#1_H \\ &= a\omega^{-1}(h_{(1)}S(h_{(9)}), h_{(10)})\omega(h_{(2)}, S(h_{(8)})h_{(11)})\omega^{-1}(h_{(4)}, S(h_{(7)})) \\ &\qquad \omega(h_{(5)}, S(h_{(6)}))\#1_H \\ &= a\omega^{-1}(h_{(1)}S(h_{(6)}), h_{(7)})\omega(h_{(2)}, S(h_{(5)})h_{(8)})(h_{(3)} \cdot (S(h_{(4)}) \cdot 1_A))\#1_H \\ &= a\omega^{-1}(h_{(1)}S(h_{(2)}), h_{(7)})\omega(h_{(3)}, S(h_{(5)})h_{(6)})(h_{(4)} \cdot 1_A)\#1_H \\ &= a(h \cdot 1_A)\#1_H \\ &= i(\widetilde{\epsilon}_l(a\#h)). \end{aligned}$$

Also, we have

$$
\begin{aligned}
& \mu \circ (\overline{\widetilde{\gamma}} \otimes_{E(A),\triangleright\triangleleft} \widetilde{\gamma}) \circ \widetilde{\Delta}_l(a\#h) = \\
= \; & \mu \circ (\overline{\widetilde{\gamma}} \otimes_{E(A),\triangleright\triangleleft} \widetilde{\gamma})(a\#h_{(1)} \otimes 1_A\#h_{(2)}) \\
= \; & \mu \circ (\overline{\widetilde{\gamma}} \otimes_{E(A),\triangleright\triangleleft} \widetilde{\gamma})((1_A\#h_{(1)}) \triangleleft a \otimes 1_A\#h_{(2)}) \\
= \; & \mu \circ (\overline{\widetilde{\gamma}} \otimes_{E(A),\triangleright\triangleleft} \widetilde{\gamma})(1_A\#h_{(1)} \otimes a \triangleright (1_A\#h_{(2)})) \\
= \; & \mu \circ (\overline{\widetilde{\gamma}} \otimes_{E(A),\triangleright\triangleleft} \widetilde{\gamma})(1_A\#h_{(1)} \otimes a\#h_{(2)}) \\
= \; & \overline{\widetilde{\gamma}}(1_A\#h_{(1)})\widetilde{\gamma}(a\#h_{(2)}) \\
= \; & (\omega^{-1}(S(h_{(2)}), h_{(3)})\#S(h_{(1)}))(a\#h_{(4)}) \\
= \; & a\omega^{-1}(S(h_{(4)}), h_{(5)})(S(h_{(3)}) \cdot a)\omega(S(h_{(2)}), h_{(6)})\#S(h_{(1)})h_{(7)} \\
= \; & a\omega^{-1}(S(h_{(5)}), h_{(6)})\omega(S(h_{(4)}), h_{(7)})(S(h_{(3)}) \cdot a)\#S(h_{(1)})h_{(2)} \\
= \; & (S(h_{(2)}) \cdot (h_{(3)} \cdot 1_A))(S(h_{(1)}) \cdot a)\#1_H \\
= \; & (S(h) \cdot a)\#1_H = i(\widetilde{\epsilon}_r(a\#h)).
\end{aligned}
$$

Therefore, $\widetilde{A} \subset \underline{\widetilde{A}\#_\omega H}$ is a $\underline{E(A)\#H}$-cleft extension.

■

The following is an immediate consequence of the last theorem and Theorem 6.11, and show us that partially H-cleft extensions are related with the theory of $\mathcal{H}$-Galois extensions if H is a cocommutative Hopf algebra and A is a commutative algebra.

Corollary 6.13 *Let $\mathcal{H} = E(A)\#H$ be the Hopf algebroid given above and $\widetilde{A} \subset \underline{\widetilde{A}\#_\omega H}$ a right $\mathcal{H}$-extension. Then, the following statements are equivalent:*

1. *$\widetilde{A} \subset \underline{\widetilde{A}\#_\omega H}$ is an $\mathcal{H}$-cleft extension;*

2. (a) The extension $\widetilde{A} \subset \underline{\widetilde{A}\#_\omega H}$ is $\mathcal{H}$-Galois;

   (b) $\underline{\widetilde{A}\#_\omega H} \simeq \widetilde{A} \otimes_{E(A)} \underline{E(A)\#H}$ as left $\widetilde{A}$-modules and right $\mathcal{H}$-comodules.

□

Chapter 7

Conclusions and outlook

In this work, we introduced a cohomology theory for partial actions of Hopf algebras, extending the results of [30] and [19]. Also we are able to give a cohomological notion for the partial crossed product introduced in [5], since that we considered H a cocommutative Hopf algebra and A a commutative algebra. Furthermore, we have unexpectedly shown that the theory of partial cleft extensions for Hopf algebras [5] can be understood in the context of cleft extensions theory for Hopf algebroids in [12].

In addition, in association with Professor Joost Vercruisse (ULB), we tried to investigate whether there would be a general cohomological theory, with arbitrary H and A, but, as well as for the cohomological theory for Hopf algebras, we did not obtain some results.

However, we observe that all the cohomology theory done in this tesis was done over cocommutative Hopf algebras acting partially over commutative algebras. This can be generalized for cocommutative Hopf algebra objects and commutative algebra objects in braided monoidal categories.

Lastly, we present some directions for next works:

- One topic of interest is to relate this cohomology for partial actions and the cohomology for its globalization, then constructing a bridge between this theory and the classical Sweedler's theory.

- The last theorem placed the notion of a partially cleft extension within the context of cleft extensions for Hopf algebroids. This suggests, perhaps, that this entire cohomological theory can be understood properly as a cohomological theory of Hopf algebroids.

- Another topic to be explored in further research can be the obstruction theory for the existence of partially cleft extensions and its relation with the third cohomology group in the same spirit of [29].

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