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Asymptotic properties of evolution models with a logarithmic-Laplacian type operator

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Asymptotic properties of evolution models with a logarithmic-Laplacian type operator

O presente trabalho em nível de doutorado foi avaliado e aprovado por banca examinadora composta pelos seguintes membros:

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To my family.

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RESUMO

Neste trabalho são considerados alguns problemas de Cauchy em $\boldsymbol{R}^n$ associados a novos modelos de evolução do tipo ondas baseados em um operador Laplaciano-logarítimico introduzido por Charão-Ikehata em [6]. Esse operador que é a composição da função logarítmica com $I - \Delta^\theta$, $\theta > 0$, é mais fraco para dissipar a energia associada à equação da onda com dissipação estrutural mas produzindo mesmo tipo de estimativas como se observa nos trabalhos [4, 6]. Essa interessante consequência do uso desse novo operador também ocorre nos problemas estudados neste trabalho. Outra vantagem em usar esse tipo de operador é poder tomar dados iniciais em espaços mais gerais para certos modelos. Para os modelos considerados são estudados perfis assintóticos que ajudam a provar taxas ótimas de decaimento ou *blow-up* em tempo infinito para a norma L^2 das soluções dependendo da dimensão espacial. O problema considerado no Capítulo 3 possui perfil assintótico do tipo oscilatório. Taxa ótima de decaimento para a solução quando $n \geq 3$ é obtida e nos casos $n = 1, 2$ mostra-se que a solução explode em tempo infinito exibindo taxa ótima de crescimento. O segundo problema apresenta propriedade de perda de regularidade e devido a isso o seu perfil assintótico é do tipo difusivo para alta regularidade dos dados iniciais, do tipo oscilatório para baixa regularidade e é combinação dos dois tipos para uma regularidade limiar. Também são derivadas taxas ótimas de decaimento dependendo da regularidade imposta nos dados iniciais. O problema considerado no Capítulo 5 apresenta o fenômeno de dupla difusão e obtêm-se taxas ótimas de decaimento para a solução nos casos $n \geq 2$. Quando $n = 1$ um parâmetro crítico $\theta^* = 1/4$ aparece de modo que a solução do problema decai com certa taxa ótima para $\theta \in (0, \theta^*)$ e explode em tempo infinito se $\theta \in [\theta^*, 1/2)$ com taxa ótima de crescimento. Ao que parece este tipo de resultado para $\theta \geq \theta^*$ ainda não tinha sido descoberto em trabalhos de outros autores.

Palavras-chave: Equações do tipo ondas. Operador Laplaciano-logarítmico. Dissipação logarítmica. Perfil assintótico. Taxas de decaimento em L^2. Estimativas ótimas.

RESUMO EXPANDIDO

Introdução

Neste trabalho consideramos problemas de Cauchy associados a três novos modelos de evolução do tipo ondas em $\boldsymbol{R}^n$ dados por

$$u_{tt} + Lu + Lu_t = 0, \tag{0.1}$$

$$u_{tt} + Lu + (I + L)^{-1} u_t = 0, \tag{0.2}$$

$$u_{tt} - \Delta u + L_\theta u_t = 0, \quad 0 < \theta < \frac{1}{2}, \tag{0.3}$$

onde L_θ é o operador Laplaciano-logarítmico formalmente definido por $\log(I + (-\Delta)^\theta)$ e $L = L_1$. O operador para $\theta = 1$ foi introduzido inicialmente como termo dissipativo em uma equação da onda por Charão-Ikehata em [6]. Em seguida Charão-D'Abbicco-Ikehata generalizaram o problema para dissipação $L_\theta u_t$ com $\theta > \frac{1}{2}$ em [4]. Este operador L está definido em espaços mais gerais que os espaços de Sobolev e é mais fraco do que o operador usual $-\Delta$ no sentido que mantém quase a mesma regularidade de funções em $H^s(\boldsymbol{R}^n)$, $s > 0$.

Objetivos

O objetivo deste trabalho é provar o comportamento assintótico no sentido L^2, quando $t \to \infty$, das soluções dos Problemas de Cauchy associados às equações (0.1), (0.2) e (0.3). Deseja-se investigar perfis assintóticos para tais soluções e taxas ótimas de decaimento e/ou crescimento quando $t \to \infty$.

Metodologia

Em problemas como os considerados neste trabalho cujos domínios são o espaço $\boldsymbol{R}^n$ podemos considerar um problema associado no chamado espaço de Fourier. Um método conhecido na literatura para provar comportamento assintótico de soluções e também da energia associada a uma equação chama-se *método de multiplicadores*. Usamos os multiplicadores de Ikehata-Natsume [27] (see also [46]) para os problemas associados às equações (0.1) e (0.2). Este método pode ser eficaz para que taxas de decaimento da solução nos casos $n \geq 3$ e também da energia total do sistema para qualquer dimensão sejam derivadas. Através deste método, no entanto, não é possível verificar se as taxas de decaimento obtidas são ótimas.

Nos três problemas considerados também se pode encontrar uma fórmula explícita para a solução do problema no espaço de Fourier. Taxas de decaimento podem ser obtidas considerando-se uma decomposição adequada para a solução no espaço de Fourier que é chamada de *expansão assintótica*. Um dos termos da expansão assintótica possui a mesma taxa que a solução do problema e é chamado de *perfil assintótico* para esta solução. Mostra-se que a norma L^2 da diferença entre a solução e este perfil assintótico converge para zero quando $t \to \infty$ com determinada taxa de decaimento. Em outras palavras se mostra que a solução do problema se comporta como o perfil assintótico em tempo infinito. Conhecer um perfil assintótico em uma forma simples pode ser mais importante que conhecer a própria solução, que pode ter uma forma complicada. Além disso, taxas ótimas de decaimento e/ou crescimento do perfil assintótico correspondem a taxas ótimas de decaimento e de crescimento para solução.

Resultados, discussão e considerações finais

Assumindo dados iniciais no espaço $L^2(\boldsymbol{R}^n) \cap L^{1,1}(\boldsymbol{R}^n)$ provamos que o problema associado a (0.1) possui perfil assintótico do tipo oscilatório. Derivamos taxas ótimas de decaimento da ordem $t^{-\frac{n-2}{4}}$ para as soluções nas dimensões $n \geq 3$. Nos casos unidimensional e bidimensional mostramos que a solução explode em tempo infinito com taxas ótimas de crescimento da ordem $\sqrt{t}$ e $\sqrt{\log t}$, respectivamente. Estas taxas ótimas obtidas já eram conhecidas do problema da onda usual, mas sendo o operador L mais fraco que o operador $-\Delta$, podemos dizer que o operador Laplaciano-logarítmico é mais eficiente que o operador Laplaciano.

Para o problema associado a (0.2), foi necessário impor mais regularidade nos dados iniciais além de $L^2(\boldsymbol{R}^n) \cap L^{1,1}(\boldsymbol{R}^n)$, a saber $(u_0, u_1) \in Y^{l+1} \times Y^l$ com $l \geq 0$, para obter taxas de decaimento na região de alta frequência no espaço de Fourier e, portanto, dizemos que o problema apresenta perda de regularidade. Para altas regularidades dos dados iniciais correspondentes a $l > \frac{n}{2} - 1$ obtemos perfil assintótico do tipo difusivo, para baixa regularidade dos dados iniciais com $l < \frac{n}{2} - 1$ o perfil assintótico é do tipo onda, enquanto para a regularidade limiar $l = \frac{n}{2} - 1$ o perfil assintótico é a combinação dos dois tipos. Também encontramos duas possibilidades de taxas de decaimento para as soluções de ordens $t^{-\frac{l+1}{2}}$ e $t^{-\frac{n}{4}}$ que valem dependendo da regularidade dos dados iniciais e da dimensão considerada.

Estudamos o problema associado a (0.3) para valores de $\theta \in [0, \frac{1}{2})$. Para dados iniciais em $L^2(\boldsymbol{R}^n) \cap L^{1,2\theta}(\boldsymbol{R}^n)$, mostramos que a equação apresenta o fenômeno que chamamos de *dupla difusão*, pois derivamos um perfil assintótico que é a diferença de duas soluções de equações de difusão. Nos casos em que $n > 4\theta$ observamos que o perfil assintótico poderia ser tomado como sendo apenas um dos termos que tem a pior taxa de decaimento dada por $t^{-\frac{n-4\theta}{4(1-\theta)}}$, neste caso a solução apresenta fenômeno difusivo e tem a mesma taxa ótima

de decaimento. Estas taxas de decaimento já eram conhecidas em problemas da onda usual com dissipação $(-\Delta)^{\theta}u_t$. No entanto, no caso $n=1$ com $\frac{1}{4} \leq \theta < \frac{1}{2}$ o fenômeno de dupla difusão é essencial para que taxas de crescimento fossem obtidas. Provamos que a solução tem *blow-up* em tempo infinito com taxa ótima de crescimento $t^{\frac{4\theta-1}{4\theta}}$ se $\frac{1}{4} < \theta < \frac{1}{2}$ e $\sqrt{\log t}$ para $\theta = \frac{1}{4}$. Tal resultado, até onde sabemos, ainda não tinha sido descoberto em outros trabalhos e estas novas estimativas apresentadas também podem ser verificadas para a equação da onda com dissipação usual.

Palavras-chave: Equações do tipo ondas. Operador Laplaciano-logarítmico. Dissipação logarítmica. Perfil assintótico. Taxas de decaimento em L^2. Estimativas ótimas.

ABSTRACT

In this work, some Cauchy problems in $\boldsymbol{R}^n$ associated to new wave-like evolution models based on logarithmic-Laplacian operator, which was introduced by Charão-Ikehata in [6], are considered. This operator is the composition of the logarithmic function with $I - \Delta^{\theta}$, $\theta > 0$, and it is weaker to dissipate the energy associated to wave equation with structural damping, but it produces estimates of the same type as observed in works [4, 6]. This interesting consequence of using this operator also appears in the problems studied in this work. Another advantage of using this operator is that initial data in more general spaces can be taken to certain models. For the considered models, asymptotic profiles are studied and they help to prove optimal decay or infinite time blow-up rates to the L^2-norm of the solution depending on the spacial dimension. The problem considered in Chapter 3 has wave-like asymptotic profile. Optimal decay rate to the solution is obtained when $n \geq 3$ and in the cases when $n = 1, 2$ it is shown that the solution blows up in infinite time and optimal growth estimates are showed. The second problem in Chapter 4 presents regularity-loss property and because of that its asymptotic profile is diffusive-like for high regularity of initial data, it is wave-like for low regularity and it is combination of both for a threshold regularity. Optimal decay rates are also derived depending on the regularity imposed on the initial data. The problem considered in Chapter 5 presents double diffusion phenomenon and optimal decay rates are obtained if $n \geq 2$. When $n = 1$ a critical parameter $\theta^* = 1/4$ appears such that the solution decays with certain optimal estimate for $\theta \in (0, \theta^*)$ and blows up in infinite time for $\theta \in [\theta^*, 1/2)$ with optimal growth rate. This type of result for $\theta \in [\theta^*, 1/2)$ seems to have not been discovered in studies by other authors.

Keywords: Wave-like equation. Logarithmic-Laplacian operator. Logarithmic damping. Asymptotic profile. L^2-decay. Optimal estimates.

SUMARY

1 INTRODUCTION

Quite recently Charão-Ikehata [6] introduced in a pioneer work a new type of damping mechanism of a logarithm-Laplacian type L to dissipate energy of solutions of the wave equation. Later Charão-D'Abbicco-Ikehata in [4] consider an operator L_θ by introducing a parameter $\theta > 1/2$.

In our work we introduce new models of evolution wave equations based on operators L and L_θ as follows

$$u_{tt} + Lu + Lu_t = 0, \tag{1.1}$$

$$u_{tt} + Lu + (I+L)^{-1}u_t = 0, \tag{1.2}$$

$$u_{tt} - \Delta u + L_\theta u_t = 0, \quad 0 < \theta < \frac{1}{2}, \tag{1.3}$$

where $u = u(t,x)$ for $(t,x) \in (0,\infty) \times \boldsymbol{R}^n$. The operator L_θ is defined as

$$L_\theta : D(L_\theta) \subset L^2(\boldsymbol{R}^n) \to L^2(\boldsymbol{R}^n),$$

which combines the composition of logarithm function with the Laplace operator. Its domain is

$$D(L_\theta) := \left\{ f \in L^2(\boldsymbol{R}^n) \;\middle|\; \int_{\boldsymbol{R}^n} (\log(1+|\xi|^{2\theta}))^2 |\hat{f}(\xi)|^2 d\xi < +\infty \right\}, \quad \theta > 0 \tag{1.4}$$

and it is defined, via Fourier transform, as follows

$$(L_\theta f)(x) := \mathcal{F}^{-1}\left(\log(1+|\xi|^{2\theta})\hat{f}(\xi)\right)(x), \quad f \in D(L_\theta). \tag{1.5}$$

We note that the operator L_θ is a generalization of the original operator $L = L_1$. Here $\mathcal{F}(f)(\xi)$ denotes the Fourier transform of $f(x)$ and $\mathcal{F}^{-1}$ expresses its inverse Fourier transform. We refer to L_θ as *logarithmic-Laplacian operator*.

The operator L_θ is nonnegative and self-adjoint in $L^2(\boldsymbol{R}^n)$, because it is unitary equivalent to a multiplication operator in $L^2(\boldsymbol{R}^n_\xi)$. Therefore the square root

$$L_\theta^{1/2} : D(L_\theta^{1/2}) \subset L^2(\boldsymbol{R}^n) \to L^2(\boldsymbol{R}^n)$$

can be defined and it is also nonnegative and self-adjoint with domain

$$D(L_\theta^{1/2}) = \left\{ f \in L^2(\boldsymbol{R}^n) \;\middle|\; \int_{\boldsymbol{R}^n} \log(1+|\xi|^{2\theta}) |\hat{f}(\xi)|^2 d\xi < +\infty \right\}.$$

We notice that

$$H^s(\boldsymbol{R}^n) \hookrightarrow D(L_\theta) \hookrightarrow D(L_\theta^{1/2}) \hookrightarrow L^2(\boldsymbol{R}^n)$$

for any $s > 0$, because

$$\lim_{|\xi|\to\infty} \frac{\log(1+|\xi|^{2\theta})}{|\xi|^s} = 0, \quad s > 0.$$

We say that the operator logarithmic-Laplacian L, when $\theta = 1$, is weaker than Laplacian operator in the following sense. Let $f \in H^2(\boldsymbol{R}^n) \subset D(L)$, then $(-\Delta)^{1/2} f \in H^1(\boldsymbol{R}^n)$, whereas $L^{1/2} f \in H^s(\boldsymbol{R}^n)$ for each $s \in (0,2)$. In fact, for $0 < s < 2$ there exists $M > 0$ such that

$$\log(1 + |\xi|^2) \leq |\xi|^{4-2s}, \quad |\xi| \geq M.$$

Then

$$(1 + |\xi|^2)^s \log(1 + |\xi|^2) |\hat{f}|^2 \leq C(1 + |\xi|^2)^2 |\hat{f}|^2, \quad |\xi| \geq M,$$

where $C > 0$ is constant. The above inequality implies that $L^{1/2} f \in H^s(\boldsymbol{R}^n)$ for each $s \in (0,2)$ and $f \in H^2(\boldsymbol{R}^n)$.

The symbol $\log(1 + |\xi|^{2\theta})$ appears in Lévy process [33, 43], more specifically, in the process called rotationally invariant geometric strictly α-stable when the so-called characteristic exponent has the form

$$\psi(\xi) = \log(1 + |\xi|^\alpha),$$

with $\alpha \in (0,2]$. The particular case $\alpha = 2$ is called symmetric variance gamma process, which has some applications in financial models.

It should be noted that the space $D(L)$ is closely related with the so-called generalized Bessel potential spaces $H_p^{s,b}(\boldsymbol{R}^n)$, one can refer [16], where

$$H_p^{s,b}(\boldsymbol{R}^n) := \left\{ f \in \mathcal{S}'(\boldsymbol{R}^n) \,|\, \|\mathcal{F}^{-1}\left((1 + |\xi|^2)^{s/2} (1 + \log(1 + |\xi|^2))^b \hat{f} \right)(\cdot)\|_p < +\infty \right\}.$$

Indeed, $D(L) \hookrightarrow H_2^{0,1}(\boldsymbol{R}^n)$.

Symbolically writing, we may see that

$$L_\theta = \log(I + (-\Delta)^\theta),$$

where Δ is the usual Laplace operator defined on $H^2(\boldsymbol{R}^n)$.

Formally, we can notice that

$$\log(I + (-\Delta)^\theta)(u)(x) = \frac{d}{ds}|_{s=0} [(I + (-\Delta)^\theta)^s u](x)$$

for $u \in C_0^\infty(\boldsymbol{R}^n)$. This (formal) relation comes from a modified idea of Chen-Weth [10], where they studied a Dirichlet problem for a logarithmic-Laplacian operator whose symbol is $2\log(|\xi|)$.

It seems that an independent study was developed on the same operator L from another point of view. In [17] the integrodifferential operator defined by

$$(I - \Delta)^{\log} u(x) := \frac{2}{(2\pi)^{\frac{n}{2}}} \int_{R^n} \frac{u(x) - u(x+y)}{|y|^{\frac{n}{2}}} K_{\frac{n}{2}}(|y|) dy, \tag{1.6}$$

where K_ν is the Bessel modified function of second kind with index ν, is presented. The author prove that the operator $(I - \Delta)^{\log}$ is the same that the operator L defined above.

Indeed, the Fourier transform of this two operators is the same. Definition (1.6) relates the operator L to Lévy process. Finally, we emphasize that in [17] Dirichlet problems associated to the operator $(I-\Delta)^{\log}$ are studied.

We can express Cauchy problems associated to equations (1.1), (1.2) and (1.3) as the abstract formulation

$$u_{tt} + A_1 u + A_2 u_t = 0 \tag{1.7}$$

$$u(0) = u_0, \quad u_t(0) = u_1 \tag{1.8}$$

where $A_j : D(A_j) \subset H \to H$, $j = 1, 2$, are two nonnegative self-adjoint operator in a Hilbert space H.

The abstract problems as in (1.7) with $A_1 = A_2 =: A$ was studied by Ikehata-Todorova-Yordanov [31]. The authors prove that

$$e^{-t\frac{A}{2}} \left(\cos(tA^{1/2})u_0 + A^{-1/2}\sin(tA^{1/2})u_1 \right) \tag{1.9}$$

is the leading term to its solution by employing an abstract energy method in the Fourier space combined with the spectral analysis. The especial abstract problem with $A_2 = I$, was studied by Chill-Haraux [11] with initial data in $D(A^{1/2}) \times H$ and they prove that the difference, in $D(A^{1/2})$-sense, between the solution $u(t)$ of the problem (1.7) and the solution v of the heat problem

$$v_t + A_1 v = 0$$

$$v(0) = u_0 + u_1$$

decay to zero as $t \to \infty$. This means that $u(t)$ behaves as $v(t)$ for large $t > 0$ and we say that the considered wave equation has *diffusion phenomenon* [42].

Now we comment on several models associated to the generalized wave equation

$$u_{tt} + (-\Delta)^{\sigma} u + (-\Delta)^{\theta} u_t = 0, \tag{1.10}$$

which have an abstract formulation as in (1.7).

The case $\sigma = \theta = 1$ is the classical wave equation under effects of a strong damping, and it describes waves with viscoelastic damping. We can cite the two pioneering works of G. Ponce [41] and Y. Shibata [44] where they study decay estimates of the solution to the associated Cauchy problem in the general L^p-L^q sense. In the work [25] an *diffusive wave-like* asymptotic profile as in (1.9) to the solution in L^2-sense has been derived and optimal decay rates (for $n \geq 3$) are obtained. In the case $n = 1, 2$ there is a strong singularity in the leading term and it is proved an infinity time blow up for the solution. Then the optimallity of the growth rates for $n = 1, 2$ is proved in [28].

The case $\sigma = 1$ and $\theta = 0$ described waves with external damping and it was studied in [37]. In this work, Matsumura prove that the solution decay in L^2-sense with decay rate $t^{-\frac{n}{4}}$ under $L^1(\boldsymbol{R}^n) \cap L^2(\boldsymbol{R}^n)$ assumptions on initial data. This result represents an important "parameter" when we studying decay rates for problems as (1.10) with θ near 0.

The intermediate case $\sigma = 1$ and $0 < \theta < 1$ is called the wave equation with structural (or fractional) damping. Applying energy methods, this case was studied in [27] in L^2-sense and the obtained result for $0 < \theta < \frac{1}{2}$ was improved in [7]. More generally L^p-L^q estimates, $1 \leq p, q \leq \infty$, was obtained in [38]. The case $\sigma = 1$ with $\theta > 1$ was studied in [26]. The authors obtained a diffusive-like asymptotic profile and optimal decay estimates, however a *regularity-loss* property appears, that is additional regularity on initial data is required to obtain optimal decay estimates.

A more general model of (1.10) with a super damping, i.e. $\theta > \sigma$, was studied in [5]. In this work, energy decay rates and an asymptotic profile and optimal estimates were obtained. We can also cite [19], where the authors studied an abstract equation similar to (1.10) with $\theta = 1$. They investigate the regularity of solutions to the homogeneous problem depending on σ.

In the recent work [6] above mentioned, which have introduced a new mechanism of damping based on the logarithm-Laplacian type operator L, Charão-Ikehata studied the following Cauchy problem to the wave equation

$$u_{tt} - \Delta u + L u_t = 0, \quad t > 0, \quad x \in \boldsymbol{R}^n \tag{1.11}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \boldsymbol{R}^n. \tag{1.12}$$

The authors derived an diffusive wave-like asymptotic profile as $t \to \infty$ in a simple form:

$$\left(\int_{R^n} u_1(x)dx \right) \mathcal{F}^{-1} \left((1 + |\xi|^2)^{-\frac{t}{2}} \frac{\sin(|\xi|t)}{|\xi|} \right) \tag{1.13}$$

and they also obtained the same optimal estimates as in [25, 28] to the solution of this problem. To obtain such estimates, the authors found some sharp estimates to integrals as

$$I_p(t) := \int_0^1 (1 + r^2)^{-t} r^p dr, \quad t > 0$$

for $p > 0$.

Integrals as $I_p(t)$ are related to hypergeometric functions and, in [4], it was possible to find estimates $I_p(t)$ for more general case $p > -1$. In such work, Charão-D'Abbicco-Ikehata study the Cauchy problem associated to

$$u_{tt} - \Delta u + L_\theta u_t = 0 \tag{1.14}$$

with $\theta > 1/2$. The case $1/2 < \theta \leq 1$ present the same behavior as the solution of the problem associated to the wave equation

$$u_{tt} - \Delta u - \Delta^\theta u_t = 0 \tag{1.15}$$

studying in [25]. However, they noticed that for $\theta > 1$ no regularity-loss property appears. This fact occurs, because

$$\lim_{|\xi|\to\infty} \frac{\log(1+|\xi|^{2\theta})}{|\xi|^2}$$

is finite for any $\theta > 0$.

On the other hand, if we consider equation (1.14) with $\theta = 0$, we obtain the equation

$$u_{tt} - \Delta u + u_t = 0$$

as well as the one study abstractly by Chill-Haraux [11]. As already mentioned, this equation has diffusion phenomenon.

In this sense, we investigate the model (1.14) for $0 < \theta < 1/2$ in order to identify for which value of θ the asymptotic behavior changes. Moreover, we also study the asymptotic behavior of the solution of (1.7) when we work with the operator L instead of the abstracts operators in equation (1.7).

It is important to note that equation (1.2) is equivalent to

$$u_{tt} + Lu_{tt} + L^2u + Lu + u_t = 0,$$

if we consider sufficiently regular initial data. That model is a type of plate equation as

$$u_{tt} + (-\Delta)^{\delta} u_{tt} - \alpha\Delta u + \beta\Delta^2 u + (-\Delta)^{\theta} u_t = 0, \quad (\theta \geq 0). \tag{1.16}$$

The equation (1.16) with $\delta = 1$, $\alpha = 0$ and $\theta = 0$ is known as plate equation under effects of rotational inertia term Δu_{tt} and it has a frictional dissipation u_t. This particular case was studied by Luz-Charão in [34] and the authors also proved the global existence of solution and asymptotic behavior to a semilinear problem. In [45] Sugitani-Kawashima investigated decay rates to the solution of that particular case and they observed that this equation presents an regularity-loss property. In this connection, such a regularity-loss structure has been first discovered and named by S. Kawashima through the analysis for the dissipative Timoshenko system (see e.g. [23]). For the more general case $\delta = 1$, $\alpha = 0$ and $0 \leq \theta \leq 1$ energy decay rates were obtained by employing the so-called Haraux-Komornik inequality in [8] and the authors noted that the regularity-loss property becomes weaker as θ increases and it disappears when $\theta = 1$.

The equation (1.16) has also been studied for general parameters δ and θ by Horbach-Ikehata-Charão in [22]. In that work the authors obtained decay rates depending on the parameters of solutions to Cauchy problems based on the multiplier method. They also got asymptotic profiles and optimal decay rate of the solutions for some cases where $\theta > 1/2$. However, for the case $\theta \leq 1/2$ (in particular, $\theta = 0$), no asymptotic profile or optimal estimates were obtained.

In this sense, Fukushima-Ikehata-Michihisa [18] investigated the asymptotic profiles of the solution to (1.16) with $\delta = 1$ and $\theta = 0$. Such profiles are divided into two parts:

one is the diffusive like for high regularity initial data and the other is the wave-like for low regularity initial data from the viewpoint of regularity-loss structure.

It should be noted that knowing optimal estimates allows to investigate the so-called critical exponents for semi-linear equations. These critical exponents have been studied a lot and we can cite, for example, [15] and [12–14].

The main topic of this work is to introduce an asymptotic profile for the solution in the L^2 framework to the Cauchy Problem associated to each equation in (1.1), (1.2) and (1.3) in a simple form. Then we use this asymptotic profiles to obtain optimal decay rates to the L^2-norm of solutions. Our interest to study these equations is only from a pure mathematical point of view.

This work is organized as follows. In chapter 2 we state already known results and we improve the proof based originally on hypergeomtric and Gamma functions of an asymptotic lemma that appears in [4]. Moreover we also introduce new elementary results that we used in subsequent chapters. The notations used in this work are also described at the beginning of Chapter 2. In Chapter 3, we study the Cauchy problem associated to equation (1.1). The problem associated to equation (1.2) is studied in Chapter 4. Finally, we study the problem for equation (1.3) in Chapter 5.

The main results obtained in Chapter 3 of this Doctoral Thesis was published in 2022 in *Journal of Mathematical Analysis and Applications* (see [2]). The results of Chapter 4 will be published in May 2022 in *Discrete and Continuous Dynamical Systems* (see [3]). The Chapter 5 of this work was published in 2022 in *Journal of Differential Equations* (see [40]). These three papers were made with the supervision by Professor Ruy Coimbra Charão with collaboration of Professor Ryo Ikehata from Hiroshima University, Japan.

2 GENERAL BASIC RESULTS

In this chapter we have collected important results that are used in the next chapters. In the first section we introduce some results that we use to obtain existence and uniqueness of solutions. In section 2, we remember some facts about the Fourier transform in $L^2(\boldsymbol{R}^n)$. The purpose of third section is to ensure that the spaces like $D(L)$ (see (1.4)) have suitable properties in order to apply the results of Section 1. In Section 4 we state some technical lemmas which are mainly used to estimate the asymptotic profiles. In the last section we prepare some asymptotic lemmas that we use later to obtain sharp estimates.

Throughout this paper, $\|\cdot\|_q$ stands for the usual $L^q(\boldsymbol{R}^n)$-norm. For simplicity of notation, in particular, we use $\|\cdot\|$ instead of $\|\cdot\|_2$. The relation $f(t) \sim g(t)$ as $t \to \infty$ means that there exist positive constants C_1, C_2 such that

$$C_1 g(t) \leq f(t) \leq C_2 g(t), \quad (t \gg 1).$$

For $\Omega \subset \boldsymbol{R}^n$ we denote $f \approx g$ on Ω if, and only if, there are constants $K_1, K_2 > 0$ such that

$$K_1 f(y) \leq g(y) \leq K_2 f(y), \quad \forall y \in \Omega.$$

In this case, we say that f is equivalent to g on Ω.

Finally, we denote the surface area of the n-dimensional unit ball by $\omega_n := \int_{|\omega|=1} d\omega$.

2.1 ABSTRACT CAUCHY PROBLEM: EXISTENCE AND UNIQUENESS OF SOLUTION

Let X be a Banach space. For a linear operator $A : D(A) \subset X \to X$, we consider the abstract Cauchy problem

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t), \quad t > 0 \tag{2.1}$$

$$u(0) = u_0, \tag{2.2}$$

where $u_0 \in X$.

Definition 2.1. *Let $u : [0, \infty) \to X$ be a continuous function. We say that $u = u(t)$ is a strong solution of (2.1)–(2.2) if it is continuously differentiable for all $t > 0$, $u(t) \in D(A)$ for all $t > 0$ and $u(t)$ satisfies the two conditions (2.1) and (2.2).*

In this section, we enunciate some definitions and useful results to prove existence and uniqueness of solution to the above abstract Cauchy problem. We denote by $\mathcal{L}(X)$ the set of all bounded linear operators $S : X \to X$. These definitions and results appear, for example, in references [21] and [39].

Definition 2.2. *A semigroup of bounded linear operators on X is a family $T = \{T(t); t \geq 0\} \subset \mathcal{L}(X)$ that satisfies:*

i. $T(0) = I$, *where $I : X \to X$ is the identity operator;*

ii. $T(s+t) = T(s)T(t)$ *for all $s, t \geq 0$.*

If, in addition, $\lim_{t\to 0^+} \|S(t)x - x\|_X = 0$, for all $x \in X$, we say that the semigroup T is strongly continuous or a C^0-semigroup. Moreover, if $\|T(t)\|_{\mathcal{L}(X)} \leq 1$ for all $t \geq 0$, T is called a C^0-semigroup of contractions.

Theorem 2.3. *Let T be a C^0-semigroup. For $x \in X$, the function $u : [0, \infty) \to X$ defined by $u(t) = T(t)x$ is continuous.*

Definition 2.4. *The infinitesimal generator of the C^0-semigroup T is the operator $A : D(A) \to X$ defined as follows:*

$$D(A) = \left\{ x \in X;\ \lim_{t\to 0^+} \frac{T(t)x - x}{t} \ \textit{exists} \right\},$$

and for $x \in D(A)$

$$Ax := \lim_{t\to 0^+} \frac{T(t)x - x}{t}.$$

Theorem 2.5. *Let T be a C^0-semigroup and $A : D(A) \to X$ its infinitesimal generator. Then A is a closed linear operator and $D(A)$ is dense in X. Furthermore, for $x \in D(A)$, $T(t)x \in D(A)$ for all $t \geq 0$, the application $t \mapsto T(t)x$ is differentiable*

$$\frac{\mathrm{d}}{\mathrm{d}t} T(t)x = AT(t)x = T(t)Ax.$$

If A is the infinitesimal generator of a C^0-semigroup T, the above theorem prove that the function defined by $u(t) := S(t)u_0$ is a strong solution to the problem (2.1)–(2.2) for $u_0 \in D(A)$. Under the same conditions, the uniqueness of solution is guaranteed by the next theorem.

Theorem 2.6. *Suppose $A : D(A) \to X$ is the infinitesimal generator of the C^0-semigroup T. Then, for $u_0 \in D(A)$, the problem (2.1)–(2.2) has a unique strong solution which is given by*

$$u(t) = T(t)u_0 \in C^1([0, \infty), D(A)).$$

Definition 2.7. *Let T be a C^0-semigroup and $A : D(A) \to X$ its infinitesimal generator. If the initial data $u_0 \in X$, we say that the function $u(t) := T(t)u_0$ is a weak solution to the problema (2.1)–(2.2). In this case,*

$$u \in C([0, \infty), X).$$

The next results are sufficient conditions for an linear operator to be an infinitesimal generator of a C^0-semigroup.

Remember that the set X' represents the (topological) dual of the Banach space X. For $f \in X$ and $\varphi \in X'$, we define

$$\langle \varphi, f \rangle := \varphi(f).$$

We define the *duality map* of X, $J : X \to 2^{X'}$, as

$$J(f) = \left\{ \varphi \in X'; \|\varphi\|_{X'}^2 = \|f\|_X^2 = \langle f, \varphi \rangle \right\}.$$

The set $J(f)$ is nonempety, due to Hanh Banach Theorem.

Definition 2.8. *Let $A : D(A) \to X$ be an linear operator. We say that A is dissipative if for each $f \in D(A)$, there exists $\varphi \in J(f)$ such that*

$$Re \langle A(f), \varphi \rangle \leq 0.$$

Remark 2.9. Let H be a Hilbert space with inner product $(\cdot,\cdot)_H : H \times H \to \boldsymbol{R}$ (or $\boldsymbol{C}$). From the Riesz Representation Theorem, we may prove that $A : D(A) \subset H \to H$ is dissipative if, and only if,

$$\mathrm{Re}(Ax, x)_H \leq 0$$

for all $x \in H$.

Theorem 2.10 (Lumer-Phillips)**.** *Let $A : D(A) \to X$ be a linear operator such that $D(A)$ is dense in X.*

i. If A is the infinitesimal generator of a C^0-semigroup of contractions, then A is dissipative and $Im(\lambda I - A) = X$ for all $\lambda > 0$. Furthermore, if $f \in D(A)$, then $Re \langle A(f), \varphi \rangle \leq 0$ for every $\varphi \in J(f)$.

ii. If A is dissipative and there exists $\lambda_0 > 0$ such that $Im(\lambda_0 I - A) = X$, then A is the infinitesimal generator of a C^0-semigroup of contractions on X.

Theorem 2.11. *[Theorem 6.4, [21]] Let A be generator of a C^0-semigroup in X and $B \in \mathcal{L}(X)$, that is, $B : X \to X$ is a bounded linear operator. Then $A + B$ generates a C^0-semigroup.*

Although the *Lax-Milgram Theorem* is not a result of semigroup theory, it is a very important tool to solve linear partial elliptic differential equations. It is also useful to prove that the conditions of the Lumer-Phillips Theorem (see 2.10) are satisfied. Therefore, we close this section by stating this result whose proof can be found in [1].

Let H be a real Hilbert space. Let $a : H \times H \to \boldsymbol{R}$ a bilinear form, that is, an application such that $a(\cdot, v) : H \to \boldsymbol{R}$ and $a(u, \cdot) : H \to \boldsymbol{R}$ are linear. If there exists a constant $C > 0$ such that

$$|a(u,v)| \leq C\|u\|_H \|v\|_H, \quad \forall u, v \in H,$$

we say that a is *continuous*. The bilinear form a is said to be *coercive*, if there is a constant $\alpha > 0$ such that

$$a(u,u) \geq \alpha \|u\|_H^2, \quad \forall u \in H.$$

Theorem 2.12 (Lax-Milgram)**.** *Let H be a real Hilbert space. If $a : H \times H \to \boldsymbol{R}$ is a continuous and coercive bilinear form and $\varphi \in H'$, then there exists a unique $u \in H$ such that*

$$a(u,v) = \langle \varphi, v \rangle, \quad \forall v \in H.$$

2.2 THE FOURIER TRANSFORM

In this section, we consider the Schwartz Space $\mathcal{S}(\boldsymbol{R}^n)$ of all C^∞ functions $f : \boldsymbol{R}^n \to \boldsymbol{R}$(or $\boldsymbol{C}$) of rapidly decreasing. More details and results about Fourier transform appear in reference [32].

Let f be a $L^1(\boldsymbol{R}^n)$ function. We define the *Fourier Transform* as

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) := (2\pi)^{-\frac{n}{2}} \int_{\boldsymbol{R}^n} e^{-ix\cdot\xi} f(x)dx, \quad \xi \in \boldsymbol{R}^n, \tag{2.3}$$

where $i := \sqrt{-1}$. It is easy to check that $\hat{f} \in L^\infty(\boldsymbol{R}^n)$ and that the map $L^1(\boldsymbol{R}^n) \ni f \mapsto \hat{f} \in L^\infty(\boldsymbol{R}^n)$ is a continuous linear function satisfying

$$\|\hat{u}\|_\infty \leq \|u\|_1.$$

Since $\mathcal{S}(\boldsymbol{R}^n) \subset L^1(\boldsymbol{R}^n)$, the Fourier Transform $\hat{u}$ is defined for all $u \in S(\boldsymbol{R}^n)$. It is possible to prove that if $u \in \mathcal{S}(\boldsymbol{R}^n)$, then $\hat{u} \in \mathcal{S}(\boldsymbol{R}^n)$. Therefore, the function

$$\begin{aligned} \mathcal{F} :\ & \mathcal{S}(\boldsymbol{R}^n) \to \mathcal{S}(\boldsymbol{R}^n) \\ & u \mapsto \hat{u} \end{aligned}$$

is well defined.

Proposition 2.13. *The function $\mathcal{F} : \mathcal{S}(\boldsymbol{R}^n) \to \mathcal{S}(\boldsymbol{R}^n)$ is bijective and*

$$\mathcal{F}^{-1}(u)(x) = (2\pi)^{-\frac{n}{2}} \int_{\boldsymbol{R}^n} e^{ix\cdot\xi} u(\xi)d\xi.$$

Furthermore, if $f \in \mathcal{S}(\boldsymbol{R}^n)$, then

$$\|u\| = \|\mathcal{F}u\| = \|\hat{u}\|,$$

where $\|\cdot\|$ is the L^2-norm.

Since $\mathcal{S}(\boldsymbol{R}^n)$ is dense in $L^2(\boldsymbol{R}^n)$ and the map $\mathcal{F} : \mathcal{S}(\boldsymbol{R}^n) \to \mathcal{S}(\boldsymbol{R}^n)$ is a L^2-isometry, there exists a unique extension of $\mathcal{F}$ to the space $L^2(\boldsymbol{R}^n)$, which is also L^2-isometry. For simplicity, we denote such extension by $\mathcal{F}$. We state the following theorem about Fourier transform in $L^2(\boldsymbol{R}^n)$.

Theorem 2.14 (Plancherel)**.** *There exists a unique isometry*

$$\mathcal{F} : L^2(\boldsymbol{R}^n) \to L^2(\boldsymbol{R}^n)$$

which is bijective and that satisfies

$$\mathcal{F}(u) = \hat{u}, \quad u \in \mathcal{S}(\boldsymbol{R}^n).$$

Furthermore, Parseval's identity also holds:

$$\int_{\boldsymbol{R}^n} f(x)\overline{g(x)}dx = \int_{\boldsymbol{R}^n} \mathcal{F}(f)(\xi)\overline{\mathcal{F}(g)(\xi)}d\xi.$$

We consider the dual space of $\mathcal{S}(\boldsymbol{R}^n)$, which is called the space of *tempered distributions* and denoted by $\mathcal{S}'(\boldsymbol{R}^n)$. The Fourier transform on $\mathcal{S}'(\boldsymbol{R}^n)$ is defined by duality: Let $T \in \mathcal{S}'(\boldsymbol{R}^n)$, then

$$\left\langle \widehat{T}, f \right\rangle = \left\langle T, \hat{f} \right\rangle, \quad f \in \mathcal{S}(\boldsymbol{R}^n).$$

2.3 THE SPACES Y^s

In this work, we study problems based on the logarithmic-Laplacian operator defined in (1.5). This section is devoted to discuss some results regarding the domain $D(L)$ (see (1.4)) of this operator.

Definition 2.15. *Let $s \in \boldsymbol{R}$. We define the space Y^s by*

$$Y^s = \left\{ f \in L^2(\boldsymbol{R}^n); \int_{\boldsymbol{R}^n} (1 + \log(1 + |\xi|^2))^s |\hat{f}(\xi)|^2 d\xi < \infty \right\}. \tag{2.4}$$

In this space we define the norm

$$\|f\|_{Y^s} := \left(\int_{\boldsymbol{R}^n} (1 + \log(1 + |\xi|^2))^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2}, \quad f \in Y^s. \tag{2.5}$$

Remark 2.16. Due to the fact that $\log(1 + |\xi|^2) \leq |\xi|^2$ for all $\xi \in \boldsymbol{R}^n$, one notices $H^s(\boldsymbol{R}^n) \subset Y^s \subset L^2(\boldsymbol{R}^n)$ for $s \geq 0$.

It is standard to verify that Y^s is vectorial space and that $\| \cdot \|_{Y^s} : Y^s \to \boldsymbol{R}$ is norm. The following lemma allows us to define another equivalent norm in Y^s.

Lemma 2.17. *Let $s \geq 0$ and $\xi \in \boldsymbol{R}^n$, then*

i. $\frac{1}{2}(1 + \log^s(1 + |\xi|^2)) \leq (1 + \log(1 + |\xi|^2))^s \leq 2^s(1 + \log^s(1 + |\xi|^2))$;

ii. $2^{-s}(1 + \log^s(1 + |\xi|^2))^{-1} \leq (1 + \log(1 + |\xi|^2))^{-s} \leq 2(1 + \log^s(1 + |\xi|^2))^{-1}$.

Proof. *i.* For $|\xi| \leq \sqrt{e-1}$, we have

$$1 \leq 1 + \log^r(1+|\xi|^2) \leq 2, \quad r \geq 0.$$

Then

$$\frac{1}{2}(1+\log^s(1+|\xi|^2)) \leq 1 \leq (1+\log(1+|\xi|^2))^s \leq 2^s \leq 2^s(1+\log^s(1+|\xi|^2)).$$

In the case $|\xi| \geq \sqrt{e-1}$, it holds that

$$1 + \log(1+|\xi|^2) \leq 2\log(1+|\xi|^2).$$

Thus,

$$\begin{aligned}\frac{1}{2}(1+\log^s(1+|\xi|^2)) \leq \log^s(1+|\xi|^2) \leq (1+\log(1+|\xi|^2))^s \leq (2\log(1+|\xi|^2))^s\\ \leq 2^s(1+\log^s(1+|\xi|^2)).\end{aligned}$$

ii. For $|\xi| \leq \sqrt{e-1}$,

$$1 \leq 1 + \log^r(1+|\xi|^2) \leq 2, \quad r \geq 0.$$

Then

$$2^{-s}(1+\log^s(1+|\xi|^2))^{-1} \leq 2^{-s} \leq (1+\log(1+|\xi|^2))^{-s} \leq 1 \leq 2(1+\log^s(1+|\xi|^2))^{-1}.$$

For $|\xi| \geq \sqrt{e-1}$, we first observe that

$$1 + \log^r(1+|\xi|^2) \leq 2\log^r(1+|\xi|^2), \quad r \geq 0.$$

So, for $s \geq 0$

$$\begin{aligned}(1+\log(1+|\xi|^2))^s \leq 2^s\log^s(1+|\xi|^2) \leq 2^s(1+\log^s(1+|\xi|^2)) \leq 2^s(2\log^s(1+|\xi|^2))\\ \leq 2^{s+1}(1+\log(1+|\xi|^2))^s.\end{aligned}$$

Thus,

$$2^{-s}(1+\log^s(1+|\xi|^2))^{-1} \leq (1+\log(1+|\xi|^2))^{-s} \leq 2(1+\log^s(1+|\xi|^2))^{-1}.$$

□

Therefore, we proved that

$$\|f\|_{Y^s} = \left(\int_{\boldsymbol{R}^n}(1+\log^s(1+|\xi|^2))|\hat{f}(\xi)|^2 d\xi\right)^{1/2}, \quad f \in Y^s$$

is equivalent to the norm (2.5).

In particular, we notice that $D(L) = Y^2$. As we have already mentioned in introduction chapter, the operator L is nonnegative and self-adjoint in $L^2(\boldsymbol{R}^n)$. Therefore, for $s > 0$, we may define the operator $L^{s/2} : Y^s \to L^2(\boldsymbol{R}^n)$ as

$$(L^{s/2}f)(x) := \mathcal{F}^{-1}\left(\log^{s/2}(1+|\xi|^2)\hat{f}(\xi)\right)(x), \quad f \in Y^s.$$

We may still observe that the norm $\|f\|_{Y^s}$ comes from the inner product

$$(f,g)_{Y^s} = \int_{\boldsymbol{R}^n} (1+\log^s(1+|\xi|^2))\hat{f}\overline{\hat{g}}d\xi. \tag{2.6}$$

It is easy to see that Y^s is complete and, therefore, Y^s is a Hilbert space.

Proposition 2.18. *Let r, s be positive real numbers with $r \le s$. Then the inclusion $Y^s \subset Y^r$ is dense.*

Proof. It is easy to verify that $Y^s \subset Y^r$. Let $f \in Y^r$, that is $(1+\log^{r/2}(1+|\xi|^2))\hat{f} \in L^2(\boldsymbol{R}^n)$. Then, the function g given by

$$\hat{g}(\xi) = \frac{1+\log^{r/2}(1+|\xi|^2)}{\sqrt{1+|\xi|^{2r}}}\hat{f}(\xi)$$

is an element of $H^r(\boldsymbol{R}^n)$, due to

$$\int_{R^n} (1+|\xi|^{2r})|\hat{u}(\xi)|^2 d\xi \approx \int_{R^n} (1+|\xi|^2)^r |\hat{u}(\xi)|^2 d\xi, \quad u \in H^r(R^n).$$

We know that the space $C_0^\infty(\boldsymbol{R}^n)$ is dense in $H^r(\boldsymbol{R}^n)$ for all $r \ge 0$. Then we consider a sequence $\{u_m\} \subset C_0^\infty(\boldsymbol{R}^n)$ such that

$$\|u_m - g\|_{H^r} \to 0.$$

Since $u_m \in C_0^\infty(\boldsymbol{R}^n) \subset H^r(\boldsymbol{R}^n)$ for all $m \in \boldsymbol{N}$, we have

$$\begin{aligned}
\int_{\boldsymbol{R}^n} &(1+\log^{s/2}(1+|\xi|^2))^2 \frac{1+|\xi|^{2r}}{(1+\log^{r/2}(1+|\xi|^2))^2}|\hat{u}_m|^2 d\xi \\
&\approx \int_{\boldsymbol{R}^n} (1+\log(1+|\xi|^2))^s \frac{(1+|\xi|^2)^r}{(1+\log(1+|\xi|^2))^r}|\hat{u}_m|^2 d\xi \\
&= \int_{\boldsymbol{R}^n} (1+\log(1+|\xi|^2))^{s-r}(1+|\xi|^2)^r |\hat{u}_m|^2 d\xi \\
&\le \int_{\boldsymbol{R}^n} (1+|\xi|^2)^{s-r}(1+|\xi|^2)^r |\hat{u}_m|^2 d\xi \\
&= \int_{\boldsymbol{R}^n} (1+|\xi|^2)^s |\hat{u}_m|^2 d\xi < \infty.
\end{aligned}$$

Then we may define, via Fourier transform, the sequence $\{v_m\} \subset Y^s$ by

$$\hat{v}_m := \frac{\sqrt{1+|\xi|^{2r}}}{1+\log^{r/2}(1+|\xi|^2)}\hat{u}_m.$$

Now, we observe that

$$\begin{aligned}
\|v_m - f\|_{Y^r}^2 &= \int_{\boldsymbol{R}^n} (1+\log^{r/2}(1+|\xi|^2))^2 \left| \frac{\sqrt{1+|\xi|^{2r}}}{1+\log^{r/2}(1+|\xi|^2)}\hat{u}_m - \hat{f} \right|^2 d\xi \\
&= \int_{\boldsymbol{R}^n} \left| \sqrt{1+|\xi|^{2r}}\hat{u}_m - (1+\log^{r/2}(1+|\xi|^2))\hat{f} \right|^2 d\xi.
\end{aligned}$$

Then,

$$\|v_m - f\|_{Y^r}^2 = \int_{\boldsymbol{R}^n} (1 + |\xi|^{2r}) \left| \hat{u}_m - \frac{1 + \log^{r/2}(1 + |\xi|^2)}{\sqrt{1 + |\xi|^{2r}}} \hat{f} \right|^2 d\xi$$
$$= \|u_m - g\|_{H^r}^2 \to 0, \quad m \to \infty.$$

Therefore, Y^s is dense in Y^r. □

In (1.2) and also when we study existence and uniqueness of solution of Cauchy problems associated to (1.1) the operator $(I + L)^{-1}$ appears. The next result shows that such operator is well defined in $L^2(\boldsymbol{R}^n)$.

Lemma 2.19. *Let $g \in L^2(\boldsymbol{R}^n)$. Then there exists a unique $f \in Y^2$ such that $(I + L)f = g$. In particular, we may define $(I + L)^{-1} g := f$.*

Proof. First we consider the linear functional $F : Y^1 \to \boldsymbol{R}$ given by

$$\langle F, \psi \rangle = (g, \psi).$$

We have

$$|\langle F, \psi \rangle| = |(g, \psi)| \leq \|g\| \|\psi\| \leq \|g\| \|\psi\|_{Y^1}.$$

Thus, F is continuous.

Now, we consider the symmetrical bilinear form $a : Y^1 \times Y^1 \to \boldsymbol{R}$

$$a(\varphi, \psi) = (\varphi, \psi) + (L^{1/2}\varphi, L^{1/2}\psi).$$

We observe that a is continuous, because

$$\begin{aligned} |a(\varphi, \psi)| &\leq |(\varphi, \psi)| + |(L^{1/2}\varphi, L^{1/2}\psi)| \\ &\leq \|\varphi\| \|\psi\| + \|L^{1/2}\varphi\| \|L^{1/2}\psi\| \\ &\leq 2\|\varphi\|_{Y^1} \|\psi\|_{Y^1}. \end{aligned}$$

Moreover, a is coercive. Indeed,

$$a(\varphi, \varphi) = (\varphi, \varphi) + (L^{1/2}\varphi, L^{1/2}\varphi) = \|\varphi\|_{Y^1}^2.$$

From the Lax-Milgram Theorem, there exists a unique $f \in Y^1$ such that

$$a(f, \psi) = \langle F, \psi \rangle, \tag{2.7}$$

for all $\psi \in Y^1$. In particular, (2.7) holds for all $\psi \in \mathcal{S}(\boldsymbol{R}^n)$ and we have the following equality em $\mathcal{S}'(\boldsymbol{R}^n)$:

$$f + Lf = g.$$

By applying Fourier transform, we obtain

$$\hat{f} + \log(1 + |\xi|^2)\hat{f} = \hat{g}$$

and

$$\int_{\boldsymbol{R}^n} (1 + \log(1 + |\xi|^2))|\hat{f}|d\xi = \int_{\boldsymbol{R}^n} |\hat{g}|d\xi < \infty.$$

Therefore, $f \in Y^2$ and (2.7) implies that

$$(f, \psi) + (Lf, \psi) = (g, \psi), \forall \psi \in C_0^\infty(\boldsymbol{R}^n).$$

From the density of $C_0^\infty(\boldsymbol{R}^n)$ in $L^2(\boldsymbol{R}^n)$, we have

$$(I + L)f = g$$

and the result is proved. □

2.4 TECHNICAL LEMMAS

In this section we introduce some lemmas to derive estimates of several quantities related to the solution of problems studied in this work.

The first lemma is very important to get estimates in Chapter 4 on the high frequency zone $|\xi| \geq \delta$, $\delta > 0$, in the Fourier space. It is similar to Lemma 2.2 in [22].

Lemma 2.20. *Let c, ν be positive real numbers and $a \in \boldsymbol{R}$. Then, there exists a constant $C > 0$ such that*

$$t^\nu e^{-c(1+\log(1+|\xi|^2))^a t} \leq C(1 + \log(1 + |\xi|^2))^{-a\nu}.$$

Proof. We set $s := c(1 + \log(1 + |\xi|^2))^a t$. Then $t = c^{-1}(1 + \log(1 + |\xi|^2))^{-a} s$ and

$$t^\nu = c^{-\nu}(1 + \log(1 + |\xi|^2))^{-a\nu} s^\nu.$$

The definition of s implies

$$t^\nu e^{-c(1+\log(1+|\xi|^2))^a t} = c^{-\nu}(1 + \log(1 + |\xi|^2))^{-a\nu} s^\nu e^{-s}.$$

Since the function $\boldsymbol{R} \ni s \mapsto s^\nu e^{-s}$ is bounded, there exists $C > 0$ such that

$$t^\nu e^{-c(1+\log(1+|\xi|^2))^a t} \leq C(1 + \log(1 + |\xi|^2))^{-a\nu}.$$

□

Lemma 2.21. *It holds that*

$$\frac{\sinh x}{x} \leq e^x,$$

for $x > 0$.

Proof. Let $x > 0$. From the mean value theorem, there exists $c \in (0, x)$ such that

$$\sinh(x) = x\cosh(c) \le xe^c \le xe^x.$$

□

The following lemma is used to get sharp estimate in Chapter 3.

Lemma 2.22. *The inequalities*

$$-1 \le \int_2^{2\sqrt{t}} \frac{\cos y}{y} dy \le 1$$

hold for all $t > 1$.

Proof. Using integration by parts we obtain for $t > 1$

$$\begin{aligned}
\Big|\int_2^{2\sqrt{t}} \frac{\cos y}{y} dy\Big| &= \Big|\frac{1}{y}\sin y\Big|_2^{2\sqrt{t}} + \int_2^{2\sqrt{t}} \frac{1}{y^2}\sin y\, dy\Big| \\
&\le \frac{|\sin(2\sqrt{t})|}{2\sqrt{t}} + \frac{|\sin 2|}{2} + \int_2^{2\sqrt{t}} \frac{1}{y^2}|\sin y| dy \\
&\le \frac{1}{2\sqrt{t}} + \frac{1}{2} + \int_2^{2\sqrt{t}} \frac{1}{y^2} dy \\
&= \frac{1}{2\sqrt{t}} + \frac{1}{2} + \frac{1}{2} - \frac{1}{2\sqrt{t}} = 1,
\end{aligned}$$

which implies the desired estimate. □

Remark 2.23. We note that a more precise estimate than that in Lemma 2.22 is

$$-1 < \int_2^{2\sqrt{t}} \frac{\cos y}{y} dy < 0, \quad t > 1.$$

However, it is a little more difficult to be proved. For our propose in this work, it is sufficient to use the rough estimate of Lemma 2.22.

Let $f \in L^1(\boldsymbol{R}^n)$. We may decompose the Fourier transform of f as follows:

$$\hat{f}(\xi) = A_f(\xi) - iB_f(\xi) + P_f, \tag{2.8}$$

for all $\xi \in \boldsymbol{R}^n$, where $i := \sqrt{-1}$ and

- $A_f(\xi) = \int_{\boldsymbol{R}^n} (\cos(x \cdot \xi) - 1) f(x) dx,$
- $B_f(\xi) = \int_{\boldsymbol{R}^n} \sin(x \cdot \xi) f(x) dx,$
- $P_f = \int_{\boldsymbol{R}^n} f(x) dx.$

Let $\kappa > 0$. We define the weighted L^1-space, $L^{1,\kappa}(\boldsymbol{R}^n)$, by

$$L^{1,\kappa}(\boldsymbol{R}^n) := \left\{ f \in L^1(\boldsymbol{R}^n) : \int_{\boldsymbol{R}^n} (1 + |x|^\kappa)|f(x)|dx < +\infty \right\}.$$

Then we state the next lemma about the decomposition (2.8). It can be proved in a standard way (see [24]).

Lemma 2.24. *i) If $f \in L^1(\boldsymbol{R}^n)$, then for all $\xi \in \boldsymbol{R}^n$ it is true that*

$$|A_f(\xi)| \leq L\|f\|_{L^1} \quad \textit{and} \quad |B_f(\xi)| \leq N\|f\|_{L^1}.$$

ii) If $0 < \kappa \leq 1$ and $f \in L^{1,\kappa}(\boldsymbol{R}^n)$, then for all $\xi \in \boldsymbol{R}^n$ it is true that

$$|A_f(\xi)| \leq K|\xi|^\kappa\|f\|_{L^{1,\kappa}} \quad \textit{and} \quad |B_f(\xi)| \leq M|\xi|^\kappa\|f\|_{L^{1,\kappa}}$$

with L, N, K and M positive constants depending only on the dimension n and/or κ.

2.5 ASYMPTOTIC LEMMAS

In the final part of this chapter, we discuss about the integrals below, which are already studied and developed in the works [4, 6].

$$I_p(t) = \int_0^1 (1 + r^2)^{-t} r^p dr, \quad p > -1, \tag{2.9}$$

$$J_p(t) = \int_1^\infty (1 + r^2)^{-t} r^p dr, \quad p \in \boldsymbol{R}. \tag{2.10}$$

In [6] Charão-Ikehata found sharp estimates to (2.9) for $p \geq 0$ by integral calculus. Then, after that, Charão-D'Abbicco-Ikehata [4] generalized such results for $p > -1$. They used theory of hypergeometric functions associated to the Beta and Gamma functions combined with the Gautschi inequality (see Watson [47] to definition and properties of Hypergeometric functions).

The next lemma is important to get estimates on the zone of high frequency for the problems studied in this work. It implies that $J_p(t)$ decays exponentially.

Lemma 2.25. *Let $p \in \boldsymbol{R}$. Then it holds that*

$$J_p(t) \sim \frac{2^{-t}}{t-1}, \quad t \gg 1.$$

Proof. The following proof of this lemma was made by Charão-Ikehata [6].
Due to $(1 + r^2)^{-t} = e^{-t\log(1+r^2)}$, we may rewrite $J_p(t)$ as

$$J_p(t) = \int_1^\infty e^{-t\log(1+r^2)} r^p dr, \quad t > 1.$$

Applying the change of variable $u = \log(1 + r^2)$, we obtain

$$J_p(t) = \frac{1}{2} \int_{\log 2}^{\infty} e^{-(t-1)u} (e^u - 1)^{\frac{p-1}{2}} du. \tag{2.11}$$

For $u \geq \log 2$, it holds that

$$1 \leq e^u - 1 \leq e^u. \tag{2.12}$$

Assuming $p < 1$, we have

$$e^{\frac{p-1}{2}u} \leq (e^u - 1)^{\frac{p-1}{2}} \leq 1.$$

Using both inequalities, from (2.11) we get

$$2^{\frac{p-1}{2}} \frac{2^{-t}}{t - \frac{p+1}{2}} = \frac{1}{2} \int_{\log 2}^{\infty} e^{-(t - \frac{p+1}{2})u} du \leq J_p(t) \leq \frac{1}{2} \int_{\log 2}^{\infty} e^{-(t-1)u} du = \frac{2^{-t}}{t-1}, \quad t > 1.$$

Now, for $p \geq 1$, from (2.12) we have

$$1 \leq (e^u - 1)^{\frac{p-1}{2}} \leq e^{\frac{p-1}{2}u}.$$

Therefore,

$$\frac{2^{-t}}{t-1} \leq J_p(t) \leq 2^{\frac{p-1}{2}} \frac{2^{-t}}{t - \frac{p+1}{2}}, \quad t > 1.$$

Then the result is proved for all $p \in \boldsymbol{R}$. □

The next step is to get estimates as in [4,6] for $I_p(t)$. We derive the same estimates by using the idea from [4, Lemma 2.1], but without using hypergeometric functions.

Lemma 2.26. *Let $\mu > 0$. Then there exist positive constants C_1, C_2 depending only on μ such that*

$$C_1 t^{-\mu} \leq \int_0^{\infty} \frac{x^{\mu-1}}{(1+x)^t} dx \leq C_2 t^{-\mu}, \quad t \gg 1.$$

Proof. For $\mu > 0$ and $t > \mu$, by combining [8.380] with [8.384] in [20], we have

$$\int_0^{\infty} \frac{x^{\mu-1}}{(1+x)^t} dx = \frac{\Gamma(\mu)\Gamma(t-\mu)}{\Gamma(t)}, \tag{2.13}$$

where Γ is the Gamma function:

$$\Gamma(z) = \int_0^{\infty} y^{z-1} e^{-y} dy, \quad z > 0.$$

In [48], Wendel proved that, for real numbers μ and t, the limit holds

$$\lim_{t\to\infty} t^{\mu} \frac{\Gamma(t-\mu)}{\Gamma(t)} = 1. \tag{2.14}$$

Thus, from (2.13) and (2.14), we have

$$\lim_{t\to\infty} t^{\mu} \int_0^{\infty} \frac{x^{\mu-1}}{(1+x)^t} dx = \Gamma(\mu), \quad \mu > 0, t > \mu.$$

By definition of limit, there exists $T > \mu$ such that

$$\frac{1}{2}\Gamma(\mu) \le t^{\mu}\int_0^{\infty}\frac{x^{\mu-1}}{(1+x)^t}dx \le \frac{3}{2}\Gamma(\mu), \quad t \ge T.$$

Therefore,

$$C_1 t^{-\mu} \le \int_0^{\infty}\frac{x^{\mu-1}}{(1+x)^t}dx \le C_2 t^{-\mu}, \quad t > T,$$

where C_1 and C_2 are multiples of $\Gamma(\mu) > 0$. □

Lemma 2.27. *Let $p > -1$. There exist positive constants C_1, C_2 depending only on p such that*

$$C_1 t^{-\frac{p+1}{2}} \le \int_0^{\infty}(1+r^2)^{-t}r^p dr \le C_2 t^{-\frac{p+1}{2}}, \quad t \gg 1.$$

Proof. We notice that

$$\int_0^{\infty}(1+r^2)^{-t}r^p dr = \frac{1}{2}\int_0^{\infty}\frac{x^{\frac{p-1}{2}}}{(1+x)^t}dx = \frac{1}{2}\int_0^{\infty}\frac{x^{\frac{p+1}{2}-1}}{(1+x)^t}dx.$$

Since $p > -1$, we may apply Lemma 2.26 for $\mu = \frac{p+1}{2} > 0$. Therefore,

$$C_1 t^{-\frac{p+1}{2}} \le \int_0^{\infty}(1+r^2)^{-t}r^p dr \le C_2 t^{-\frac{p+1}{2}},$$

where C_1, C_2 are constants depending only on p. □

Lemma 2.28. *Let $p > -1$ be a real number. Then*

$$I_p(t) \sim t^{-\frac{p+1}{2}}, \quad t \gg 1.$$

Proof. The non elementary proof of this lemma was also made originally in Charão-Ikehata [6] to the case $p \ge 0$ by using differential and integral calculus.

The upper estimate is immediate from Lemma 2.26, because

$$\int_0^{1}(1+r^2)^{-t}r^p dr \le \int_0^{\infty}(1+r^2)^{-t}r^p dr \le C_2 t^{-\frac{p+1}{2}}, \quad t \gg 1.$$

On the other hand, from Lemmas 2.25 and 2.26, we have

$$\begin{aligned}\int_0^{1}(1+r^2)^{-t}r^p dr + C_3\frac{2^{-t}}{t-1} &\ge \int_0^{1}(1+r^2)^{-t}r^p dr + \int_1^{\infty}(1+r^2)^{-t}r^p dr \\ &\ge C_1 t^{-\frac{p+1}{2}}, \quad t \gg 1.\end{aligned}$$

Then

$$\int_0^{1}(1+r^2)^{-t}r^p dr \ge t^{-\frac{p+1}{2}}\left(C_1 - C_3\frac{t^{\frac{p+1}{2}}}{t-1}2^{-t}\right), \quad t \gg 1. \tag{2.15}$$

By noticing that

$$\lim_{t\to\infty}\left(C_1 - C_3\frac{t^{\frac{p+1}{2}}}{t-1}2^{-t}\right) = C_1,$$

there exists $T > 0$ such that

$$C_1 - C_3\frac{t^{-\frac{p+1}{2}}}{t-1}2^{-t} \geq \frac{C_1}{2}, \quad t \geq T.$$

Using this inequality in (2.15), we conclude

$$\int_0^1 (1+r^2)^{-t} r^p dr \geq \frac{C_1}{2} t^{-\frac{p+1}{2}}, \quad t \gg 1.$$

□

For later use we prepare the following simple lemma, which implies the exponential decay estimates of the middle frequency part.

Lemma 2.29. *Let $p \in \boldsymbol{R}$, and $\eta \in (0,1]$. Then there is a constant $C > 0$ such that*

$$\int_\eta^1 (1+r^2)^{-t} r^p dr \leq C(1+\eta^2)^{-t}, \quad t \geq 0.$$

Lemma 2.30. *Let $0 \leq \theta < 1$ and $q > -1$. Then*

$$\int_0^1 (1+r^{2-2\theta})^{-t} r^q dr \sim \frac{1}{1-\theta} t^{-\frac{q+1}{2(1-\theta)}}, \quad t \gg 1.$$

In particular, for $0 \leq \theta \leq 1/2$ and $q > -1$ it holds that

$$\int_0^1 (1+r^{2-2\theta})^{-t} r^q dr \sim t^{-\frac{q+1}{2(1-\theta)}}, \quad t \gg 1.$$

Proof. Let $s = r^{1-\theta}$. Then

$$\int_0^1 (1+r^{2-2\theta})^{-t} r^q dr = \frac{1}{1-\theta}\int_0^1 (1+s^2)^{-t} s^{\frac{q+\theta}{1-\theta}} ds.$$

Since $0 \leq \theta < 1$ and $q > -1$, we have $\frac{q+\theta}{1-\theta} > -1$. Thus, we can apply the Lemma 2.28 to obtain the result. □

Remark 2.31. Actually, for $\eta > 0$, $0 \leq \theta \leq 1/2$ and $q > -1$, it holds that

$$\int_0^\eta (1+r^{2-2\theta})^{-t} r^q dr \geq C t^{-\frac{q+1}{2(1-\theta)}}, \quad t \gg 1$$

for some constant $C > 0$ depending on each $\eta > 0$.

Indeed, it suffices to check the case of $0 < \eta < 1$. In this case, one notices

$$\int_0^\eta (1+r^{2-2\theta})^{-t} r^q dr = \int_0^1 (1+r^{2-2\theta})^{-t} r^q dr - \int_\eta^1 (1+r^{2-2\theta})^{-t} r^q dr,$$

and one has

$$\int_\eta^1 (1+r^{2-2\theta})^{-t} r^q dr \leq \frac{1}{1+q}(1-\eta^{q+1})(1+\eta^{2-2\theta})^{-t}.$$

Since the last term implies the exponential decay, the desired estimate can be derived soon via Lemma 2.30. □

Lemma 2.32. *Let $\theta > 0$ and $q > -1$. Then*

$$\int_0^1 (1+r^{2\theta})^{-t} r^q dr \sim \frac{1}{\theta} t^{-\frac{q+1}{2\theta}}, \quad t \gg 1.$$

Proof. We consider the change of variable $s = r^\theta$. Then

$$\int_0^1 (1+r^{2\theta})^{-t} r^q dr = \frac{1}{\theta} \int_0^1 (1+s^2)^{-t} s^{\frac{q+1-\theta}{\theta}} ds$$

for $t \geq 0$. Finally, $\frac{q+1-\theta}{\theta} > -1$ because of $q > -1$. From Lemma 2.28 the desired result follows. □

Lemma 2.33. *Let $\theta > 0$ and $q \in \boldsymbol{R}$. Then*

$$\int_1^\infty (1+r^{2\theta})^{-t} r^q dr \sim \frac{1}{\theta} \frac{2^{-t}}{t-1}, \quad t \gg 1.$$

Lemma 2.34. *Let $0 \leq \theta < 1$ and $q \in \boldsymbol{R}$. Then*

$$\int_1^\infty (1+r^{2-2\theta})^{-t} r^q dr \sim \frac{2^{-t}}{t-1}, \quad t \gg 1.$$

Proof of Lemmas 2.33 and 2.34. From Lemma 2.25 and the change of variables as in Lemmas 2.32 and 2.30, the result now follows. □

3 A DISSIPATIVE LOGARITHMIC TYPE EVOLUTION EQUATION

In this chapter we introduce the following Cauchy problem associated to a new wave-like model with a damping mechanism of logarithmic-Laplacian type

$$u_{tt} + Lu + Lu_t = 0, \quad (t,x) \in (0,\infty) \times \boldsymbol{R}^n, \tag{3.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \boldsymbol{R}^n, \tag{3.2}$$

where L is the operator defined in (1.4)-(1.5).

We prove that the problem (3.1)–(3.2) has an unique weak solution in the class

$$C([0,\infty), D(L^{1/2})) \cap C^1([0,\infty), L^2(\boldsymbol{R}^n))$$

by employing the Lumer-Phillips theorem.

The unique solution of problem (3.1)–(3.2) satisfies the energy identity

$$\frac{d}{dt}E_u(t) + \|L^{1/2}u_t\| = 0, \tag{3.3}$$

where the total energy is

$$E_u(t) := \frac{1}{2}\left(\|u_t(t,\cdot)\|_{L^2}^2 + \|L^{1/2}u(t,\cdot)\|_{L^2}^2\right). \tag{3.4}$$

The inequality (3.3) implies that the total energy is a non increasing function in time, due to the existence of some kind of dissipative term Lu_t. Based on multiplier method of (cf. [27, 46]), we obtain energy decay rates for $n \geq 1$. The same method allows to obtain decay rates of solution for $n \geq 3$. However this method is not effective to get estimates to the solution when $n = 1, 2$, because a strong regularity near the origin appear in these cases.

We also introduce an asymptotic profile as $t \to \infty$ to the solution of (3.1)–(3.2) in a simple form. Then based on this asymptotic profile we prove that the decay rate obtained via multiplier method for $n \geq 3$ is optimal. Through the asymptotic profile we were also able to get optimal estimates for the cases $n = 1, 2$ and we prove that the solution blows up in these cases.

Although we obtain the same estimates to the solution as in the classical wave equation with $L = -\Delta$, the above problem is a little more effective in the sense that the operator L is weaker than the Laplacian. Moreover due to the domain of the operator L contain $H^2(\boldsymbol{R}^n)$, the initial data in (3.1)–(3.2) can be more general.

The results obtained in sections 3.2, 3.3 and 3.4 of this chapter was published in 2022 in *Journal of Mathematical Analysis and Applications* (see [2]).

This chapter is organized as follows. In Section 3.1 we study the existence and uniqueness of solution of problem (3.1)–(3.2). In Section 3.2 we employ the energy method to get some energy and solution estimates. In Section 3.3 we derive the leading term (as $t \to \infty$) of the solution to problem (3.1)-(3.2). The final Section 3.4 is devoted to the derivation of the optimal decay rate of the L^2-norm of the solution in case of $n \geq 3$ and the infinite time blow-up in L^2-sense for solutions in dimension spaces $n = 1, 2$.

3.1 EXISTENCE AND UNIQUENESS

In this section, we study the existence and uniqueness of solution to the problem (3.1)–(3.2) based on Ikehata-Todorova-Yordanov idea [31], where the authors proved the existence and uniqueness of solutions to an abstract problem as (3.1)–(3.2). The total energy associated to this problem is

$$E(t) = \frac{\|u_t(t,\cdot)\|^2 + \|L^{1/2}u(t,\cdot)\|^2}{2}$$

and it satisfies the energy identity

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) + \|L^{1/2}u_t(t,\cdot)\|^2 = 0.$$

Thus, we define the energy space as

$$X := Y^1 \times L^2(\boldsymbol{R}^n),$$

where the space Y^1 is defined in (2.4) and it is the domain of the operator $L^{1/2}$. We remember that Y^2 is the domain of the operator L and we state the theorem of existence and uniqueness as follows.

Theorem 3.1. *Let $n \geq 1$. For initial data $u_0, u_1 \in Y^1$ that satisfy $u_0 + u_1 \in Y^2$, the problem (3.1)–(3.2) admits an unique strong solution $u = u(t)$ such that*

$$(u, u_t) \in C^1([0,\infty), D(B)),$$

where $D(B) = \{(u,v) \in Y^1 \times Y^1; u + v \in Y^2\}$. In particular,

$$u \in C^1([0,\infty), Y^1) \cap C^2([0,\infty), Y^1).$$

Furthermore, for initial data $(u_0, u_1) \in Y^1 \times L^2(\boldsymbol{R}^n)$, the problem admits an unique weak solution in the class

$$C([0,\infty), Y^1) \cap C^1([0,\infty), L^2(\boldsymbol{R}^n)).$$

Proof of Theorem 3.1

In order to prove Theorem 1, we need some results that we prove below. Our goal is to apply Lumer-Phillips theorem to some suitable operator. For this we determine this operator and prove that it satisfies the necessary hypothesis.

We set $v = u_t$, $U = \begin{pmatrix} u \\ v \end{pmatrix}$ and we have

$$\frac{\mathrm{d}}{\mathrm{d}t}U = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_t \\ u_{tt} \end{pmatrix}$$

Then,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}U &= \begin{pmatrix} u_t \\ -Lu - Lu_t \end{pmatrix} \\
&= \begin{pmatrix} 0 & I \\ -L & -L \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\
&= \begin{pmatrix} 0 & I \\ -L-I & -L \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ u \end{pmatrix} \\
&= BU + FU
\end{aligned}$$

where formally

$$BU = \begin{pmatrix} 0 & I \\ -L-I & -L \end{pmatrix} U$$

and

$$FU = \begin{pmatrix} 0 \\ u \end{pmatrix}. \tag{3.5}$$

We define the domain $D(B)$ of B as

$$D(B) = \{(u,v) \in Y^1 \times Y^1;\ u+v \in Y^2\},$$

where $Y^2 = D(L)$. We observe that $Y^2 \times Y^2 \subset D(B)$. Since $Y^2 \times Y^2$ is dense in X (see Proposition 2.18), the set $D(B)$ is dense in X.

The choices of B, $D(B)$ and F were made in order to prove that B is dissipative, $I - B$ is surjective and $F \in \mathcal{L}(X)$. Thus we may apply the Lumer-Phillips Theorem to prove that B is infinitesimal generator of a C^0-semigroup of contractions and then the perturbation generator Theorem 2.11 can also be applied.

Lemma 3.2. *The operator B is dissipative.*

Proof. In the space $X = Y^1 \times L^2(\boldsymbol{R}^n)$, we consider the following inner product

$$((u_1,v_1),(u_2,v_2))_X = (u_1,u_2)_{Y^1} + (v_1,v_2)$$

for $(u_j,v_j) \in Y^1 \times L^2(\boldsymbol{R}^n)$, $j = 1,2$. We remember that $(\cdot,\cdot)_{Y^1}$ is given by

$$(u_1,u_2)_{Y^1} = \int_{\boldsymbol{R}^n} (1+\log(1+|\xi|^2))\hat{u}_1\overline{\hat{u}_2}d\xi.$$

For $(u,v) \in D(B)$, we have

$$\begin{aligned}
(B(u,v),(u,v))_X &= ((v, -u - L(u+v)),(u,v))_X \\
&= (v,u)_{Y^1} + (-u - L(u+v), v).
\end{aligned}$$

Then,

$$
\begin{aligned}
(B(u,v),(u,v))_X &= \int_{R^n}(1+\log(1+|\xi|^2))\hat{v}\overline{\hat{u}}d\xi - \int_{R^n}\log(1+|\xi|^2)(\hat{u}+\hat{v})\overline{\hat{v}}d\xi \\
&- \int_{R^n}\hat{u}\overline{\hat{v}}d\xi \\
&= \int_{R^n}(1+\log(1+|\xi|^2))\hat{v}\overline{\hat{u}}d\xi - \int_{R^n}(1+\log(1+|\xi|^2))\hat{u}\overline{\hat{v}}d\xi \\
&- \int_{R^n}\log(1+|\xi|^2)|\hat{v}|^2 d\xi \\
&= 2i\int_{R^n}(1+\log(1+|\xi|^2))\mathrm{Im}(\hat{v}\overline{\hat{u}})d\xi - \int_{R^n}\log(1+|\xi|^2)|\hat{v}|^2 d\xi
\end{aligned}
$$

where the notation Imz indicates the imaginary part of z. Thus

$$
\mathrm{Re}\Big((B(u,v),(u,v))_X\Big) - \int_{R^n}\log(1+|\xi|^2)|\hat{v}|^2 d\xi
$$

and this concludes that B is dissipative from Remark 2.9. □

Lemma 3.3. *The operator $I-B : D(B) \to X$ is surjective.*

Proof. Let $(u,v) \in D(B)$. Then

$$
\begin{aligned}
&u - v \in Y^1 \\
&u + v + L(u+v) \in L^2(\boldsymbol{R}^n).
\end{aligned}
$$

Thus, $(I-B)D(B) \subset X$.

On the other hand, let $(f,g) \in X$. Let us prove that there exists a pair $(u,v) \in D(B)$ that satisfies

$$
u - v = f \tag{3.6}
$$

$$
u + v + L(u+v) = g. \tag{3.7}
$$

From Lemma 2.19,

$$
u + v = (I+L)^{-1}g \in Y^2.
$$

Then

$$
2u = f + (I+L)^{-1}g.
$$

Applying the Fourier transform, we have

$$
2\hat{u} = \hat{f} + \frac{\hat{g}}{1+\log(1+|\xi|^2)}.
$$

Then,

$$
2\sqrt{1+\log(1+|\xi|^2)}\hat{u} = \sqrt{1+\log(1+|\xi|^2)}\hat{f} + \frac{\hat{g}}{\sqrt{1+\log(1+|\xi|^2)}}.
$$

And we may conclude, from Young's inequality, that

$$\begin{aligned}\int_{\boldsymbol{R}^n}(1+\log(1+|\xi|^2))|\hat{u}|^2 d\xi &\leq \int_{\boldsymbol{R}^n}(1+\log(1+|\xi|^2))|\hat{f}|^2 d\xi + \int_{\boldsymbol{R}^n}\frac{|\hat{g}|^2}{1+\log(1+|\xi|^2)}d\xi \\ &\leq \int_{\boldsymbol{R}^n}(1+\log(1+|\xi|^2))|\hat{f}|^2 d\xi + \int_{\boldsymbol{R}^n}|\hat{g}|^2 \\ &= \|f\|^2_{Y^1} + \|g\| < \infty.\end{aligned}$$

Thus $u \in Y^1$. Since $f \in Y^1$, we have $v = u - f \in Y^1$.

Therefore we prove that there exists a pair $(u, v) \in Y^1 \times Y^1$ such that $u + v \in Y^2$ that satisfies (3.6) and (3.7), that is,

$$X \subset (I - B)(D(B)).$$

□

Theorem 3.4. *The operator $B : D(B) \to X$ is infinitesimal generator of a C^0-semigroup of contractions.*

Proof. We know that $D(B)$ is dense in X, B is dissipative and $(I - B)(D(B)) = X$. The result follows from Lumer-Phillips Theorem. □

In order to prove the existence and uniqueness of solution to the problem (3.1)–(3.2), we still have to prove a result on the operador F given in (3.5) whose domain is X.

Lemma 3.5. *$F : D(F) \to X$ is a bounded linear operator.*

Proof. It is easy to see that the operator F is linear. Let $(u, v) \in X = D(L^{1/2}) \times L^2(\boldsymbol{R}^n)$. Then,

$$\|F(u, v)\|_X = \|(0, u)\|_X = \|u\| \leq \|u\|_{D(L^{1/2})} + \|v\| = \|(u, v)\|_X.$$

Therefore, the operator F is bounded. □

Since $B : D(B) \to X$ is infinitesimal generator of a C^0-semigroup of contractions and $F : X \to X$ is a bounded linear operator, we may apply Theorem 2.11 to obtain the following result.

Theorem 3.6. *The operator $B + F : D(B) \to X$ is infinitesimal generator of a C^0-semigroup $S(t)$ in X.*

Finally, we conclude from Theorem 2.6 that for initial data $(u_0, u_1) \in D(B)$ the problem (3.1)–(3.2) has a unique strong solution $u = u(t)$ such that

$$(u, u_t) \in C^1([0, \infty), D(B)).$$

Furthermore, for initial data $(u_0, u_1) \in Y^1 \times L^2(\boldsymbol{R}^n)$

$$(u, u_t)(t) = S(t)(u_0, u_1)$$

define the weak solution to the problem. Then Theorem 3.1 is proved.

□

3.2 ASYMPTOTIC BEHAVIOR VIA MULTIPLIER METHOD

In this section, we obtain estimates of the total energy of the following Fourier transformed equation together with initial data of the original system (3.1)-(3.2). To do so we employ the so-called energy method in the Fourier space developed in [46] and [27].

$$\widehat{u}_{tt} + \log(1+|\xi|^2)\widehat{u} + \log(1+|\xi|^2)\widehat{u}_t = 0, \quad (t,\xi) \in (0,\infty) \times \boldsymbol{R}^n, \tag{3.8}$$

$$\widehat{u}(0,\xi) = \widehat{u}_0(\xi), \quad \widehat{u}_t(0,\xi) = \widehat{u}_1(\xi), \quad \xi \in \boldsymbol{R}^n. \tag{3.9}$$

Multiplying the equation (3.8) by $\bar{\widehat{u}}_t$ one can get the following point wise energy identity

$$\frac{\mathrm{d}E_0(t,\xi)}{\mathrm{d}t} + \log(1+|\xi|^2)|\widehat{u}_t(t,\xi)|^2 = 0, \tag{3.10}$$

where

$$E_0(t,\xi) = \frac{|\widehat{u}_t(t,\xi)|^2}{2} + \log(1+|\xi|^2)\frac{|\widehat{u}(t,\xi)|^2}{2},$$

for $t > 0$ and $\xi \in \boldsymbol{R}^n$, is the total density of energy of the system (3.8)-(3.9). Note from (3.10) that $E_0(t,\xi)$ is a decreasing function of t for each ξ.

Now we define the following function of ξ. The way to choose the best $\rho(\xi)$-function is showed in the work by Luz-Ikehata-Charão [35]:

$$\rho(\xi) = \begin{cases} \frac{1}{2}\log(1+|\xi|^2) & \text{if } |\xi| \le \sqrt{e-1}, \\ \frac{1}{2} & \text{if } |\xi| \ge \sqrt{e-1}. \end{cases} \tag{3.11}$$

By multiplying the equation (3.8) by $\rho(\xi)\bar{\widehat{u}}$ we obtain the identity

$$\rho(\xi)\frac{\mathrm{d}}{\mathrm{d}t}\left(\widehat{u}_t\bar{\widehat{u}}\right) - \rho(\xi)|\widehat{u}_t|^2 + \log(1+|\xi|^2)\rho(\xi)|\widehat{u}|^2 + \log(1+|\xi|^2)\rho(\xi)\frac{\mathrm{d}}{\mathrm{d}t}\frac{|\widehat{u}|^2}{2} = 0,$$

for all $t > 0$ and $\xi \in \boldsymbol{R}^n$. Taking the real part on the last identity we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\rho(\xi)\mathrm{Re}\left(\widehat{u}_t\bar{\widehat{u}}\right) + \rho(\xi)\log(1+|\xi|^2)\frac{|\widehat{u}|^2}{2}\right] + \rho(\xi)\log(1+|\xi|^2)|\widehat{u}|^2 = \rho(\xi)|\widehat{u}_t|^2, \tag{3.12}$$

which holds for $t > 0$ and $\xi \in \boldsymbol{R}^n$.

To proceed further we define the following functions on $(0,\infty) \times \boldsymbol{R}^n$:

$$\begin{aligned} E(t,\xi) &= E_0(t,\xi) + \rho(\xi)\mathrm{Re}\left(\widehat{u}_t(t,\xi)\bar{\widehat{u}}(t,\xi)\right) + \frac{\rho(\xi)}{2}\log(1+|\xi|^2)|\widehat{u}(t,\xi)|^2, \\ F(t,\xi) &= \log(1+|\xi|^2)|\widehat{u}_t(t,\xi)|^2 + \rho(\xi)\log(1+|\xi|^2)|\widehat{u}(t,\xi)|^2, \\ R(t,\xi) &= \rho(\xi)|\widehat{u}_t(t,\xi)|^2. \end{aligned} \tag{3.13}$$

Then, adding (3.10) and (3.12), we get the following identity

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t,\xi)+F(t,\xi)=R(t,\xi), \tag{3.14}$$

which also holds for $t>0$ and $\xi\in\boldsymbol{R}^n$. Before continuing our argument, we need the next lemma.

Lemma 3.7. *The function $\rho(\xi)$ defined in (3.11) satisfies the estimates*

$$\rho(\xi)\leq\frac{1}{2}$$

for all $\xi\in\boldsymbol{R}^n$. Moreover,

$$\rho^2(\xi)\leq\frac{1}{4}\log(1+|\xi|^2)$$

for all $\xi\in\boldsymbol{R}^n$.

Proof. Indeed, for $|\xi|\leq\sqrt{e-1}$ we have $\log(1+|\xi|^2)|\leq 1$ which implies

$$\log(1+|\xi|^2)\leq\log^{\frac{1}{2}}(1+|\xi|^2).$$

Thus, $\rho^2(\xi)\leq\frac{1}{4}\log(1+|\xi|^2)$ and $\rho(\xi)\leq\frac{1}{2}$ according to the definition of $\rho(\xi)$ in (3.11).

For $|\xi|\geq\sqrt{e-1}$ one has $\log(1+|\xi|^2)\geq 1$. Thus, $\rho^2(\xi)=\frac{1}{4}\leq\frac{1}{4}\log(1+|\xi|^2)$. □

Lemma 3.8. *It holds that*

$$\frac{1}{2}E_0(t,\xi)\leq E(t,\xi)\leq 3E_0(t,\xi),\quad t>0,\ \xi\in\boldsymbol{R}^n.$$

Proof. Using the inequality $\rho(\xi)\mathrm{Re}\left(\widehat{u}_t\bar{\widehat{u}}\right)\geq-\frac{|\widehat{u}_t|^2}{4}-\rho^2(\xi)|\widehat{u}|^2$ and Lemma 3.7, one has

$$\begin{aligned}
E(t,\xi)&=E_0(t,\xi)+\rho(\xi)\mathrm{Re}\left(\widehat{u}_t\bar{\widehat{u}}\right)+\frac{\rho(\xi)}{2}\log(1+|\xi|^2)|\widehat{u}|^2\\
&\geq E_0(t,\xi)-\frac{|\widehat{u}_t|^2}{4}-\rho^2(\xi)|\widehat{u}|^2\\
&=\frac{|\widehat{u}_t|^2}{2}+\log(1+|\xi|^2)\frac{|\widehat{u}|^2}{2}-\frac{|\widehat{u}_t|^2}{4}-\rho^2(\xi)|\widehat{u}|^2\\
&=\frac{1}{4}|\widehat{u}_t|^2+\left(\frac{\log(1+|\xi|^2)}{2}-\rho^2(\xi)\right)|\widehat{u}|^2\\
&\geq\frac{1}{4}|\widehat{u}_t|^2+\frac{\log(1+|\xi|^2)}{4}|\widehat{u}|^2=\frac{1}{2}E_0(t,\xi),
\end{aligned}$$

which holds for $t>0$ and $\xi\in\boldsymbol{R}^n$.

On the other hand, using Lemma 3.7 one has the estimates

$$\begin{aligned}
E(t,\xi)&=E_0(t,\xi)+\rho(\xi)\mathrm{Re}\left(\widehat{u}_t\bar{\widehat{u}}\right)+\frac{\rho(\xi)}{2}\log(1+|\xi|^2)|\widehat{u}|^2\\
&\leq E_0(t,\xi)+\frac{|\widehat{u}_t|^2}{2}+\frac{\rho^2(\xi)}{2}|\widehat{u}|^2+\frac{\rho(\xi)}{2}\log(1+|\xi|^2)|\widehat{u}|^2\\
&\leq E_0(t,\xi)+\frac{|\widehat{u}_t|^2}{2}+\frac{\log(1+|\xi|^2)}{8}|\widehat{u}|^2+\frac{1}{4}\log(1+|\xi|^2)|\widehat{u}|^2\\
&\leq 3E_0(t,\xi),
\end{aligned}$$

which also holds for $t>0$ and $\xi\in\boldsymbol{R}^n$. □

Lemma 3.9. *Let $t > 0$ and $\xi \in \boldsymbol{R}^n$. Then*

$$\frac{d}{dt}E(t,\xi) + \frac{\rho(\xi)}{2}E(t,\xi) \leq 0.$$

Proof. The expressions (3.14), (3.13) and Lemma 3.8 imply that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E(t,\xi) + \frac{\rho(\xi)}{2}E(t,\xi) &= R(t,\xi) - F(t,\xi) + \frac{\rho(\xi)}{2}E(t,\xi) \\
&\leq R(t,\xi) - F(t,\xi) + \frac{3\rho(\xi)}{2}E_0(t,\xi) \\
&= \rho(\xi)|\widehat{u}_t|^2 - \log(1+|\xi|^2)|\widehat{u}_t|^2 - \rho(\xi)\log(1+|\xi|^2)|\widehat{u}|^2 \\
&+ \frac{3\rho(\xi)}{4}|\widehat{u}_t|^2 + \frac{3\rho(\xi)}{4}\log(1+|\xi|^2)|\widehat{u}|^2 \\
&= \left(\frac{7\rho(\xi)}{4} - \log(1+|\xi|^2)\right)|\widehat{u}_t|^2 - \frac{1}{4}\rho(\xi)\log(1+|\xi|^2)|\widehat{u}|^2 \\
&\leq 0,
\end{aligned}$$

where we have just used the fact that

$$\frac{7\rho(\xi)}{4} - \log(1+|\xi|^2) = \begin{cases} \frac{-1}{8}\log(1+|\xi|^2) & \text{if } |\xi| \leq \sqrt{e-1}, \\ \frac{7}{8} - \log(1+|\xi|^2) & \text{if } |\xi| > \sqrt{e-1}, \end{cases}$$

and the fact that $\log(1+|\xi|^2) \geq 1$ for $|\xi| > \sqrt{e-1}$. Therefore,

$$\frac{7}{8} - \log(1+|\xi|^2) < \frac{-1}{8}$$

for $|\xi| > \sqrt{e-1}$. □

We note that Lemma 3.9 implies

$$E(t,\xi) \leq E(0,\xi)e^{-\frac{\rho(\xi)}{2}t}.$$

Combining the last estimate with Lemma 3.8 we arrive at the important estimate:

$$E_0(t,\xi) \leq 6E_0(0,\xi)e^{-\frac{\rho(\xi)}{2}t},$$

for all $t > 0$ and $\xi \in \boldsymbol{R}^n$. Therefore using the definition of $E_0(t,\xi)$ we have obtained the important pointwise estimates in the Fourier space stated below.

Proposition 3.10. *It holds that*

$$|\widehat{u}_t(t,\xi)|^2 + \log(1+|\xi|^2)|\widehat{u}(t,\xi)|^2 \leq 6\left(|\widehat{u}_1(\xi)|^2 + \log(1+|\xi|^2)|\widehat{u}_0(\xi)|^2\right)e^{-\frac{\rho(\xi)}{2}t}, \quad (3.15)$$

for all $t > 0$ and $\xi \in \boldsymbol{R}^n$, and

$$|\widehat{u}(t,\xi)|^2 \leq 6\left(\frac{1}{\log(1+|\xi|^2)}|\widehat{u}_1(\xi)|^2 + |\widehat{u}_0(\xi)|^2\right)e^{-\frac{\rho(\xi)}{2}t}, \quad (3.16)$$

for all $t > 0$ and $\xi \in \boldsymbol{R}^n, \xi \neq 0$.

Then we use the results obtained in Proposition 3.10 to obtain decay rates for the energy of the system (3.8)–(3.9) and also for the solution of the problem.

Proposition 3.11. *Let $u(t,x)$ be the solution to problem (3.1)-(3.2) with initial data*

$$(u_0, u_1) \in \left(D(L^{1/2}) \cap L^1(\boldsymbol{R}^n)\right) \times \left(L^2(\boldsymbol{R}^n) \cap L^1(\boldsymbol{R}^n)\right).$$

Then, the total energy of this system satisfies

$$\begin{aligned}\|u_t(t,\cdot)\|^2 + \left\|L^{1/2}u(t,\cdot)\right\|^2 &\leq C\left(\|u_1\|^2_{L^1} t^{-\frac{n}{2}} + \|u_0\|^2_{L^1} t^{-\frac{n+2}{2}}\right)\\ &\quad + C2^{-\frac{t}{4}}\left(\|u_1\|^2_{L^2} + \|u_0\|^2_{L^2}\right) + Ce^{-\frac{t}{4}}E_u(0),\end{aligned}$$

for $t \gg 0$, where C is a positive constant depending only on n.

Proof. To begin with, applying the Plancherel Theorem and integrating the inequality (3.15) over $\boldsymbol{R}^n$ one has

$$\begin{aligned}\|u_t(t,\cdot)\|^2 + \left\|L^{1/2}u(t,\cdot)\right\|^2 &= \|\widehat{u}_t(t,\cdot)\|^2 + \left\|\log^{1/2}(1+|\cdot|^2)|\hat{u}(t,\cdot)\right\|^2\\ &= \int_{R^n}\left(|\widehat{u}_t|^2 + \log(1+|\xi|^2)|\widehat{u}|^2\right)d\xi\\ &\leq 6\int_{R^n}\left(|\widehat{u}_1|^2 + \log(1+|\xi|^2)|\widehat{u}_0|^2\right)e^{-\frac{\rho(\xi)}{2}t}d\xi\\ &= 6\int_{|\xi|\leq 1}|\widehat{u}_1|^2 e^{-\frac{\rho(\xi)}{2}t}d\xi + 6\int_{|\xi|\leq 1}\log(1+|\xi|^2)|\widehat{u}_0|^2 e^{-\frac{\rho(\xi)}{2}t}d\xi\\ &\quad + 6\int_{1<|\xi|\leq\sqrt{e-1}}|\widehat{u}_1|^2 e^{-\frac{\rho(\xi)}{2}t}d\xi + 6\int_{1<|\xi|\leq\sqrt{e-1}}\log(1+|\xi|^2)|\widehat{u}_0|^2 e^{-\frac{\rho(\xi)}{2}t}d\xi\\ &\quad + 6\int_{|\xi|>\sqrt{e-1}}|\widehat{u}_1|^2 e^{-\frac{\rho(\xi)}{2}t}d\xi + 6\int_{|\xi|>\sqrt{e-1}}\log(1+|\xi|^2)|\widehat{u}_0|^2 e^{-\frac{\rho(\xi)}{2}t}d\xi\\ &= 6(A_1 + A_2 + A_3),\end{aligned} \tag{3.17}$$

with A_i $(i = 1, 2, 3)$ according to the integrals on low, middle and high frequencies, respectively.

1) Estimate on the zone $|\xi| \leq 1$:

At this stage we assume that the initial data $u_0, u_1 \in L^1(\boldsymbol{R}^n)$. Then $\widehat{u}_0, \widehat{u}_1 \in L^\infty(\boldsymbol{R}^n)$ and

$$\|\widehat{u}_0\|_\infty \leq \|u_0\|_1 \quad \text{and} \quad \|\widehat{u}_1\|_\infty \leq \|u_1\|_1 .$$

On this zone we have $\rho(\xi) = \frac{1}{2}\log(1+|\xi|^2)$. Then, using the definition of $\rho(\xi)$ we may estimate the integrals on the low frequency region as follows.

$$\begin{aligned}A_1 &= \int_{|\xi|\leq 1}|\widehat{u}_1|^2 e^{-\frac{\log(1+|\xi|^2)}{4}t}d\xi + \int_{|\xi|\leq 1}\log(1+|\xi|^2)|\widehat{u}_0|^2 e^{-\frac{\log(1+|\xi|^2)}{4}t}d\xi\\ &= \int_{|\xi|\leq 1}|\widehat{u}_1|^2(1+|\xi|^2)^{-\frac{t}{4}}d\xi + \int_{|\xi|\leq 1}\log(1+|\xi|^2)|\widehat{u}_0|^2(1+|\xi|^2)^{-\frac{t}{4}}d\xi.\end{aligned}$$

Thus,

$$
\begin{aligned}
A_1 &\leq \|\widehat{u}_1\|_\infty^2 \int_{|\xi|\leq 1} (1+|\xi|^2)^{-\frac{t}{4}} d\xi + \|\widehat{u}_0\|_\infty^2 \int_{|\xi|\leq 1} \log(1+|\xi|^2)(1+|\xi|^2)^{-\frac{t}{4}} d\xi \\
&\leq \|u_1\|_1^2 \int_{|\xi|\leq 1} (1+|\xi|^2)^{-\frac{t}{4}} d\xi + \|u_0\|_1^2 \int_{|\xi|\leq 1} \log(1+|\xi|^2)(1+|\xi|^2)^{-\frac{t}{4}} d\xi \\
&= \|u_1\|_1^2 \, \omega_n \int_0^1 (1+r^2)^{-\frac{t}{4}} r^{n-1} dr + \|u_0\|_1^2 \, \omega_n \int_0^1 \log(1+r^2)(1+r^2)^{-\frac{t}{4}} r^{n-1} dr \\
&\leq \|u_1\|_1^2 \, \omega_n \int_0^1 (1+r^2)^{-\frac{t}{4}} r^{n-1} dr + \|u_0\|_1^2 \, \omega_n \int_0^1 (1+r^2)^{-\frac{t}{4}} r^{n+1} dr,
\end{aligned}
$$

because of the fact that $\log(1+r^2) \leq r^2$ for all $r \geq 0$. From Lemma 2.28 with (2.9), we may obtain

$$
\begin{aligned}
A_1 &\leq \|u_1\|_1^2 \, \omega_n I_{n-1}(t/4) + \|u_0\|_1^2 \, \omega_n I_{n+1}(t/4) \\
&\leq C_n \left(\|u_1\|_1^2 \, t^{-\frac{n}{2}} + \|u_0\|_1^2 \, t^{-\frac{n+2}{2}} \right), \quad t \gg 1,
\end{aligned}
$$

where C_n is a positive constant depending only on n.

2) Estimate on the middle frequency zone $1 \leq |\xi| \leq \sqrt{e-1}$**:**

In this middle zone we also have $\rho(\xi) = \frac{1}{2}\log(1+|\xi|^2)$ and we may estimate $\log(1+|\xi|^2)$ by

$$\log 2 \leq \log(1+|\xi|^2) \leq 1.$$

Thus, one has

$$
\begin{aligned}
A_2 &= \int_{1\leq|\xi|\leq\sqrt{e-1}} |\widehat{u}_1|^2 e^{-\frac{\log(1+|\xi|^2)}{4}t} d\xi + \int_{1\leq|\xi|\leq\sqrt{e-1}} \log(1+|\xi|^2)|\widehat{u}_0|^2 e^{-\frac{\log(1+|\xi|^2)}{4}t} d\xi \\
&\leq \int_{1\leq|\xi|\leq\sqrt{e-1}} |\widehat{u}_1|^2 e^{-\frac{\log 2}{4}t} d\xi + \int_{1\leq|\xi|\leq\sqrt{e-1}} |\widehat{u}_0|^2 e^{-\frac{\log 2}{4}t} d\xi \\
&\leq 2^{-\frac{t}{4}} \|\widehat{u}_1\|_2^2 + 2^{-\frac{t}{4}} \|\widehat{u}_0\|_2^2 \\
&= 2^{-\frac{t}{4}} \left(\|u_1\|_2^2 + \|u_0\|_2^2 \right), \quad t > 0.
\end{aligned}
$$

3) Estimate on the high frequency zone $|\xi| \geq \sqrt{e-1}$**:**

On this region we have $\rho(\xi) = \frac{1}{2}$. Thus we obtain the estimate

$$
\begin{aligned}
A_3 &= \int_{|\xi|\geq\sqrt{e-1}} |\widehat{u}_1|^2 e^{-\frac{t}{4}} d\xi + \int_{|\xi|\geq\sqrt{e-1}} \log(1+|\xi|^2)|\widehat{u}_0|^2 e^{-\frac{t}{4}} d\xi \\
&\leq e^{-\frac{t}{4}} \|\widehat{u}_1\|_2^2 + e^{-\frac{t}{4}} \int_{\boldsymbol{R}^n} \log(1+|\xi|^2)|\widehat{u}_0|^2 d\xi \\
&= e^{-\frac{t}{4}} \left(\|u_1\|_2^2 + \|L^{1/2}u_0\|_2^2 \right) = 2e^{-\frac{t}{4}} E_u(0), \quad t > 0.
\end{aligned}
$$

By combining the estimates for A_1, A_2, A_3 with (3.17) the proof is now complete. □

Remark 3.12. The above proposition says that the total energy of the system decays as $t^{-n/2}$, that is

$$E_u(t) \leq C_{1,n}\left(E_u(0) + \|u_0\|_{L^2}^2 + \|u_0\|_{L^1}^2 + \|u_1\|_{L^1}^2\right) t^{-\frac{n}{2}}, \quad t \gg 1,$$

with a constant $C_{1,n} > 0$ depending only on n.

Proposition 3.13. *Let $n \geq 3$ and $u(t,x)$ be the solution to problem (3.1)-(3.2) with initial data*

$$u_0, u_1 \in L^2(\boldsymbol{R}^n) \cap L^1(\boldsymbol{R}^n).$$

Then

$$\|u(t,\cdot)\|_{L^2} \leq C_n \left(\|u_0\|_{L^2} + \|u_1\|_{L^2} + \|u_0\|_{L^1} + \|u_1\|_{L^1}\right) t^{-\frac{n-2}{4}}, \quad t \gg 1,$$

with a constant $C_n > 0$ depending only on n.

Proof. To estimate the L^2-norm of $u(t,x)$, we first observe that

$$\lim_{r \to 0} \frac{r^2}{\log(1+r^2)} = 1.$$

Thus, there exists a small $\delta \in (0,1)$ such that

$$\frac{1}{2} \leq \frac{r^2}{\log(1+r^2)} \leq \frac{3}{2}$$

for $0 < r \leq \delta$.

By integrating the inequality (3.16) on $\boldsymbol{R}^n$ and using the Plancherel theorem we obtain

$$\begin{aligned} \|u(t,\cdot)\|^2 &\leq 6 \int_{\boldsymbol{R}^n} \left(\frac{1}{\log(1+|\xi|^2)}|\widehat{u}_1|^2 + |\widehat{u}_0|^2\right) e^{-\frac{\rho(\xi)}{2}t} d\xi \\ &= 6 \int_{|\xi| \leq \delta} \left(\frac{1}{\log(1+|\xi|^2)}|\widehat{u}_1|^2 + |\widehat{u}_0|^2\right) e^{-\frac{\rho(\xi)}{2}t} d\xi \\ &+ 6 \int_{|\xi| > \delta} \left(\frac{1}{\log(1+|\xi|^2)}|\widehat{u}_1|^2 + |\widehat{u}_0|^2\right) e^{-\frac{\rho(\xi)}{2}t} d\xi \\ &=: 6(B_1 + B_2), \end{aligned} \tag{3.18}$$

where B_1 and B_2 are the integrals on the low and high frequency, respectively.

Analogous to the estimates for the energy we may obtain exponential decay to the integral B_2 on the high frequency zone $|\xi| > \delta$, that is,

$$B_2 \leq C\left(||u_0||_2^2 + ||u_1||_2^2\right) e^{-kt}, \quad t > 0,$$

where $k = \frac{\log(1+\delta^2)}{2}$.

On the low frequency region $|\xi| \leq \delta$, by using Lemma 2.28 together with (2.9) one has

$$
\begin{aligned}
B_1 &= \int_{|\xi|\leq\delta} \frac{1}{\log(1+|\xi|^2)} |\widehat{u}_1|^2 e^{-\frac{\log(1+|\xi|^2)}{4}t} d\xi + \int_{|\xi|\leq\delta} |\widehat{u}_0|^2 e^{-\frac{\log(1+|\xi|^2)}{4}t} d\xi \\
&\leq \|u_1\|_1^2 \int_{|\xi|\leq\delta} \frac{1}{\log(1+|\xi|^2)} e^{-\frac{\log(1+|\xi|^2)}{4}t} d\xi + \|u_0\|_1^2 \int_{|\xi|\leq\delta} (1+|\xi|^2)^{-\frac{t}{4}} d\xi \\
&\leq \|u_1\|_1^2 \omega_n \int_0^1 \frac{1}{\log(1+r^2)} (1+r^2)^{-\frac{t}{4}} r^{n-1} dr + \|u_0\|_1^2 \omega_n \int_0^1 (1+r^2)^{-\frac{t}{4}} r^{n-1} dr \\
&\leq \|u_1\|_1^2 \frac{3\omega_n}{2} \int_0^1 (1+r^2)^{-\frac{t}{4}} r^{n-3} dr + \|u_0\|_1^2 \omega_n \int_0^1 (1+r^2)^{-\frac{t}{4}} r^{n-1} dr \\
&= \|u_1\|_1^2 \frac{3\omega_n}{2} I_{n-3}(t/4) + \|u_0\|_1^2 \omega_n I_{n-1}(t/4) \\
&\leq C_n \left(\|u_1\|_1^2 t^{-\frac{n-2}{2}} + \|u_0\|_1^2 t^{-\frac{n}{2}} \right), \quad t \gg 1
\end{aligned}
$$

for $n \geq 3$, where $C_n > 0$ depends only on n. By combining estimates for B_1, B_2 with (3.18), we have just proved Proposition 3.13. □

Remark 3.14. The decay rate of the quantity $\|u(t,\cdot)\|$ can be derived only for the spatial dimension $n \geq 3$ under the L^1-regularity on the initial data. The cases $n = 1, 2$ have a strong singularity near 0-frequency region. In Subsection 3.4.2 we prove that, for $n = 1, 2$, the solution blows up on infinite time.

3.3 ASYMPTOTIC PROFILE OF SOLUTIONS

In order to investigate the optimality of decay rate obtained in Proposition 3.13 we do study the asymptotic profile of the solution $u(t,x)$ as $t \to \infty$ in L^2-sense. The asymptotic profile helps us to find optimal estimates to the solution of (3.8)–(3.9), which we cannot control in the cases $n = 1, 2$ using multipliers method (see Proposition 3.13).

To obtain an asymptotic profile we consider, without loss of generality, the case of initial amplitude $u_0 = 0$ (see Remark 3.17). Then, the corresponding Cauchy problem to problem (3.1)-(3.2) in the Fourier space is given by

$$
\begin{aligned}
&\hat{u}_{tt}(t,\xi) + \log(1+|\xi|^2)\hat{u}_t(t,\xi) + \log(1+|\xi|^2)\hat{u}(t,\xi) = 0, \quad t > 0, \ \xi \in \boldsymbol{R}^n, \qquad (3.19)\\
&\hat{u}(0,\xi) = 0, \quad \hat{u}_t(0,\xi) = \hat{u}_1(\xi), \quad \xi \in \boldsymbol{R}^n.
\end{aligned}
$$

The characteristics roots λ_+ and λ_- of the characteristic polynomial

$$\lambda^2 + \log(1+|\xi|^2)\lambda + \log(1+|\xi|^2) = 0, \quad \xi \in \boldsymbol{R}^n$$

associated to the equation (3.19) are given by

$$\lambda_{\pm} = \frac{-\log(1+|\xi|^2) \pm i\sqrt{4\log(1+|\xi|^2) - \log^2(1+|\xi|^2)}}{2}, \qquad (3.20)$$

for $|\xi| \le \sqrt{e^4-1}$. The solution formula can be expressed by

$$\hat{u}(t,\xi) = \frac{\hat{u}_1}{b(\xi)} e^{-a(\xi)t} \sin(b(\xi)t) \tag{3.21}$$

for small frequency region such that $|\xi| \le \sqrt{e^4-1}$, where $a(\xi)$ and $b(\xi)$ are the real and imaginary parts of the characteristics roots, that is

$$a(\xi) = \frac{\log(1+|\xi|^2)}{2} \text{ and } b(\xi) = \frac{\sqrt{4\log(1+|\xi|^2) - \log^2(1+|\xi|^2)}}{2}. \tag{3.22}$$

We note that $a(\xi)$ and $b(\xi)$ are well defined for $|\xi| \le 1$. In fact, it is easy to see that

$$4\log(1+|\xi|^2) - \log^2(1+|\xi|^2) > 0$$

for $0 \le |\xi| < \sqrt{e^4-1}$.

Remark 3.15. It holds that

$$\sqrt{\log(1+|\xi|^2)} \le 2b(\xi) \le 2\sqrt{\log(1+|\xi|^2)}$$

for $|\xi| \le 1$. To see this, we observe that

$$b(\xi) \;=\; \frac{\sqrt{4\log(1+|\xi|^2) - \log^2(1+|\xi|^2)}}{2} \le \frac{\sqrt{4\log(1+|\xi|^2)}}{2} = \sqrt{\log(1+|\xi|^2)},$$

for $|\xi| < \sqrt{e^4-1}$. On the other hand, for $|\xi| \le 1 \le \sqrt{e^3-1}$, we have

$$\begin{aligned} 1 \le |\xi|^2 + 1 \le e^3 \quad &\Leftrightarrow \quad 0 \le \log(1+\xi^2) \le 3 \Leftrightarrow \log^2(1+\xi^2) - 3\log(1+\xi^2) \le 0 \\ &\Leftrightarrow \quad \log(1+\xi^2) \le 4\log(1+\xi^2) - \log^2(1+\xi^2). \end{aligned}$$

Thus

$$\frac{\sqrt{\log(1+\xi^2)}}{2} \le b(\xi), \quad |\xi| \le 1 \le \sqrt{e^3-1}.$$

Let us capture a leading term of the solution based on (3.21) and decomposition of initial data given by (2.8). Assuming that the initial data $u_1 \in L^1(\boldsymbol{R}^n)$, we may write

$$\hat{u_1}(\xi) = A(\xi) - iB(\xi) + P_1,$$

where $A(\xi) := A_{u_1}(\xi)$, $B(\xi) := B_{u_1}(\xi)$ and $P_1 := P_{u_1}$ are defined in (2.8).

We may apply the mean value theorem to get

$$\sin(b(\xi)t) - \sin\left(t\sqrt{\log(1+|\xi|^2)}\right) = t\cos(\mu(\xi)t)\left[b(\xi) - \sqrt{\log(1+|\xi|^2)}\right], \tag{3.23}$$

where $\mu(\xi) = \theta b(\xi) + (1-\theta)\sqrt{\log(1+|\xi|^2)}$ for some $0 < \theta < 1$. For this reason, we can rewrite the solution formula (3.21) as

$$\begin{aligned} \hat{u}(t,\xi) = \frac{A(\xi) - iB(\xi)}{b(\xi)} e^{-a(\xi)t} \sin(b(\xi)t) + \frac{P_1}{b(\xi)} e^{-a(\xi)t} \sin\left(t\sqrt{\log(1+|\xi|^2)}\right) \\ + P_1 \frac{\left(b(\xi) - \sqrt{\log(1+|\xi|^2)}\right)}{b(\xi)} e^{-a(\xi)t} t\cos(\mu(\xi)t). \end{aligned} \tag{3.24}$$

Now, we define the following function

$$\varphi(t,\xi) = \frac{P_1}{\sqrt{\log(1+|\xi|^2)}} e^{-a(\xi)t} \sin\left(t\sqrt{\log(1+|\xi|^2)}\right), \tag{3.25}$$

which is equivalent to the second term of the right hand side of the above expression on the zone $|\xi| \le 1$ according to Remark 3.15. Subtracting $\varphi(t,\xi)$ from both sides of (3.24), we have

$$\hat{u}(t,\xi) - \varphi(t,\xi) = F_1(t,\xi) + F_2(t,\xi) + F_3(t,\xi), \tag{3.26}$$

where

$$\begin{aligned}
F_1(t,\xi) &= \frac{A(\xi) - iB(\xi)}{b(\xi)} e^{-a(\xi)t} \sin(b(\xi)t),\\
F_2(t,\xi) &= P_1 \frac{\left(b(\xi) - \sqrt{\log(1+|\xi|^2)}\right)}{b(\xi)} e^{-a(\xi)t} t \cos(\mu(\xi)t),\\
F_3(t,\xi) &= \frac{P_1}{b(\xi)} e^{-a(\xi)t} \sin\left(t\sqrt{\log(1+|\xi|^2)}\right) - \varphi(t,\xi).
\end{aligned}$$

The next step is to get decay estimates in time to the three terms defined above and so we assume $u_1 \in L^{1,1}(\boldsymbol{R}^n)$.

We know that

$$\lim_{r\to+0} \frac{r^2}{\log(1+r^2)} = 1,$$

so, there exists $0 < \delta_1 < 1$ such that

$$\frac{r^2}{\log(1+r^2)} < 2$$

for all $0 < r < \delta_1$. By using this fact, Remark 3.15, Lemma 2.24 with $\kappa = 1$, we obtain

$$\begin{aligned}
\int_{|\xi|\le\delta_1} |F_1(t,\xi)|^2 d\xi &\le 4\int_{|\xi|\le\delta_1} \frac{|A(\xi) - iB(\xi)|^2}{\log(1+|\xi|^2)} e^{-2a(\xi)t} \sin^2(b(\xi)t) d\xi\\
&\le 4\int_{|\xi|\le\delta_1} \frac{(|A(\xi)| + |B(\xi)|)^2}{\log(1+|\xi|^2)} e^{-2a(\xi)t} d\xi\\
&\le 4\int_{|\xi|\le\delta_1} \frac{(K+M)^2|\xi|^2\|u_1\|_{1,1}^2}{\log(1+|\xi|^2)} (1+|\xi|^2)^{-t} d\xi\\
&= 4\omega_n(K+M)^2\|u_1\|_{1,1}^2 \int_0^{\delta_1} \frac{r^{n+1}}{\log(1+r^2)} (1+r^2)^{-t} dr\\
&\le 4\omega_n(K+M)^2\|u_1\|_{1,1}^2 \int_0^{\delta_1} \frac{r^2}{\log(1+r^2)} (1+r^2)^{-t} r^{n-1} dr\\
&\le 8\omega_n(K+M)^2\|u_1\|_{1,1}^2 \int_0^{\delta_1} (1+r^2)^{-t} r^{n-1} dr\\
&\le 8\omega_n(K+M)^2\|u_1\|_{1,1}^2 \int_0^{1} (1+r^2)^{-t} r^{n-1} dr\\
&= 8\omega_n\|u_1\|_{1,1}^2 (K+M)^2 I_{n-1}(t)\\
&\le C_{1,n}\|u_1\|_{1,1}^2 t^{-\frac{n}{2}}, \quad t \gg 1,
\end{aligned} \tag{3.27}$$

where we just used Lemma 2.28.

Now, we observe that for $0 < r := |\xi| < \sqrt{e^4 - 1}$, we have

$$b(r) - \sqrt{\log(1+r^2)} = -\sqrt{\log(1+r^2)} \frac{\frac{\log^2(1+r^2)}{4\log(1+r^2)}}{1 + \sqrt{1 - \frac{\log^2(1+r^2)}{4\log(1+r^2)}}}.$$

Due to the fact that the denominator in the above equality is greater than 1, we have

$$\left| b(r) - \sqrt{\log(1+r^2)} \right| \le \sqrt{\log(1+r^2)} \frac{\log^2(1+r^2)}{4\log(1+r^2)},$$

for $0 < r < \sqrt{e^4 - 1}$. By combining this fact with Remark 3.15, we obtain

$$\frac{\left| b(r) - \sqrt{\log(1+r^2)} \right|^2}{|b(r)|^2} \le \log^2(1+r^2),$$

for $0 < r \le 1$. Also, we know that

$$\lim_{r \to +0} \frac{\log^2(1+r^2)}{r^4} = 1.$$

Thus there exists $0 < \delta < \delta_1$ such that

$$\frac{1}{2} \le \frac{\log^2(1+r^2)}{r^4} \le \frac{3}{2} \tag{3.28}$$

for all $0 \le r \le \delta$. These informations combined with Lemma 2.28 provide us the following sequence of estimates.

$$\begin{aligned}
\int_{|\xi| \le \delta} |F_2(t,\xi)|^2 d\xi &= \int_{|\xi| \le \delta} |P_1|^2 \frac{\left| b(\xi) - \sqrt{\log(1+|\xi|^2)} \right|^2}{|b(\xi)|^2} e^{-2a(\xi)t} t^2 \cos^2(\mu(\xi)t) d\xi \\
&\le |P_1|^2 t^2 \int_{|\xi| \le \delta} \frac{\left| b(\xi) - \sqrt{\log(1+|\xi|^2)} \right|^2}{|b(\xi)|^2} (1+|\xi|^2)^{-t} d\xi \\
&= |P_1|^2 t^2 \omega_n \int_0^\delta \frac{\left| b(r) - \sqrt{\log(1+r^2)} \right|^2}{|b(r)|^2} (1+r^2)^{-t} r^{n-1} dr \\
&\le |P_1|^2 t^2 \omega_n \int_0^\delta (1+r^2)^{-t} \frac{\log^2(1+r^2)}{r^4} r^{n+3} dr \\
&\le \frac{3\omega_n}{2} |P_1|^2 t^2 \int_0^\delta (1+r^2)^{-t} r^{n+3} dr \\
&\le \frac{3\omega_n}{2} |P_1|^2 t^2 \int_0^1 (1+r^2)^{-t} r^{n+3} dr \\
&\le \frac{3\omega_n}{2} |P_1|^2 t^2 I_{n+3}(t) \\
&\le C_{2,n} |P_1|^2 t^{-\frac{n}{2}}, \quad t \gg 1.
\end{aligned} \tag{3.29}$$

Finally, we estimate the function $F_3(t,\xi)$. First, we observe that

$$\frac{1}{b(\xi)} - \frac{1}{\sqrt{\log(1+|\xi|^2)}} = \frac{\sqrt{\log(1+|\xi|^2)}}{\sqrt{4-\log(1+|\xi|^2)}(2+\sqrt{4-\log(1+|\xi|^2)})}.$$

Due to $0 \le \log(1+|\xi|^2) \le \log 2$, for $|\xi| \le 1$, we have

$$\frac{1}{8}\sqrt{\log(1+|\xi|^2)} \le \frac{1}{b(\xi)} - \frac{1}{\sqrt{\log(1+|\xi|^2)}} \le K\sqrt{\log(1+|\xi|^2)} \tag{3.30}$$

for $0 < |\xi| \le 1$, where $K = \frac{1}{\sqrt{4-\log 2}(2+\sqrt{4-\log 2})}$.

Remembering that $\log(1+r^2) \le r^2$ for all $r \ge 0$ and $\sin^2(a) \le 1$ for all $a \in \boldsymbol{R}$, we obtain

$$\begin{aligned}\int_{|\xi|\le\delta} |F_3(t,\xi)|^2 d\xi &\le K|P_1|^2 \int_{|\xi|\le\delta} (1+|\xi|^2)^{-t}\log(1+|\xi|^2)\sin^2(t\sqrt{\log(1+|\xi|^2)})d\xi \\ &\le K|P_1|^2 \int_{|\xi|\le\delta}(1+|\xi|^2)^{-t}|\xi|^2 d\xi \\ &= K|P_1|^2\omega_n \int_0^\delta (1+r^2)^{-t} r^{n+1} dr.\end{aligned}$$

From Lemma 2.28,

$$\int_{|\xi|\le\delta} |F_3(t,\xi)|^2 d\xi \le C|P_1|^2 t^{-\frac{n+2}{2}}, \quad t \gg 1. \tag{3.31}$$

Then we have the following result, which implies that the leading term of the Fourier transformed solution is the very $\varphi(t,\xi)$. The result holds for all $n \ge 1$.

Theorem 3.16. *Let $n \ge 1$, $u_0 = 0$ and $u_1 \in \left(L^2(\boldsymbol{R}^n) \cap L^{1,1}(\boldsymbol{R}^n)\right)$. Then, the unique solution $u(t,x)$ to problem (3.1)-(3.2) satisfies*

$$\left\| u(t,\cdot) - \left(\int_{\boldsymbol{R}^n} u_1(x)dx\right)\mathcal{F}^{-1}\left((1+|\xi|^2)^{-\frac{t}{2}}\frac{\sin\left(t\sqrt{\log(1+|\xi|^2)}\right)}{\sqrt{\log(1+|\xi|^2)}}\right)\right\| \le I_0 t^{-\frac{n}{4}},$$

for $t \gg 1$, with

$$I_0 := \|u_1\| + \|u_1\|_{1,1}.$$

Proof. Let $\delta < 1$ be a positive number as in (3.28). From (3.26), (3.27), (3.29) and (3.31), we may derive

$$\begin{aligned}\int_{|\xi|\le\delta} |\hat{u}(t,\xi) - \varphi(t,\xi)|^2 d\xi &\le 4\int_{|\xi|\le\delta}\left(|F_1(t,\xi)|^2 + |F_2(t,\xi)|^2 + |F_3(t,\xi)|^2\right)d\xi \\ &\le C\left(\|u_1\|_{1,1}^2 t^{-\frac{n}{2}} + |P_1|^2 t^{-\frac{n}{2}} + |P_1|^2 t^{-\frac{n+2}{2}}\right) \\ &\le 2C\left(\|u_1\|_{1,1}^2 t^{-\frac{n}{2}} + |P_1|^2 t^{-\frac{n}{2}}\right), \quad t \gg 1 \end{aligned} \tag{3.32}$$

with some generous constant $C > 0$.

In the zone of high frequency $\{|\xi| \geq \delta\}$ we have the following estimates. From Proposition 3.10, it follows that

$$|\hat{u}(t,\xi)|^2 \leq 6\frac{|\hat{u}_1(\xi)|^2}{\log(1+|\xi|^2)}e^{-\frac{t}{4}} \leq 6|\hat{u}_1(\xi)|^2 e^{-\frac{t}{4}}, \quad |\xi| \geq \sqrt{e-1}. \tag{3.33}$$

Additionally,

$$\begin{aligned}|\varphi(t,\xi)|^2 &\leq |P_1|^2 \frac{1}{\log(1+|\xi|^2)}(1+|\xi|^2)^{-t}\sin^2(t\sqrt{\log(1+|\xi|^2)}) \\ &\leq |P_1|^2(1+|\xi|^2)^{-t}, \quad |\xi| \geq \sqrt{e-1}.\end{aligned} \tag{3.34}$$

Thus, (2.10), (3.33), (3.34) and Lemma 2.25 imply

$$\begin{aligned}\int_{|\xi|\geq\sqrt{e-1}} |\hat{u}(t,\xi)-\varphi(t,\xi)|^2 d\xi &\leq 2\int_{|\xi|\geq\sqrt{e-1}} (|\hat{u}(t,\xi)|^2 + |\varphi(t,\xi)|^2)d\xi \\ &\leq 12e^{-\frac{t}{4}}\|u_1\|^2 + 2|P_1|^2\omega_n \int_1^\infty (1+r^2)^{-t} r^{n-1}dr \\ &= 12e^{-\frac{t}{4}}\|u_1\|^2 + 2|P_1|^2\omega_n J_{n-1}(t) \\ &\leq 12e^{-\frac{t}{4}}\|u_1\|^2 + 2|P_1|^2\omega_n \frac{2^{-t}}{t-1}, \quad t \gg 1.\end{aligned} \tag{3.35}$$

Similarly to the derivation of (3.35), if $\delta \leq |\xi| \leq \sqrt{e-1}$, then

$$\log(1+\delta^2) \leq \log(1+r^2) \leq 1,$$

so that from Proposition 3.10 one can get

$$|\hat{u}(t,\xi)|^2 \leq 6\frac{|\hat{u}_1(\xi)|^2}{\log(1+|\xi|^2)}e^{-\frac{\rho(\xi)}{2}t} \leq 6\frac{|\hat{u}_1(\xi)|^2}{\log(1+\delta^2)}e^{-\frac{\log(1+\delta^2)}{4}t}, \tag{3.36}$$

and

$$|\varphi(t,\xi)|^2 \leq \frac{|P_1|^2}{\log(1+|\xi|^2)}(1+|\xi|^2)^{-t} \leq \frac{|P_1|^2}{\log(1+\delta^2)}(1+\delta^2)^{-t}, \tag{3.37}$$

for $\delta \leq |\xi| \leq \sqrt{e-1}$.

The estimates (3.35), (3.36) and (3.37) in the high and middle frequency zones $|\xi| \geq \sqrt{e-1}$ and $\delta \leq |\xi| \leq \sqrt{e-1}$ imply the following exponential decay estimate

$$\int_{|\xi|\geq\delta} |\hat{u}(t,\xi)-\varphi(t,\xi)|^2 d\xi \leq C(\|u_1\|^2 + |P_1|^2)e^{-\eta t}, \tag{3.38}$$

for $t \gg 1$ with positive constants C and η.

From (3.32), (3.38) and Plancherel Theorem, the result follows. □

Remark 3.17. In addition if we suppose that the initial data $u_0 \neq 0$ with $u_0 \in L^1(\boldsymbol{R}^n) \cap L^2(\boldsymbol{R}^n)$, then $\varphi(t,\cdot)$ remains as a leading term. In fact, the part of the solution in the low frequency region to the problem in the Fourier space (3.8)–(3.9) that corresponds to the initial data $\hat{u}_0$ is given by

$$v(t,\xi) := e^{-a(\xi)t}\cos(b(\xi)t)\hat{u}_0 + \frac{a(\xi)}{b(\xi)}e^{-a(\xi)t}\sin(b(\xi)t)\hat{u}_0.$$

From Remark 3.15, we have that $b(\xi) \approx \sqrt{\log(1+|\xi|^2)}$ and the singularity $\frac{a(\xi)}{b(\xi)}$ is removable. Thus it is easy to obtain

$$\int_{|\xi|\leq 1} |v(t,\xi)|^2 d\xi \leq C\|u_0\|_1 t^{-\frac{n}{2}}, \quad t \gg 1.$$

It is exactly the same decay rate as in (3.32).

Remark 3.18. If we apply the general theory developed in [31] to the abstract evolution equation

$$u_{tt} + Au + Au_t = 0, \tag{3.39}$$

where A is a nonnegative self-adjoint operator in a real Hilbert space, at least one can observe the asymptotic profile of the solution to problem (3.1)-(3.2) is

$$e^{-tL/2}\frac{\sin(L^{1/2}t)}{L^{1/2}}u_1. \tag{3.40}$$

By restricting the initial data further to the class $L^{1,1}(\boldsymbol{R}^n)$, one can obtain the statement of Theorem 3.16.

3.4 OPTIMAL ESTIMATES: DECAY RATES AND BLOW-UP ON INFINITE TIME

In the previous section, we find the leading term for the solution to the problem (3.1)–(3.2):

$$\mathcal{F}^{-1}(\varphi(t,\xi)) = P_1\mathcal{F}^{-1}\left((1+|\xi|^2)^{-\frac{t}{2}}\frac{\sin\left(t\sqrt{\log(1+|\xi|^2)}\right)}{\sqrt{\log(1+|\xi|^2)}}\right), \tag{3.41}$$

where $\varphi(t,\xi)$ is given by (3.25). It is possible to get L^2-estimates to the solution $u(t,\cdot)$ from Theorem 3.16 as long as we know estimates to the leading term (3.41). In this section, we work to find sharp estimates in t for the following improper integrals

$$\mathcal{I}_n(t) := \int_{\boldsymbol{R}^n} \frac{(1+|\xi|^2)^{-t}\sin^2(t\sqrt{\log(1+|\xi|^2)})}{\log(1+|\xi|^2)}d\xi \tag{3.42}$$

for any dimension n. As a consequence, we were able to find optimal estimates for the solutions to the problem (3.1)–(3.2) even in cases $n = 1, 2$, which we could not obtain via the energy method (see Proposition (3.13)).

3.4.1 Optimal decay rate for $n \geq 3$

In this subsection, we investigate the precise decay rate of the leading term (3.25) in L^2-sense as $t \to \infty$. The case of $n \geq 3$ is first treated.

Lemma 3.19. *Let $n \geq 3$. Then there exists $t_0 > 0$ such that for $t \geq t_0$ it holds that*

$$C_{1,n} t^{-\frac{n-2}{2}} \leq \int_{\boldsymbol{R}^n} \frac{(1+|\xi|^2)^{-t} \sin^2(t\sqrt{\log(1+|\xi|^2)})}{\log(1+|\xi|^2)} d\xi \leq C_{2,n} t^{-\frac{n-2}{2}},$$

where $C_{1,n}$ and $C_{2,n}$ are positive constants depending only on n.

Proof. We first observe that

$$\lim_{r\to 0} \frac{r^2}{\log(1+r^2)} = 1.$$

Then, we can obtain $0 < \delta < 1$ such that

$$\frac{1}{2} \leq \frac{r^2}{\log(1+r^2)} \leq \frac{3}{2}$$

for $0 < r \leq \delta$. We have

$$\begin{aligned}
\int_{\boldsymbol{R}^n} \frac{(1+|\xi|^2)^{-t} \sin^2(t\sqrt{\log(1+|\xi|^2)})}{\log(1+|\xi|^2)} d\xi &= \int_0^\infty \frac{(1+r^2)^{-t} \sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)} r^{n-1} dr \\
&\leq \int_0^\infty \frac{(1+r^2)^{-t}}{\log(1+r^2)} r^{n-1} dr \\
&= A_1(t) + A_2(t) + A_3(t), \qquad (3.43)
\end{aligned}$$

where

$$\begin{aligned}
A_1(t) &= \int_0^\delta \frac{(1+r^2)^{-t}}{\log(1+r^2)} r^{n-1} dr, \\
A_2(t) &= \int_\delta^1 \frac{(1+r^2)^{-t}}{\log(1+r^2)} r^{n-1} dr, \\
A_3(t) &= \int_1^\infty \frac{(1+r^2)^{-t}}{\log(1+r^2)} r^{n-1} dr.
\end{aligned}$$

Thus, from Lemma 2.28, we obtain

$$\begin{aligned}
A_1(t) &= \int_0^\delta \frac{r^2}{\log(1+r^2)} (1+r^2)^{-t} r^{n-3} dr \leq \frac{3}{2} \int_0^\delta (1+r^2)^{-t} r^{n-3} dr \\
&\leq \frac{3}{2} \int_0^1 (1+r^2)^{-t} r^{n-3} dr \leq C_{1,n} t^{-\frac{n-2}{2}}, \quad t \gg 1.
\end{aligned}$$

The estimate on the middle frequency zone $[\delta, 1]$ also follows from Lemma 2.28.

$$\begin{aligned}
A_2(t) &= \int_\delta^1 \frac{(1+r^2)^{-t}}{\log(1+r^2)} r^{n-1} dr \leq \frac{1}{\log(1+\delta^2)} \int_\delta^1 (1+r^2)^{-t} r^{n-1} dr \\
&\leq \frac{1}{\log(1+\delta^2)} \int_\delta^1 (1+r^2)^{-t} r^{n-1} dr \leq \frac{1}{\log(1+\delta^2)} \int_0^1 (1+r^2)^{-t} r^{n-1} dr \\
&\leq C_{2,n,\delta} t^{-\frac{n}{2}}.
\end{aligned}$$

Finally,

$$
\begin{aligned}
A_3(t) &= \int_1^\infty \frac{(1+r^2)^{-t}}{\log(1+r^2)} r^{n-1} dr \le \frac{1}{\log 2} \int_1^\infty (1+r^2)^{-t} r^{n-1} dr \\
&\le C_{3,n} \frac{2^{-t}}{t-1}, \quad t \gg 1,
\end{aligned}
$$

due to Lemma 2.25.

The three estimates above combined with (3.43) imply that there exists $t_0 > 0$ such that

$$
\int_0^\infty \frac{(1+r^2)^{-t} \sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)} r^{n-1} dr \le C_{4,n} t^{-\frac{n-2}{2}} \tag{3.44}
$$

for all $t \ge t_0$.

On the other hand, one notices the following computation

$$
\begin{aligned}
M(t) &:= \int_0^\infty \frac{(1+r^2)^{-t} \sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)} r^{n-1} dr \\
&= \int_0^\infty \frac{(1+r^2)^{-t}(1+r^2) \sin^2(t\sqrt{\log(1+r^2)}) r^{n-2} \sqrt{t} r}{\sqrt{t}\sqrt{\log(1+r^2)}(1+r^2)\sqrt{\log(1+r^2)}} dr \\
&\ge \int_0^\infty \frac{(1+r^2)^{-t} \sin^2(t\sqrt{\log(1+r^2)}) r^{n-2} \sqrt{t} r}{\sqrt{t}\sqrt{\log(1+r^2)}(1+r^2)\sqrt{\log(1+r^2)}} dr, \quad t > 0.
\end{aligned}
$$

And also, it is known that $r^2 \ge \log(1+r^2)$ for $r \ge 0$, so that by using the change of variable $y = \sqrt{t}\sqrt{\log(1+r^2)}$ we have

$$
\begin{aligned}
M(t) &\ge \int_0^\infty \frac{(1+r^2)^{-t} \sin^2(t\sqrt{\log(1+r^2)})(\log(1+r^2))^{\frac{n-2}{2}} \sqrt{t} r}{\sqrt{t}\sqrt{\log(1+r^2)}(1+r^2)\sqrt{\log(1+r^2)}} dr \\
&= \int_0^\infty \frac{e^{-y^2} \sin^2(\sqrt{t} y) y^{n-2}}{t^{\frac{n-2}{2}} y} dy \\
&= \frac{t^{-\frac{n-2}{2}}}{2} \int_0^\infty e^{-y^2} y^{n-3} dy - \frac{t^{-\frac{n-2}{2}}}{2} \int_0^\infty e^{-y^2} y^{n-3} \cos(2\sqrt{t} y) dy \\
&= \frac{t^{-\frac{n-2}{2}}}{2} (A_n - F_n(t)),
\end{aligned}
$$

where

$$
A_n := \int_0^\infty e^{-y^2} y^{n-3} dy \text{ and } F_n(t) := \int_0^\infty e^{-y^2} y^{n-3} \cos(2\sqrt{t} y) dy.
$$

Due to the fact $e^{-y^2} y^{n-3} \in L^1(\boldsymbol{R})$ for $n \ge 3$, we can apply the Riemann-Lebesgue Lemma to get

$$
F_n(t) \to 0, \quad (t \to \infty).
$$

Then there exists $t_1 > t_0$ such that $F_n(t) \le \frac{A_n}{2}$ for all $t \ge t_1$, that is

$$
A_n - F_n(t) \ge \frac{A_n}{2} \text{ for all } t \ge t_1.
$$

Thus, one has

$$\int_0^\infty \frac{(1+r^2)^{-t}\sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)} r^{n-1} dr \geq \frac{A_n}{4} t^{-\frac{n-2}{2}} \tag{3.45}$$

for $t \geq t_1$.

Finally, the desired statement can be obtained from (3.44) and (3.45). □

3.4.2 Blow-up on infinite time for $n = 1$ and $n = 2$

In this subection we study the optimal blow-up rate in the sense of L^2-norm of the solution to problem (3.1)-(3.2).

We first derive the following lemma to the case of dimension $n = 1$.

Lemma 3.20. *There exists $T > 2$ such that*

$$\frac{(64+49\pi^2)t}{196\pi^2} \leq \int_{\boldsymbol{R}} \frac{(1+|\xi|^2)^{-t}\sin^2(t\sqrt{\log(1+|\xi|^2)})}{\log(1+|\xi|^2)} d\xi \leq 12t$$

for all $t \geq T$.

Proof. We have

$$\int_{\boldsymbol{R}} \frac{(1+|\xi|^2)^{-t}\sin^2(t\sqrt{\log(1+|\xi|^2)})}{\log(1+|\xi|^2)} d\xi = \int_0^\infty \frac{(1+r^2)^{-t}\sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)} dr.$$

Initially, we obtain a lower bound for this integral as follows. Set

$$Q_l(t) := \int_0^{\frac{1}{t}} \frac{(1+r^2)^{-t}\sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)} dr,$$

$$Q_h(t) := \int_{\frac{1}{t}}^\infty \frac{(1+r^2)^{-t}\sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)} dr.$$

This implies that

$$\mathcal{I}_1(t) = Q_l(t) + Q_h(t),$$

where $\mathcal{I}_1(t)$ is defined in (3.42).

From mean value theorem, we may obtain

$$\sin(t\sqrt{\log(1+r^2)}) \geq \frac{t}{2}\sqrt{\log(1+r^2)},$$

for $0 \leq r \leq \frac{1}{t}$. Thus

$$\begin{aligned} Q_l(t) &= \int_0^{\frac{1}{t}} \frac{(1+r^2)^{-t}\sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)} dr \\ &\geq \frac{t^2}{4}\int_0^{\frac{1}{t}} \frac{(1+r^2)^{-t}\log(1+r^2)}{\log(1+r^2)} dr \\ &= \frac{t^2}{4}\int_0^{\frac{1}{t}} (1+r^2)^{-t} dr \geq \frac{t^2}{4}\left(1+\frac{1}{t^2}\right)^{-t}\int_0^{\frac{1}{t}} dr \\ &= \frac{t}{4}\left(1+\frac{1}{t^2}\right)^{-t}, \quad t > 0. \end{aligned}$$

Now, since

$$\lim_{t\to\infty}\left(1+\frac{1}{t^2}\right)^{-t}=1,$$

there exist a constant $t_1 \geq 1$ such that

$$\left(1+\frac{1}{t^2}\right)^{-t}\geq\frac{1}{2},\quad t\geq t_1.$$

Then

$$Q_l(t)\geq\frac{t}{8},\quad t\geq t_1. \tag{3.46}$$

To deal with the integral $Q_h(t)$, we consider

$$\nu_1 := \sqrt{e^{\frac{25\pi^2}{16t^2}}-1} \text{ and } \nu_2 := \sqrt{e^{\frac{49\pi^2}{16t^2}}-1}.$$

Note that for $\nu_1 \leq r \leq \nu_2$ it holds that

$$\left|\sin(t\sqrt{\log(1+r^2)})\right|\geq\frac{1}{\sqrt{2}}.$$

Then, we can estimate

$$\begin{aligned}
Q_h(t) &\geq \int_{\nu_1}^{\nu_2}\frac{(1+r^2)^{-t}\sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)}dr\\
&\geq \frac{1}{2}\int_{\nu_1}^{\nu_2}\frac{(1+r^2)^{-t}}{\log(1+r^2)}dr\\
&\geq \frac{8t^2}{49\pi^2}\int_{\nu_1}^{\nu_2}(1+r^2)^{-t}dr\\
&\geq \frac{8t^2}{49\pi^2}e^{\frac{-49\pi^2}{16t}}\int_{\nu_1}^{\nu_2}dr\\
&= \frac{8t^2}{49\pi^2}e^{\frac{-49\pi^2}{16t}}\left(\sqrt{e^{\frac{49\pi^2}{16t^2}}-1}-\sqrt{e^{\frac{25\pi^2}{16t^2}}-1}\right).
\end{aligned}$$

Note that one knows the fact that

$$\lim_{t\to\infty}t\sqrt{e^{\frac{\gamma}{t^2}}-1}=\sqrt{\gamma}\quad(\gamma>0).$$

Therefore, since one can get

$$\lim_{t\to\infty}te^{\frac{-49\pi^2}{16t}}\left(\sqrt{e^{\frac{49\pi^2}{16t^2}}-1}-\sqrt{e^{\frac{25\pi^2}{16t^2}}-1}\right)=\frac{\pi}{2},$$

there exist $t_2 \geq 1$ such that

$$te^{\frac{-49\pi^2}{16t}}\left(\sqrt{e^{\frac{49\pi^2}{16t^2}}-1}-\sqrt{e^{\frac{25\pi^2}{16t^2}}-1}\right)\geq 1,\quad t\geq t_2.$$

Therefore,

$$Q_h(t) \geq \frac{8t}{49\pi^2}, \quad t \geq t_2. \tag{3.47}$$

By adding (3.46) and (3.47), we conclude that

$$\mathcal{I}_1(t) = \int_{\boldsymbol{R}} \frac{(1+|\xi|^2)^{-t}\sin^2(t\sqrt{\log(1+|\xi|^2)})}{\log(1+|\xi|^2)} d\xi \geq \frac{t}{4} + \frac{16t}{49\pi^2}, \tag{3.48}$$

for all $t \geq \max\{t_1, t_2\}$. This estimate concludes the proof of lower bound of proposition.

In order to obtain the upper bound, we separate the integral into three parts as follows:

$$R_l(t) := \int_0^{\frac{1}{t}} \frac{(1+r^2)^{-t}\sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)} dr,$$

$$R_m(t) := \int_{\frac{1}{t}}^{\frac{1}{\sqrt{t}}} \frac{(1+r^2)^{-t}\sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)} dr,$$

$$R_h(t) := \int_{\frac{1}{\sqrt{t}}}^{\infty} \frac{(1+r^2)^{-t}\sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)} dr.$$

Then

$$\frac{1}{2}\mathcal{I}_1(t) = R_l(t) + R_m(t) + R_h(t).$$

Now, using the fact $\dfrac{|\sin x|}{x} \leq 1$ for all $x > 0$, for $t > 0$ one has

$$R_l(t) \leq \int_0^{\frac{1}{t}} (1+r^2)^{-t} t^2 dr = t^2 \int_0^{\frac{1}{t}} (1+r^2)^{-t} dr = t^2 \int_0^{\frac{1}{t}} dr = t. \tag{3.49}$$

In order to estimate the middle part, we first observe that

$$\lim_{\sigma \to 0} \frac{\sigma}{\log(1+\sigma)} = 1.$$

So, there exists $\delta_0 > 0$ such that

$$\frac{\sigma}{\log(1+\sigma)} < 2$$

for all $0 < \sigma < \delta_0$. Therefore, if $\frac{1}{t} < r < \frac{1}{\sqrt{t}}$, then $\frac{1}{t^2} < r^2 < \frac{1}{t}$ and for $t > \frac{1}{\delta_0}$, we have

$$\frac{1}{\log(1+r^2)} < \frac{2}{r^2}.$$

Therewith, using integration by parts we can get

$$\begin{aligned} R_m(t) &= \int_{\frac{1}{t}}^{\frac{1}{\sqrt{t}}} \frac{(1+r^2)^{-t}\sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)} dr \\ &\leq 2\int_{\frac{1}{t}}^{\frac{1}{\sqrt{t}}} \frac{(1+r^2)^{-t}}{r^2} dr = 2t\left(1+\frac{1}{t^2}\right)^{-t} - 2\sqrt{t}\left(1+\frac{1}{t}\right)^{-t} \\ &- 4t\int_{\frac{1}{t}}^{\frac{1}{\sqrt{t}}} (1+r^2)^{-t-1} dr \\ &\leq 2t\left(1+\frac{1}{t^2}\right)^{-t}. \end{aligned}$$

Since

$$\lim_{t\to\infty}\left(1+\frac{1}{t^2}\right)^{-t}=1,$$

there exists $t_3 \geq 1$ such that for all $t \geq t_3$

$$\left(1+\frac{1}{t^2}\right)^{-t}\leq 2,$$

which implies

$$R_m(t) \leq 4t, \quad t \geq t_3. \tag{3.50}$$

Finally, we estimate $R_l(t)$. First, we have

$$R_h(t)=\int_{\frac{1}{\sqrt{t}}}^{\infty}\frac{(1+r^2)^{-t}\sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)}dr \leq \int_{\frac{1}{\sqrt{t}}}^{\infty}\frac{(1+r^2)^{-t}}{\log(1+r^2)}dr.$$

Then, for $t \geq 2$

$$\begin{aligned}
R_h(t) &\leq \int_{\frac{1}{\sqrt{t}}}^{\infty}\frac{(1+r^2)^{-t+1}}{(1+r^2)\log(1+r^2)}dr\\
&\leq \left(1+\frac{1}{t}\right)^{-t+1}\frac{1}{\log\left(1+\frac{1}{t}\right)}\int_{\frac{1}{\sqrt{t}}}^{\infty}\frac{1}{1+r^2}dr\\
&= \left(1+\frac{1}{t}\right)^{-t+1}\frac{1}{\log\left(1+\frac{1}{t}\right)}\left(\frac{\pi}{2}-\tan^{-1}(t^{-\frac{1}{2}})\right).
\end{aligned}$$

Due to the fact

$$\lim_{t\to\infty}\frac{1}{t}\left[\left(1+\frac{1}{t}\right)^{-t+1}\frac{1}{\log\left(1+\frac{1}{t}\right)}\left(\frac{\pi}{2}-\tan^{-1}(t^{-\frac{1}{2}})\right)\right]=\frac{\pi}{2e},$$

there exist $t_4 \geq \max\{2, t_3\}$ such that

$$\left(1+\frac{1}{t}\right)^{-t+1}\frac{1}{\log\left(1+\frac{1}{t}\right)}\left(\frac{\pi}{2}-\tan^{-1}(t^{-\frac{1}{2}})\right)\leq t$$

for all $t \geq t_4$, where one has just used the fact that

$$\lim_{t\to\infty} t\log\left(1+\frac{1}{t}\right)=1.$$

Thus one has

$$R_h(t) \leq t \tag{3.51}$$

for all $t \geq t_4$. By adding (3.49), (3.50) and (3.51) one can obtain the desired upper bound:

$$\mathcal{I}_1(t)=\int_0^{\infty}\frac{(1+r^2)^{-t}\sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)}dr \leq 12t \tag{3.52}$$

for all $t \geq t_4$.

The estimates (3.48) and (3.52) prove the lemma. □

Next we study the optimal blow-up order as $t \to \infty$ of $\mathcal{I}_2(t)$ given by

$$\mathcal{I}_2(t) = \int_{\boldsymbol{R}^2} \frac{(1+|\xi|^2)^{-t}\sin^2(t\sqrt{\log(1+|\xi|^2)})}{\log(1+|\xi|^2)}d\xi.$$

In order to do this we use the Lemma 2.22.

Lemma 3.21. *There exists $T > 1$ such that*

$$\frac{\pi}{4e}\log t \le \int_{\boldsymbol{R}^2} \frac{(1+|\xi|^2)^{-t}\sin^2(t\sqrt{\log(1+|\xi|^2)})}{\log(1+|\xi|^2)}d\xi \le 6\pi\log t$$

for all $t \ge T$.

Proof. By considering the polar co-ordinate transform, we set

$$\frac{1}{2\pi}\mathcal{I}_2(t) = \int_0^\infty \frac{(1+r^2)^{-t}\sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)}rdr.$$

In order to obtain a lower bound for $\mathcal{I}_2(t)$, by using the change of variable $w = \sqrt{t\log(1+r^2)}$ and integration by parts, we observe that

$$\begin{aligned}
\frac{1}{2\pi}\mathcal{I}_2(t) &= \int_0^\infty \frac{(1+r^2)^{-t}(1+r^2)\sin^2(t\sqrt{\log(1+r^2)})\sqrt{t}r}{\sqrt{t}\sqrt{\log(1+r^2)}(1+r^2)\sqrt{\log(1+r^2)}}dr \\
&\ge \int_0^\infty \frac{(1+r^2)^{-t}\sin^2(t\sqrt{\log(1+r^2)})\sqrt{t}r}{\sqrt{t}\sqrt{\log(1+r^2)}(1+r^2)\sqrt{\log(1+r^2)}}dr \\
&= \int_0^\infty \frac{e^{-w^2}\sin^2(\sqrt{t}w)}{w}dw \\
&\ge e^{-1}\int_{\frac{1}{\sqrt{t}}}^1 \frac{\sin^2(\sqrt{t}w)}{w}dw.
\end{aligned}$$

Then we have

$$\begin{aligned}
\frac{1}{2\pi}\mathcal{I}_2(t) &\ge \frac{e^{-1}}{2}\int_{\frac{1}{\sqrt{t}}}^1 \frac{dw}{w} - \frac{e^{-1}}{2}\int_{\frac{1}{\sqrt{t}}}^1 \frac{\cos(2\sqrt{t}w)}{w}dw \\
&= \frac{e^{-1}}{4}\log t - \frac{e^{-1}}{2}\int_{\frac{1}{\sqrt{t}}}^1 \frac{\cos(2\sqrt{t}w)}{w}dw.
\end{aligned}$$

By changing variable we arrive at

$$\begin{aligned}
\frac{1}{2\pi}\mathcal{I}_2(t) &\ge \frac{e^{-1}}{4}\log t - \frac{e^{-1}}{2}\int_2^{2\sqrt{t}} \frac{\cos y}{y}dy \\
&\ge \frac{e^{-1}}{4}\log t - \frac{e^{-1}}{2} \\
&\ge \frac{e^{-1}}{8}\log t, \quad t \ge e^4.
\end{aligned}$$

The penultimate inequality above is due to Lemma 2.22.

Thus, for $t \gg 1$, one has the optimal lower bound

$$\mathcal{I}_2(t) = 2\pi \int_0^\infty \frac{(1+r^2)^{-t} \sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)} r dr \geq \frac{\pi}{4e} \log t. \tag{3.53}$$

The estimate (3.53) implies the desired estimate from below of Lemma 3.21.

Next, in order to get the upper bound for $\mathcal{I}_2(t)$ we set

$$Q_l(t) := \int_0^{\frac{1}{t}} \frac{(1+r^2)^{-t} \sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)} r dr,$$
$$Q_m(t) := \int_{\frac{1}{t}}^{\frac{1}{\sqrt{t}}} \frac{(1+r^2)^{-t} \sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)} r dr,$$
$$Q_h(t) := \int_{\frac{1}{\sqrt{t}}}^{\infty} \frac{(1+r^2)^{-t} \sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)} r dr.$$

Then

$$\frac{1}{2\pi}\mathcal{I}_2(t) = Q_l(t) + Q_m(t) + Q_h(t).$$

For $t > 1$, we first have

$$\begin{aligned} Q_l(t) &= \int_0^{\frac{1}{t}} \frac{(1+r^2)^{-t} \sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)} r dr \\ &\leq \int_0^{\frac{1}{t}} \frac{(1+r^2)^{-t} t^2 \log(1+r^2)}{\log(1+r^2)} r dr \\ &\leq t^2 \int_0^{\frac{1}{t}} (1+r^2)^{-t} r dr \\ &= \frac{t^2}{2(t-1)} \left[1 - \left(1 + \frac{1}{t^2}\right)^{1-t} \right]. \end{aligned}$$

Since

$$\lim_{t\to\infty} \frac{t^2}{t-1} \left[1 - \left(1 + \frac{1}{t^2}\right)^{1-t} \right] = 1,$$

there exists $t_2 \geq 1$ such that

$$\frac{t^2}{2(t-1)} \left[1 - \left(1 + \frac{1}{t^2}\right)^{1-t} \right] \leq 1$$

for all $t \geq t_2$. Therefore, for $t \geq t_2$ it holds that

$$Q_l(t) \leq 1. \tag{3.54}$$

Furthermore, for $t > 1$ one can get the estimate

$$\begin{aligned} Q_m(t) &= \int_{\frac{1}{t}}^{\frac{1}{\sqrt{t}}} \frac{(1+r^2)^{-t} \sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)} r dr \\ &\leq \int_{\frac{1}{t}}^{\frac{1}{\sqrt{t}}} \frac{r(1+r^2)^{-t}}{\log(1+r^2)} dr. \end{aligned}$$

And

$$Q_m(t) \leq \int_{\frac{1}{t}}^{\frac{1}{\sqrt{t}}} \frac{r(1+r^2)^{-1}}{\log(1+r^2)} dr$$
$$= \frac{1}{2}\left[\log\left(\log\left(1+\frac{1}{t}\right)\right) - \log\left(\log\left(1+\frac{1}{t^2}\right)\right)\right].$$

Now, since we have

$$\lim_{t\to\infty} \frac{1}{\log t}\left[\log\left(\log\left(1+\frac{1}{t}\right)\right) - \log\left(\log\left(1+\frac{1}{t^2}\right)\right)\right] = 1,$$

then there exists $t_3 \geq t_2$ such that

$$\frac{1}{2}\left[\log\left(\log\left(1+\frac{1}{t}\right)\right) - \log\left(\log\left(1+\frac{1}{t^2}\right)\right)\right] \leq \log t$$

for all $t > t_3$, where one has just used the facts that

$$\lim_{\sigma\to+0} \frac{\log(\log(1+\sigma^2))}{\log \sigma} = 2,$$

$$\lim_{\sigma\to+0} \frac{\log(\log(1+\sigma))}{\log \sigma} = 1.$$

Therefore, one has just arrived at the estimate:

$$Q_m(t) \leq \log t, \quad t \geq t_3. \tag{3.55}$$

Similarly, for $t > 1$ it follows that

$$Q_h(t) = \int_{\frac{1}{\sqrt{t}}}^{\infty} \frac{(1+r^2)^{-t}\sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)} rdr$$
$$\leq \int_{\frac{1}{\sqrt{t}}}^{\infty} \frac{(1+r^2)^{-t}}{\log(1+r^2)} rdr$$
$$\leq \frac{1}{\log\left(1+\frac{1}{t}\right)} \int_{\frac{1}{\sqrt{t}}}^{\infty} (1+r^2)^{-t} rdr$$
$$= \frac{1}{2(t-1)\log\left(1+\frac{1}{t}\right)}\left(1+\frac{1}{t}\right)^{1-t}.$$

We see that

$$\lim_{t\to\infty} \frac{1}{(t-1)\log\left(1+\frac{1}{t}\right)}\left(1+\frac{1}{t}\right)^{1-t} = \frac{1}{e},$$

there exists $t_4 \geq t_3 > 1$ such that

$$\frac{1}{2(t-1)\log\left(1+\frac{1}{t}\right)}\left(1+\frac{1}{t}\right)^{1-t} \leq 1$$

for all $t \geq t_4$. This implies

$$Q_h(t) \leq 1, \quad t \geq t_4. \tag{3.56}$$

By combining (3.54), (3.55) and (3.56), one can derive the crucial estimate

$$\frac{1}{2\pi}\mathcal{I}_2(t) = \int_0^\infty \frac{(1+r^2)^{-t}\sin^2(t\sqrt{\log(1+r^2)})}{\log(1+r^2)} rdr \leq 3\log t \tag{3.57}$$

for large $t \geq t_4$.

The statement of Lemma 3.21 is now proved from (3.53) and (3.57). □

3.4.3 Optimal estimates to the solution

Now that we know the L^2-estimates to the leading term (3.25), we may conclude the following important result.

Theorem 3.22. *Let $n \geq 1$, $u_0 = 0$ and $u_1 \in \left(L^2(\boldsymbol{R}^n) \cap L^{1,1}(\boldsymbol{R}^n)\right)$. Then, the unique solution $u(t,x)$ to problem (3.1)-(3.2) satisfies the following properties:*

(i) if $n = 1$, then $C_1|P_1|\sqrt{t} \leq \|u(t,\cdot)\|_{L^2} \leq C_1^{-1} I_0 \sqrt{t}$ $(t \gg 1)$,

(ii) if $n = 2$, then $C_2|P_1|\sqrt{\log t} \leq \|u(t,\cdot)\|_{L^2} \leq C_2^{-1} I_0 \sqrt{\log t}$ $(t \gg 1)$,

(iii) if $n \geq 3$, then $C_n|P_1|t^{-\frac{n-2}{4}} \leq \|u(t,\cdot)\|_{L^2} \leq C_n^{-1} I_0 t^{-\frac{n-2}{4}}$ $(t \gg 1)$.

Here I_0 is a constant defined in Theorem 3.16, and C_n $(n \in \boldsymbol{N})$ are constants independent of t and the initial data.

Proof. We prove the item (iii) and the remainders are obtained analogously.

From Young's inequality we have

$$\begin{aligned}\int_{\boldsymbol{R}^n} |\hat{u}(t,\xi)|^2 d\xi &\leq 2\int_{\boldsymbol{R}^n} |\hat{u}(t,\xi) - \varphi(t,\xi)|^2 d\xi + 2\int_{\boldsymbol{R}^n} |\varphi(t,\xi)|^2 d\xi \\ &\leq K I_0^2 t^{-\frac{n}{2}} + K P_1^2 t^{-\frac{n-2}{2}} \\ &\leq C I_0^2 t^{-\frac{n-2}{2}}, t \gg 1.\end{aligned} \tag{3.58}$$

Where we just used Theorem 3.16 and the upper estimate of Lemma 3.19.

In order to obtain the estimate from below, we first observe that

$$|\varphi(t,\xi)| \leq |\hat{u}(t,\xi) - \varphi(t,\xi)| + |\hat{u}(t,\xi)|.$$

From Young's inequality, we have

$$|\varphi(t,\xi)|^2 \leq 2|\hat{u}(t,\xi) - \varphi(t,\xi)|^2 + 2|\hat{u}(t,\xi)|^2$$

and then

$$|\hat{u}(t,\xi)|^2 \geq \frac{1}{2}|\varphi(t,\xi)|^2 - |\hat{u}(t,\xi) - \varphi(t,\xi)|^2.$$

Therefore, from Theorem 3.16 and the lower estimate of Lemma 3.19 we have

$$\begin{aligned}\int_{\boldsymbol{R}^n} |\hat{u}(t,\xi)|^2 d\xi &\geq \frac{1}{2}\int_{\boldsymbol{R}^n} |\varphi(t,\xi)|^2 d\xi - \int_{\boldsymbol{R}^n} |\hat{u}(t,\xi)-\varphi(t,\xi)|^2 d\xi \\ &\geq C|P_1|^2 t^{-\frac{n-2}{2}} - K I_0^2 t^{-\frac{n}{2}} \\ &= t^{-\frac{n-2}{2}}\left(C|P_1|^2 - K I_0^2 t^{-1}\right), \quad t \gg 1. \end{aligned} \tag{3.59}$$

But

$$\lim_{t\to\infty}\left(C|P_1|^2 - K I_0^2 t^{-1}\right) = C|P_1|^2,$$

then there exists $t_0 > 0$ such that

$$C|P_1|^2 - K I_0^2 t^{-1} \geq \frac{1}{2}C|P_1|^2, \quad t \geq t_0.$$

Thus, from (3.59), we have

$$\|u(t,\cdot)\|^2 = \int_{\boldsymbol{R}^n} |\hat{u}(t,\xi)|^2 d\xi \geq \frac{1}{2}C|P_1|^2 t^{-\frac{n-2}{2}}, \quad t \gg 1. \tag{3.60}$$

Combining Plancherel Theorem with (3.58) and (3.60) we have

$$C_1|P_1|t^{-\frac{n-2}{4}} \leq \|u(t,\cdot)\| \leq C_2 I_0 t^{-\frac{n-2}{4}}, \quad t \gg 1$$

where C_1, C_2 are positive constants depending only on n. □

Remark 3.23. Here it is important to observe that the upper bounds can be obtained only by L^1 assumption on the initial data u_1. Indeed, from (3.21) and (3.23) we have

$$\hat{u}(t,\xi) = \hat{u}_1(\xi)\left(\frac{e^{-a(\xi)t}}{b(\xi)}\sin(t\sqrt{\log(1+|\xi|^2)} + te^{-a(\xi)t}\frac{b(\xi)-\sqrt{\log(1+|\xi|^2)}}{b(\xi)}\cos(t\mu(\xi))\right).$$

Due to $\|\hat{u}_1\|_\infty \leq \|u_1\|_1$, it is easy to adapt the estimates (3.29), (3.31) and Lemmas 3.19, 3.20 and 3.21 to prove that the solution has the same estimates as in Theorem 3.22, but with $I_0 = \|u_1\|_1$. However, $u_1 \in L^{1,1}(\boldsymbol{R}^n)$ seems to be a technical condition to get lower bounds.

Remark 3.24. As a result, all estimates derived in Theorem 3.22 are overlapped already known results in [28] and/or [6], and this is quite natural because $\log(1+|\xi|^2) \approx |\xi|^2$ for small $\xi \in \boldsymbol{R}^n$, and the main contribution to the above estimates comes from the low frequency region in $\xi \in \boldsymbol{R}^n$. However, by replacing the operator $A = -\Delta$ to $L = \log(I-\Delta)$ in the equation (3.39), we encounter a big obstacle when one gets such estimates stated in Theorem 3.22 and this difficulty comes from the way that how we treat the improper integral (3.42). A big technical difficulty occurs.

Remark 3.25. We note that due to the structure of our equation (3.1) being similar to the viscoelastic equation the asymptotic profile of solutions is determined, according to the integral (3.42), by a competition between the wave type multiplier $\frac{\sin(\omega(\xi)t)}{\omega(\xi)}$, $\omega(\xi) = \sqrt{\log(1+|\xi|^2)}$, and the Gauss kernel $\exp\left(-\frac{|\xi|^2}{2}t\right)$. With respect to L^1-L^2 estimates, in the space dimension $n = 1$ the wave structure is dominant and at high dimension $n \geq 3$ the Gauss type kernel comes into play, whereas $n = 2$ is in the border line so a log-term comes into play.

Remark 3.26. We observe that our equation (3.1) is a similar model to the viscoelastic equation (i.e., (3.39) with $A = -\Delta$) studied in Y. Shibata [44]. However, the work by Shibata, different from our Theorem 3.22, gets only the upper bound estimates for all $n \geq 2$. On the dimension $n = 1$ no any upper and lower bounds can be derived. For $n = 2$, only log-order estimate from above to the L^2-norm can be obtained, namely $\log(t+2)$, but it is worse than the rate $(\log t)^{1/2}$ obtained by us. For $n \geq 3$ only optimal $t^{(n-2)/4}$ order from above can be observed. These estimates are also done in the framework of $(L^1 \cap L^2)$-initial data as in our work. It is important to emphasize that more general L^p-L^q estimates of solutions can be studied in detail in [44].

4 A DISSIPATIVE LOGARITHMIC-LAPLACIAN TYPE OF PLATE EQUATION

In this chapter we consider a new type of plate equation with the logarithmic-Laplacian operator under effects of damping mechanism. The associated Cauchy problem is

$$u_{tt} + Lu + (I+L)^{-1}u_t = 0, \quad (t,x) \in (0,\infty) \times \boldsymbol{R}^n, \tag{4.1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \boldsymbol{R}^n, \tag{4.2}$$

where $L = L_1$ is defined in (1.4)-(1.5).

By employing the standard Lumer-Phillips theorem, we prove the existence of an unique weak solution to the problem (4.1)-(4.2) in the class

$$C([0,\infty), Y^2) \cap C^1([0,\infty), L^2),$$

where the set Y^2 is defined in (2.4) and it is the domain of L.

The associated energy identity to the system (4.1)-(4.2) is

$$\frac{\mathrm{d}}{\mathrm{d}t}E_u(t) + \|\left((I+L)^{-1}\right)^{1/2}u_t\| = 0, \tag{4.3}$$

where

$$E_u(t) := \frac{1}{2}\left(\|u_t(t,\cdot)\|_{L^2}^2 + \|L^{1/2}u(t,\cdot)\|_{L^2}^2\right).$$

The identity (4.3) implies that the total energy is a non increasing function in time because of the existence of some kind of weak dissipative term $(I+L)^{-1}u_t$. In this sense, we investigate decay rates to the solution for $n \geq 3$ applying the multiplier method in the Fourier space (cf. [27, 46]). It is important emphasize that more regularity is required on the initial data to obtain decay estimates on the high frequency region $|\xi| \geq \delta$. This fact is due to the structure of regularity-loss of this type of plate equation. Such structure is characterized by

$$\lim_{|\xi|\to\infty} \mathrm{Re}\lambda_{\pm} = \lim_{|\xi|\to\infty} \frac{-1}{2(1+\log(1+|\xi|^2))} = 0,$$

where $\lambda_{\pm}$ are the associated characteristics roots.

In order to obtain optimal decay estimates of solutions for $n \geq 1$, we derived an asymptotic profile to the solution. We prove that there are three possibilities of asymptotic profile each of them depends on the regularity of the initial data: for high regularity it is diffusive-like, for low regularity the asymptotic profile is wave-like and for a threshold regularity it is the sum of both (see Theorems 4.21, 4.22 and 4.23). The ideas from [18] were very important for us to describe the three asymptotic profiles. After getting the suitable asymptotic profile, we use them to discuss the optimal decay rate of the solution in terms of the L^2-norm (see Theorems 4.24, 4.26 and 4.27). The same comments as in the introduction of Chapter 3 on the effectiveness of the operator L to get a new model of plate equation holds.

The results obtained in this chapter will be published in May 2022 in *Discrete Continuous and Dynamical Systems* (see [3]).

This chapter is organized as follows. In Section 4.1 we discuss the unique existence of solutions to problem (4.1)-(4.2). In Section 4.2 we obtain decay estimates for the L^2-norm of solutions by using the multiplier method for $n \geq 3$. The asymptotic profile and related estimates are obtained in Section 4.3 and, in particular, Theorems 4.21, 4.22 and 4.23 are proved in the final of this section. In the Section 4.4 we prove optimal decay rates.

4.1 EXISTENCE AND UNIQUENESS

In this section we study the existence and uniqueness of solutions to the problem (3.1)-(3.2). For this purpose, we follow the work by Charão-Horbach [9] (see also [36]).

The Cauchy problem (4.1)-(4.2) rewritten as

$$\begin{cases} (I+L)u_{tt} + L(I+L)u + u_t = 0, \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x). \end{cases} \tag{4.4}$$

By taking the inner product of the equation in (4.4) by u_t, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_t(t,\cdot)\|^2 + \|L^{1/2}u_t(t,\cdot)\|^2 + \|Lu(t,\cdot)\|^2 + \|L^{1/2}u(t,\cdot)\|^2\right) + \|u_t(t,\cdot)\|^2 = 0,$$

for $t > 0$. We define the total energy as

$$E(t) := \|u_t(t,\cdot)\|^2 + \|L^{1/2}u_t(t,\cdot)\|^2 + \|Lu(t,\cdot)\|^2 + \|L^{1/2}u(t,\cdot)\|^2.$$

Then, we can observe that $E(t)$ is a non-increasing function and it is well defined for weak solution of the problem (4.1)-(4.2) according to Theorem 4.6.

Associated to (4.1)-(4.2) one can choose the following energy space

$$X = Y^2 \times Y^1.$$

In this section, to study the existence of solutions is convenient to adopt the following norm

$$\|f\|_{Y^2} = \left(\int_{\boldsymbol{R}^n} \left(1 + \log(1+|\xi|^2) + \log^2(1+|\xi|^2)\right)|\hat{f}(\xi)|^2 d\xi\right)^{1/2}$$

in the space Y^2, which is equivalent to its natural norm defined in (2.5) with $\delta = 2$. The associated inner product to this norm is

$$(u,v)_{Y^2} = \int_{\boldsymbol{R}^n} \left(1 + \log(1+|\xi|^2) + \log^2(1+|\xi|^2)\right)\hat{u}\bar{\hat{v}}d\xi, \quad u,v \in Y^2. \tag{4.5}$$

Now, at least formally, from (4.4) one can write

$$u_{tt} = -(I+L)^{-1}(L^2+L)u - (I+L)^{-1}u_t = -(I+L)^{-1}(L^2+L+I)u - (I+L)^{-1}(u_t - u).$$

Then if we define $v = u_t$ and $U(t) = \begin{pmatrix} u \\ v \end{pmatrix}$ we can reduce the second order equation of (4.4) to a system of the first order as follows:

$$\begin{aligned}\frac{\mathrm{d}U}{\mathrm{d}t} &= \begin{pmatrix} u_t \\ -(I+L)^{-1}(L^2+L+I)u-(I+L)^{-1}(u_t-u)\end{pmatrix} \\ &= \begin{pmatrix} 0 & I \\ -(I+L)^{-1}(L^2+L+I) & 0\end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ (I+L)^{-1}(u-v)\end{pmatrix} \\ &= \begin{pmatrix} 0 & I \\ -A & 0\end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ (I+L)^{-1}(u-v)\end{pmatrix}.\end{aligned}$$

Thus, the first order evolution equation to U can be written as

$$\frac{\mathrm{d}U}{\mathrm{d}t} = BU + FU, \quad U(0) = (u_0, u_1), \tag{4.6}$$

where formally the operator A is given by

$$A := (I+L)^{-1}(L^2+L+I) = L + (I+L)^{-1},$$

and the operators B, J are given by

$$B = \begin{pmatrix} 0 & I \\ -A & 0\end{pmatrix}, \quad FU = \begin{pmatrix} 0 \\ (I+L)^{-1}(u-v)\end{pmatrix}, \quad U = (u,v) \in D(B).$$

We need to give a precise definition of the domains of operators A and B. To do that we set

$$\begin{aligned} D(A) = \{u \in Y^2 :\ & \text{there exists } y = y_u \in Y^1 \text{ such that} \\ & (Lu, L\psi) + (L^{1/2}u, L^{1/2}\psi) + (u,\psi) = (y,\psi) + (L^{1/2}y, L^{1/2}\psi)\ \forall\ \psi \in Y^2\}.\end{aligned}$$

We observe that $0 \in D(A)$, so $D(A) \neq \phi$ and we define

$$A : D(A) \to Y^1, \quad \text{by} \quad Au = y_u,\ u \in D(A).$$

The suitable definition to the domain of B is $D(B) := D(A) \times Y^2$. This definition implies that

$$B : D(B) \to X = Y^2 \times Y^1,$$

where X is the energy space defined above.

The following result guarantees us that A is well defined.

Lemma 4.1. *For $u \in Y^2$, there exists at most one $y \in Y^1$ that satisfies*

$$(Lu, L\psi) + (L^{1/2}u, L^{1/2}\psi) + (u,\psi) = (y,\psi) + (L^{1/2}y, L^{1/2}\psi)\ \forall\ \psi \in Y^2. \tag{4.7}$$

Proof. Suppose that $y_1, y_2 \in Y^1$ satisfy the above relation. Then $y = y_1 - y_2$ satisfies

$$(y, \psi) + (L^{1/2}y, L^{1/2}\psi) = 0 \quad \text{for each } \ \psi \in Y^2. \tag{4.8}$$

By density of Y^2 in Y^1, due to Proposition 2.18, there exists a sequence $\{y_n\} \subset Y^2$ such that

$$\|y_n - y\|_{Y^1} \to 0.$$

This implies $\|y_n\|_{Y^1} \to \|y\|_{Y^1}$ and $\|y_n - y\|_{Y^1}^2 \to 0$. Thus we have

$$\|y_n\|_{Y^1}^2 - 2(y_n, y)_{Y^1} + \|y\|_{Y^1}^2 = \|y_n - y\|_{Y^1}^2 \to 0,$$

which implies

$$\lim_{n\to\infty} (y_n, y)_{Y^1} = \|y\|_{Y^1}^2.$$

On the other hand, by (4.8) and the density argument we get

$$(y, y_n)_{Y^1} = (y, y_n) + (L^{1/2}y, L^{1/2}y_n) = 0,$$

which implies $\lim_{n\to\infty} (y, y_n)_{Y^1} = 0$. Therefore, we can conclude that $\|y\|_{Y^1} = 0$.

□

Lemma 4.2. *$D(A) \subset Y^3$ and there exists $c > 0$ such that*

$$\|u\|_{Y^3} \le c\|Au\|_{Y^1},$$

for all $u \in D(A)$.

Proof. Let $u \in D(A)$. Then there is $y \in Y^1$ that satisfies

$$(Lu, L\psi) + (L^{1/2}u, L^{1/2}\psi) + (u, \psi) = (y, \psi) + (L^{1/2}y, L^{1/2}\psi) \ \forall \ \psi \in Y^2.$$

We define $F : Y^1 \to \boldsymbol{R}$ by

$$\langle F, \psi\rangle = (y, \psi) + (L^{1/2}y, L^{1/2}\psi), \ \psi \in Y^1.$$

Then F is well defined, because $y, \psi, L^{1/2}y$ and $L^{1/2}\psi$ are in $L^2(\boldsymbol{R}^n)$ and F is linear. Furthermore, F is a continuous operator. In fact

$$\begin{aligned} |\langle F, \psi\rangle| &\le |(y, \psi)| + |(L^{1/2}y, L^{1/2}\psi)| \le \|y\|\|\psi\| + \|L^{1/2}y\|\|L^{1/2}\psi\| \\ &= \|\hat{y}\|\|\hat{\psi}\| + \|(\log(1+|\xi|^2))^{1/2}\hat{y}\|\|(\log(1+|\xi|^2))^{1/2}\hat{\psi}\| \\ &\le 2\|(1+\log(1+|\xi|^2))^{1/2}\hat{y}\|\|(1+\log(1+|\xi|^2))^{1/2}\hat{\psi}\| \\ &= 2\|y\|_{Y^1}\|\psi\|_{Y^1}. \end{aligned}$$

Since $\mathcal{S}(\boldsymbol{R}^n) \subset Y^2 \subset Y^1$, we have $(Y^1)' \subset (Y^2)' \subset \mathcal{S}'(\boldsymbol{R}^n)$. In other words, F can be seen as a tempered distribution and for all $\psi \in \mathcal{S}(\boldsymbol{R}^n)$ it holds that

$$(Lu, L\psi) + (L^{1/2}u, L^{1/2}\psi) + (u, \psi) = \langle F, \psi\rangle.$$

Thus, the equality

$$L^2u + Lu + u = F$$

holds in $\mathcal{S}'(\boldsymbol{R}^n)$ sense. By applying the Fourier transform, the definition of F and the operator L, we arrive at

$$\left[\log^2(1+|\xi|^2) + \log(1+|\xi|^2) + 1\right]\hat{u} = \hat{F} = \left[1 + \log(1+|\xi|^2)\right]\hat{y},$$

that is,

$$\hat{y} = \frac{\log^2(1+|\xi|^2) + \log(1+|\xi|^2) + 1}{1+\log(1+|\xi|^2)}\hat{u},$$

or

$$\sqrt{1+\log(1+|\xi|^2)}\hat{y} = \frac{\log^2(1+|\xi|^2) + \log(1+|\xi|^2) + 1}{\sqrt{1+\log(1+|\xi|^2)}}\hat{u}.$$

Then

$$\begin{aligned}\|y\|_{Y^1} &= \int_{\boldsymbol{R}^n}\left(1+\log(1+|\xi|^2)\right)|\hat{y}|^2 d\xi \\ &= \int_{\boldsymbol{R}^n}\frac{\left[\ 1+\log^2(1+|\xi|^2) + \log(1+|\xi|^2)\ \right]^2}{1+\log(1+|\xi|^2)}|\hat{u}|^2 d\xi.\end{aligned}$$

From Lemma 2.17, it follows that

$$(1+\log(1+|\xi|^2))^{-1}\left(\log^2(1+|\xi|^2) + \log(1+|\xi|^2) + 1\right)^2 \approx (1+\log(1+|\xi|^2)^3.$$

Then

$$\|y\|_{Y^1} \approx \int_{\boldsymbol{R}^n}(1+\log(1+|\xi|^2)^3|\hat{u}|^2 d\xi = \|u\|^2_{Y^3}.$$

Therefore, $u \in Y^3$ and, since $y = Au$, there exists constant $c > 0$ such that

$$\|u\|_{Y^3} \le c\|Au\|_{Y^1}.$$

□

Lemma 4.3. $Y^3 \subset D(A)$.

Proof. Initially, we observe that $Y^3 \subset Y^2 \subset Y^1$, because

$$1+\log(1+|\xi|^2) \le (1+\log(1+|\xi|^2))^2 \le (1+\log(1+|\xi|^2))^3.$$

Let $u \in Y^3$ and note that $L^{3/2}u \in L^2(\boldsymbol{R}^n)$. We first show that there exists $y \in Y^1$ such that

$$(L^{3/2}u, L^{1/2}\psi) + (L^{1/2}u, L^{1/2}\psi) + (u,\psi) = (y,\psi) + (L^{1/2}y, L^{1/2}\psi), \quad \text{for each} \quad \psi \in Y^1.$$

We define $a : Y^1 \times Y^1 \to \boldsymbol{R}$ by

$$a(\psi,\phi) = (\psi,\phi) + (L^{1/2}\psi, L^{1/2}\phi).$$

This function is well-defined and it is a symmetric bilinear form. Moreover,

- a is continuous. In fact,

$$\begin{aligned}|a(\psi,\phi)| &\le |(\psi,\phi)|+|(L^{1/2}\psi,L^{1/2}\phi)|\le \|\psi\|\|\phi\|+\|L^{1/2}\psi\|\|L^{1/2}\phi\|\\ &= \|\hat\psi\|\|\hat\phi\|+\|\sqrt{\log(1+|\xi|^2)}\hat\psi\|\|\sqrt{\log(1+|\xi|^2)}\hat\phi\|\\ &\le 2\|\sqrt{1+\log(1+|\xi|^2)}\hat\psi\|\|\sqrt{1+\log(1+|\xi|^2)}\hat\phi\|\\ &= 2\|\psi\|_{Y^1}\|\phi\|_{Y^1}.\end{aligned}$$

- a is coercive, because

$$a(\phi,\phi)=(\phi,\phi)+(L^{1/2}\phi,L^{1/2}\phi)=\|\phi\|^2+\|L^{1/2}\phi\|^2=\|\phi\|^2_{Y^1}.$$

On the other hand, we define $F: Y^1\to \boldsymbol{R}$ by

$$\langle F,\psi\rangle=(L^{3/2}u,L^{1/2}\psi)+(L^{1/2}u,L^{1/2}\psi)+(u,\psi).$$

This map is linear and continuous, in fact

$$\begin{aligned}|\langle F,\psi\rangle| &= |(L^{3/2}u,L^{1/2}\psi)|+|(L^{1/2}u,L^{1/2}\psi|+|(u,\psi)|\\ &\le \|L^{3/2}u\|\|L^{1/2}\psi\|+\|Lu\|\|\psi\|+\|u\|\|\psi\|\\ &= \|(\log(1+|\xi|^2))^{3/2}\hat u\|\|(\log(1+|\xi|^2))^{1/2}\hat\psi\|+\|\log(1+|\xi|^2)\hat u\|\|\hat\psi\|+\|\hat u\|\|\hat\psi\|\\ &\le \|(\log(1+|\xi|^2))^{3/2}\hat u\|\|(1+\log(1+|\xi|^2))^{1/2}\hat\psi\|\\ &\quad+\|\log(1+|\xi|^2)\hat u\|\|(1+\log(1+|\xi|^2))^{1/2}\hat\psi\|+\|\hat u\|\|(1+\log(1+|\xi|^2))^{1/2}\hat\psi\|\\ &= \Big(\|(\log(1+|\xi|^2))^{3/2}\hat u\|+\|\log(1+|\xi|^2)\hat u\|+\|\hat u\|\Big)\|(1+\log(1+|\xi|^2))^{1/2}\hat\psi\|\\ &\le \|u\|_{Y^3}\|\psi\|_{Y^1}.\end{aligned}$$

Thus, we have a continuous linear functional F and a coercive continuous bilinear form in the Hilbert Space Y^1. From the Lax-Milgram Theorem (see Theorem 2.12), there exists a unique $y\in Y^1$ such that

$$\langle F,\psi\rangle=a(y,\psi)\ \text{ for all }\quad \psi\in Y^1.$$

Due to $Y^2\subset Y^1$, this identity is valid for all $\psi\in Y^2$ and, in this case, $L\psi\in L^2(\boldsymbol{R}^n)$ and

$$(L^{3/2}u,L^{1/2}\psi)=(Lu,L\psi).$$

Therefore,

$$(Lu,L\psi)+(L^{1/2}u,L^{1/2}\psi)+(u,\psi)=(y,\psi)+(L^{1/2}y,L^{1/2}\psi),$$

for each $\psi\in Y^2$. It proves that $u\in D(A)$. □

Our goal at this section is to show that B is an infinitesimal generator of a contraction C^0-semigroup. For this, we apply the Lumer-Phillips Theorem. First, we note that $D(B) = Y^3 \times Y^2$ is dense in the energy space $X = Y^2 \times Y^1$ from Proposition 2.18.

In order to prove that B is dissipative, we consider the Hilbert space $X = Y^2 \times Y^1$ with the following inner product:

$$\big((u_1, v_1), (u_2, v_2)\big)_X = (u_1, u_2)_{Y^2} + (v_1, v_2)_{Y^1}, \quad u_1, u_2 \in Y^2,\ v_1, v_2 \in Y^1, \tag{4.9}$$

where $(u_1, u_2)_{Y^2}$ is defined in (4.5) and

$$(v_1, v_2)_{Y^1} = \int_{\boldsymbol{R}^n} (1 + \log(1 + |\xi|^2) \hat{v_1} \overline{\hat{v_2}} d\xi$$

according to the corresponding definition of norm in Y^1 given by (2.6).

Lemma 4.4. *The operator $B : D(B) \to X$ is dissipative.*

Proof. For $(u, v) \in D(B)$ one can observe that

$$\begin{aligned}
(B(u,v),(u,v))_X &= ((v,-Au),(u,v))_X = (v,u)_{Y^2} + (-Au, v)_{Y^1} \\
&= \int_{\boldsymbol{R}^n} \left(1 + \log(1+|\xi|^2) + \log^2(1+|\xi|^2)\right) \hat{v}\overline{\hat{u}} d\xi \\
&- \int_{\boldsymbol{R}^n} \left(1 + \log(1+|\xi|^2)\right) \widehat{Au}\overline{\hat{v}} d\xi \\
&= \int_{\boldsymbol{R}^n} \left(1 + \log(1+|\xi|^2) + \log^2(1+|\xi|^2)\right) \hat{v}\overline{\hat{u}} d\xi \\
&- \int_{\boldsymbol{R}^n} \left(1 + \log(1+|\xi|^2)\right) \frac{1 + \log(1+|\xi|^2) + \log^2(1+|\xi|^2)}{1 + \log(1+|\xi|^2)} \hat{u}\overline{\hat{v}} d\xi \\
&= \int_{\boldsymbol{R}^n} \left(1 + \log(1+|\xi|^2) + \log^2(1+|\xi|^2)\right) (\hat{v}\overline{\hat{u}} - \hat{u}\overline{\hat{v}}) d\xi \\
&= 2i \int_{\boldsymbol{R}^n} \left(1 + \log(1+|\xi|^2) + \log^2(1+|\xi|^2)\right) \operatorname{Im}(\hat{v}\overline{\hat{u}}) d\xi.
\end{aligned}$$

Thus, $\operatorname{Re}(B(u,v),(u,v))_X = 0$ for all $(u,v) \in D(B)$ and B is dissipative. □

Lemma 4.5. $(I - B)D(B) = X$.

Proof. Clearly $(I - B)D(B) \subset X$. In its turn, let $(f, g) \in X = Y^2 \times Y^1$. Then, we will prove that there exists $(u, v) \in D(B)$ such that $(I - B)(u, v) = (f, g)$.
We define a mapping $\mathbf{a} : Y^2 \times Y^2 \to \boldsymbol{R}$ by

$$\mathbf{a}(\varphi, \psi) = (\varphi, \psi)_{Y^1} + (\varphi, \psi)_{Y^2}.$$

Then $\mathbf{a}$ is a symmetrical bilinear form, which is
• continuous:

$$|\mathbf{a}(\varphi, \psi)| \le \|\varphi\|_{Y^1} \|\psi\|_{Y^1} + \|\varphi\|_{Y^2} \|\psi\|_{Y^2} \le 2\|\varphi\|_{Y^2} \|\psi\|_{Y^2},$$

• coercive:

$$\mathbf{a}(\psi,\psi) = (\psi,\psi)_{Y^1} + (\psi,\psi)_{Y^2} = \|\psi\|^2_{Y^1} + \|\psi\|^2_{Y^2} \geq \|\psi\|^2_{Y^2}.$$

Furthermore, we consider the linear functional $F : Y^2 \to \boldsymbol{R}$ given by

$$\langle F, \psi \rangle = (f+g, \psi)_{Y^1},$$

which is well-defined because of $Y^2 \subset Y^1$. Also, one has

$$|\langle F, \psi \rangle| \leq \|f+g\|_{Y^1}\|\psi\|_{Y^1} \leq \|f+g\|_{Y^1}\|\psi\|_{Y^2},$$

which just proves the continuity of F.

Thus, we can apply the Lax-Milgram Theorem to get the existence of a unique $u \in Y^2$ such that

$$\mathbf{a}(u,\psi) = F(\psi) \text{ for all } \psi \in Y^2.$$

In other words,

$$\begin{aligned}(u,\psi) + (L^{1/2}u, L^{1/2}\psi) + (u,\psi) + (L^{1/2}u, L^{1/2}\psi) + (Lu, L\psi)\\ = (f+g,\psi) + (L^{1/2}(f+g), L^{1/2}\psi)\end{aligned}$$

for all $\psi \in Y^2$. In particular, this equality is valid for all $\psi \in \mathcal{S}(\boldsymbol{R}^n)$ and we have the following identity in $\mathcal{S}'(\boldsymbol{R}^n)$

$$Au + u = f + g.$$

Finally, we observe that $u \in Y^3$. In fact, by applying the Fourier transform, we obtain

$$\widehat{Au} + \hat{u} = \hat{f} + \hat{g}$$

and

$$\frac{1+\log(1+|\xi|^2)+\log^2(1+|\xi|^2)}{1+\log(1+|\xi|^2)}\hat{u} + \hat{u} = \hat{f} + \hat{g},$$

which is equivalent to

$$\frac{1+\log(1+|\xi|^2)+\log^2(1+|\xi|^2)}{\sqrt{1+\log(1+|\xi|^2)}}\hat{u} = \sqrt{1+\log(1+|\xi|^2)}(\hat{f}+\hat{g}-\hat{u}).$$

From Lemma (2.17), we have

$$\frac{1+\log(1+|\xi|^2)+\log^2(1+|\xi|^2)}{\sqrt{1+\log(1+|\xi|^2)}} \approx (1+\log(1+|\xi|^2))^{3/2}.$$

Then,

$$(1+\log(1+|\xi|^2))^3|\hat{u}|^2 \approx (1+\log(1+|\xi|^2))|\hat{f}+\hat{g}-\hat{u}|^2.$$

Now, since $f, g, u \in Y^1$ we conclude that

$$\int_{\boldsymbol{R}^n}(1+\log(1+|\xi|^2))^3|\hat{u}(\xi)|^2 d\xi < \infty,$$

which proves that $u \in Y^3$.

Let $v = u - f$ and note that $v \in Y^2$, because of $f \in Y^2$ and $u \in Y^3$. Therefore, we have obtained $(u, v) \in D(B)$ such that

$$(I - B)(u, v) = (f, g).$$

Hence, we have proved that $(I - B)D(B) = X$. □

Therefore, we just have proved that B satisfies the hypothesis of Lumer-Phillips Theorem. Then B is the infinitesimal generator of a C^0-semigroup of contractions.

Next we want to prove that $F : X \to X$ given by (4.6), which is well defined from Lemma 2.19, is a linear bounded operator. The linearity is simple. The boundedness is given by the following series of inequalities:

$$\begin{aligned}
\|F(u,v)\|_X^2 &= \left\| \sqrt{1+\log(1+|\xi|^2)} \frac{\hat{u}-\hat{v}}{1+\log(1+|\xi|^2)} \right\|^2 \\
&\leq 2\int_{\boldsymbol{R}^n} \frac{|\hat{u}|^2}{1+\log(1+|\xi|^2)} d\xi + 2\int_{\boldsymbol{R}^n} \frac{|\hat{v}|^2}{1+\log(1+|\xi|^2)} d\xi \\
&\leq 2\int_{\boldsymbol{R}^n} |\hat{u}|^2 d\xi + 2\int_{\boldsymbol{R}^n} (1+\log(1+|\xi|^2))|\hat{v}|^2 d\xi \\
&\leq 2\|u\|_{Y^2}^2 + 2\|v\|_{Y^1}^2 \\
&= 2\|(u,v)\|_X^2.
\end{aligned}$$

From Theorem 2.11, $B+F : D(B) \to X$ is infinitesimal generator of a C^0-semigroup $\{T(t); t \geq 0\}$ in X, because $B : D(B) \to X$ is infinitesimal generator of a C^0-semigroup of contractions and $F : X \to X$ is a bounded linear operator. For $(u_0, u_1) \in D(B)$

$$U(t) = T(t) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \tag{4.10}$$

is the unique strong solution of the problem

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} U(t) &= (B+F) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \\
U(0) &= \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}
\end{aligned}$$

and $U \in C^1([0,\infty), D(B)) \cap C([0,\infty), X)$ (see Theorem 2.6). For initial data $(u_0, u_1) \in X = Y^2 \times Y^1$, the function given by (4.10) is the weak solution (see Definition 2.7). Therefore we obtain the following result about the existence and uniqueness of solutions of Cauchy problem (4.1)–(4.2).

Theorem 4.6. *Let $n \geq 1$ and $(u_0, u_1) \in Y^3 \times Y^2$. Then the problem (3.1)–(3.2) admits a unique strong solution in the class*

$$u \in C([0,\infty), Y^3) \cap C^1([0,\infty), Y^2) \cap C^2([0,\infty), Y^1).$$

Moreover, for initial data $(u_0, u_1) \in Y^2 \times Y^1$ *the problem* (3.1)–(3.2) *admits a unique weak solution in the class*

$$u \in C([0,\infty), Y^2) \cap C^1([0,\infty), Y^1) \cap C^2([0,\infty), L^2(\boldsymbol{R}^n)).$$

Remark 4.7. Note that for initial data $(u_0, u_1) \in Y^3 \times Y^2$ the solution given by Theorem 4.6 is just a weak solution of the plate equation

$$(I+L)u_{tt} + L(I+L)u + u_t = 0$$

whereas it is a strong solution to the wave equation with a weak dissipation term

$$u_{tt} + Lu + (I+L)^{-1}u_t = 0$$

with the same initial data.

4.2 ASYMPTOTIC BEHAVIOR VIA MULTIPLIER METHOD

We begin with this section by considering the Cauchy problem in the Fourier space associated to the problem (4.1)–(4.2) as follows

$$\begin{aligned}
&(1+\log(1+|\xi|^2))\hat{u}_{tt} + \log(1+|\xi|^2)(1+\log(1+|\xi|^2))\hat{u} + \hat{u}_t = 0, \quad t>0,\ \xi \in \boldsymbol{R}^n\\
&\hat{u}(0,\xi) = \hat{u}_0(\xi), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.11)\\
&\hat{u}_t(0,\xi) = \hat{u}_1(\xi), \quad \xi \in \boldsymbol{R}^n.
\end{aligned}$$

Multiplying the equation in (4.11) by $\overline{\hat{u}_t}$ one can get the following pointwise energy identity

$$\frac{\mathrm{d}}{\mathrm{d}t}E_0(t,\xi) + |\hat{u}_t(t,\xi)|^2 = 0, \quad t>0,\ \xi \in \boldsymbol{R}^n \tag{4.12}$$

where $E_0(t,\xi)$ is defined for $t \geq 0$ and $\xi \in \boldsymbol{R}^n$ by

$$E_0(t,\xi) = \frac{(1+\log(1+|\xi|^2))|\hat{u}_t|^2}{2} + \frac{\log(1+|\xi|^2)(1+\log(1+|\xi|^2))|\hat{u}|^2}{2},$$

is the total density of the energy for the system (4.11). From (4.12) we see that $E_0(t,\xi)$ is a non-increasing function of t for each ξ.

Lemma 4.8. *Consider the following three functions*

$$\begin{aligned}
&\varphi(\xi) = \log(1+|\xi|^2)(1+\log(1+|\xi|^2)), \quad \psi(\xi) = \frac{1}{1+\log(1+|\xi|^2)},\\
&\phi(\xi) = \sqrt{\log(1+|\xi|^2)}
\end{aligned}$$

defined for $\xi \in \boldsymbol{R}^n$*. Then, there is a unique real number* $\delta_0 \in (0,1)$ *such that*

(i). $|\xi| \leq \delta_0 \Rightarrow \varphi(\xi) \leq \psi(\xi)$ *and* $\varphi(\xi) \leq \phi(\xi)$*,*

(ii). $|\xi| \geq \delta_0 \Rightarrow \psi(\xi) \leq \varphi(\xi)$ *and* $\psi(\xi) \leq \phi(\xi)$.

Proof. Let $\theta(r) = (1+\log(1+r^2))\sqrt{\log(1+r^2)} - 1$ for $r \geq 0$. We note that $\theta(0) = -1$ and $\theta(1) = (1+\log 2)\sqrt{\log 2} - 1 > 0$. The continuity of this function implies there exists $0 < \delta_0 < 1$ such that $\theta(\delta_0) = 0$. Moreover, $\theta(r)$ is a increasing function of $r \geq 0$ and such fact implies that $\delta_0 > 0$ is a unique number satisfying $-1 \leq \theta(r) \leq 0$ for all $0 \leq r \leq \delta_0$ and $\theta(r) > 0$ for $r > \delta_0$. That is, if $0 \leq r \leq \delta_0$, then one has

$$(1+\log(1+r^2))\sqrt{\log(1+r^2)} \leq 1.$$

Thus, for $0 \leq r \leq \delta_0$,

$$(1+\log(1+r^2))\log(1+r^2) \leq \sqrt{\log(1+r^2)} \text{ and } \sqrt{\log(1+r^2)} \leq \frac{1}{1+\log(1+r^2)}.$$

Similarly, if $r > \delta_0$, then

$$(1+\log(1+r^2))\sqrt{\log(1+r^2)} > 1,$$

and one obtains

$$(1+\log(1+r^2))\log(1+r^2) > \sqrt{\log(1+r^2)} \text{ and } \sqrt{\log(1+r^2)} > \frac{1}{1+\log(1+r^2)}$$

for $r > \delta_0$. These imply the desired estimates (i) and (ii). □

For $\delta_0 > 0$ given in Lemma 4.8, we define the following function of $\xi \in \boldsymbol{R}^n$ such that

$$\rho(\xi) := \begin{cases} \frac{1}{2}\log(1+|\xi|^2)(1+\log(1+|\xi|^2)), & \text{for } |\xi| \leq \delta_0, \\ \frac{1}{2(1+\log(1+|\xi|^2))}, & \text{for } |\xi| > \delta_0. \end{cases} \tag{4.13}$$

As a consequence of Lemma 4.8, we have

$$\rho(\xi) = \min\left\{\frac{\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}{2}, \frac{1}{2(1+\log(1+|\xi|^2))}, \frac{\sqrt{\log(1+|\xi|^2)}}{2}\right\}.$$

By multiplying the equation (4.11) by $\rho(\xi)\overline{\hat{u}}$ we obtain the identity

$$\rho(\xi)(1+\log(1+|\xi|^2))\hat{u}_{tt}\overline{\hat{u}} + \rho(\xi)\log(1+|\xi|^2)(1+\log(1+|\xi|^2))|\hat{u}|^2 + \frac{\rho(\xi)}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\hat{u}|^2 = 0 \tag{4.14}$$

for all $t > 0$ and $\xi \in \boldsymbol{R}^n$. Taking the real part on the last identity we arrive at

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left[\rho(\xi)(1+\log(1+|\xi|^2))\mathrm{Re}(\hat{u}_t\overline{\hat{u}}) + \frac{\rho(\xi)}{2}|\hat{u}|^2\right] \\ &\quad + \rho(\xi)\log(1+|\xi|^2)(1+\log(1+|\xi|^2))|\hat{u}|^2 = \rho(\xi)(1+\log(1+|\xi|^2))|\hat{u}_t|^2. \end{aligned} \tag{4.15}$$

To proceed further we need to define the following functions on $(0,\infty) \times \boldsymbol{R}^n$ such that

$$\begin{aligned} E(t,\xi) &= E_0(t,\xi) + \rho(\xi)(1+\log(1+|\xi|^2))\mathrm{Re}(\hat{u}_t\overline{\hat{u}}) + \frac{\rho(\xi)}{2}|\hat{u}|^2, \\ F(t,\xi) &= |\hat{u}_t|^2 + \rho(\xi)\log(1+|\xi|^2)(1+\log(1+|\xi|^2))|\hat{u}|^2, \\ R(t,\xi) &= \rho(\xi)(1+\log(1+|\xi|^2))|\hat{u}_t|^2. \end{aligned} \tag{4.16}$$

Then, by adding (4.12) and (4.15), we get the following identity

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t,\xi)+F(t,\xi)=R(t,\xi), \quad t>0,\ \xi\in\boldsymbol{R}^n. \tag{4.17}$$

Lemma 4.9. *It holds that*

$$\frac{1}{2}E_0(t,\xi)\le E(t,\xi)\le 3E_0(t,\xi),\ t>0,\ \xi\in\boldsymbol{R}^n.$$

Proof. By Lemma 4.8 we have for $t>0$ and $\xi\in\boldsymbol{R}^n$,

$$\begin{aligned}
E(t,\xi) &\le E_0(t,\xi)+\rho(\xi)(1+\log(1+|\xi|^2))|\hat{u}_t||\hat{u}|+\frac{\rho(\xi)}{2}|\hat{u}|^2\\
&\le E_0(t,\xi)+\frac{1+\log(1+|\xi|^2)}{2}|\hat{u}_t|^2+\frac{\rho(\xi)^2(1+\log(1+|\xi|^2))}{2}|\hat{u}|^2+\frac{\rho(\xi)}{2}|\hat{u}|^2\\
&\le E_0(t,\xi)+\frac{1+\log(1+|\xi|^2)}{2}|\hat{u}_t|^2+\frac{\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}{8}|\hat{u}|^2\\
&+\frac{\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}{4}|\hat{u}|^2\\
&\le 3E_0(t,\xi),
\end{aligned}$$

according to the definition of $E_0(t,\xi)$ in (4.12).

On the other hand, one has

$$\begin{aligned}
-\rho(\xi)(1+\log(1+|\xi|^2))\mathrm{Re}(\hat{u}_t\bar{\hat{u}}) &\le \rho(\xi)(1+\log(1+|\xi|^2))|\hat{u}_t||\hat{u}|\\
&\le\frac{1+\log(1+|\xi|^2)}{4}|\hat{u}_t|^2+\rho(\xi)^2(1+\log(1+|\xi|^2))|\hat{u}|^2\\
&\le \frac{1+\log(1+|\xi|^2)}{4}|\hat{u}_t|^2\\
&+\frac{1}{4}\log(1+|\xi|^2)(1+\log(1+|\xi|^2))|\hat{u}|^2,
\end{aligned}$$

for $t>0$ and $\xi\in\boldsymbol{R}^n$.

Thus, the last estimate implies

$$\begin{aligned}
E(t,\xi) &= E_0(t,\xi)+\rho(\xi)(1+\log(1+|\xi|^2))\mathrm{Re}(\hat{u}_t\bar{\hat{u}})+\frac{\rho(\xi)}{2}|\hat{u}|^2\\
&\ge E_0(t,\xi)+\rho(\xi)(1+\log(1+|\xi|^2))\mathrm{Re}(\hat{u}_t\bar{\hat{u}})\\
&\ge\left(\frac{1}{2}-\frac{1}{4}\right)(1+\log(1+|\xi|^2))|\hat{u}_t|^2\\
&+\left(\frac{1}{2}-\frac{1}{4}\right)\log(1+|\xi|^2)(1+\log(1+|\xi|^2))|\hat{u}|^2\\
&=\frac{1}{4}(1+\log(1+|\xi|^2))|\hat{u}_t|^2+\frac{1}{4}\log(1+|\xi|^2)(1+\log(1+|\xi|^2))|\hat{u}|^2\\
&=\frac{1}{2}E_0(t,\xi),
\end{aligned}$$

for $t>0$ and $\xi\in\boldsymbol{R}^n$. These imply the desired estimates. □

Now, we need the next lemma.

Lemma 4.10. *It is true that*

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t,\xi) + \frac{\rho(\xi)}{2}E(t,\xi) \le 0, \; t > 0, \; \xi \in \boldsymbol{R}^n.$$

Proof. By definitions of $E(t,\xi)$, $F(t,\xi)$, $R(t,\xi)$, $\rho(\xi)$ and (4.17), one obtains a series of inequalities

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E(t,\xi) + \frac{\rho(\xi)}{2}E(t,\xi) &= R(t,\xi) - F(t,\xi) + \frac{\rho(\xi)}{2}E(t,\xi) \\
&\le R(t,\xi) - F(t,\xi) + \frac{3\rho(\xi)}{2}E_0(t,\xi) \\
&= \left(\frac{7}{4}\rho(\xi)(1+\log(1+|\xi|^2)) - 1\right)|\hat{u}_t|^2 \\
&- \frac{1}{4}\rho(\xi)\log(1+|\xi|^2)(1+\log(1+|\xi|^2))|\hat{u}|^2 \\
&\le -\frac{1}{8}|\hat{u}_t|^2 - \frac{1}{4}\rho(\xi)\log(1+|\xi|^2)(1+\log(1+|\xi|^2))|\hat{u}|^2 \\
&\le 0
\end{aligned}$$

for $t > 0$ and $\boldsymbol{R}^n$. The lemma is proved. □

Now, it follows from Lemma 4.10 that

$$E(t,\xi) \le E(0,\xi)e^{-\frac{\rho(\xi)}{2}t},$$

for $t > 0$ and $\xi \in \boldsymbol{R}^n$. By combining the last estimate with Lemma 4.9, one can deduce the important pointwise estimate,

$$E_0(t,\xi) \le 6E_0(0,\xi)e^{-\frac{\rho(\xi)}{2}t},$$

for $t > 0$ and $\xi \in \boldsymbol{R}^n$.

The above estimate combined with the definition of $E_0(t,\xi)$ in (4.12) implies the following crucial pointwise estimate.

Proposition 4.11. *It holds that*

$$\begin{aligned}
&(1+\log(1+|\xi|^2))|\hat{u}_t(t,\xi)|^2 + \log(1+|\xi|^2)(1+\log(1+|\xi|^2))|\hat{u}(t,\xi)|^2 \le \\
&6(1+\log(1+|\xi|^2))e^{-\frac{\rho(\xi)}{2}t}|\hat{u}_1(\xi)|^2 + 6\log(1+|\xi|^2)(1+\log(1+|\xi|^2))e^{-\frac{\rho(\xi)}{2}t}|\hat{u}_0(\xi)|^2
\end{aligned}$$

for all $t > 0$ and $\xi \in \boldsymbol{R}^n$, and

$$|\hat{u}(t,\xi)|^2 \le 6\frac{e^{-\frac{\rho(\xi)}{2}t}}{\log(1+|\xi|^2)}|\hat{u}_1(\xi)|^2 + 6e^{-\frac{\rho(\xi)}{2}t}|\hat{u}_0(\xi)|^2$$

for all $t > 0$ and $\xi \in \boldsymbol{R}^n$, $\xi \neq 0$.

As a consequence of the second estimate in Proposition 4.11 one can get the following result.

Proposition 4.12. *Let $n \geq 3$, and let $u(t, \xi)$ be the solution to problem (4.1)-(4.2). Suppose $u_0 \in L^1(\boldsymbol{R}^n) \cap Y^{\frac{n-2}{2}}$, $u_1 \in L^1(\boldsymbol{R}^n) \cap Y^{\frac{n-2}{2}}$. Then, the following estimate holds:*

$$\int_{\boldsymbol{R}^n} |u(t,x)|^2 dx \leq C_1 t^{-\frac{n-2}{2}} \left[\|u_1\|_{L^1}^2 + \|u_0\|_{Y^{\frac{n-2}{2}}}^2 \right] + C_2 t^{-\frac{n}{2}} \left[\|u_1\|_{Y^{\frac{n-2}{2}}}^2 + \|u_0\|_{L^1}^2 \right],$$

for $t > 0$, where C_1 and C_2 are positive constants depending only on n.

Proof. Let $\delta_0 > 0$ be a given real number obtained in Lemma 4.8. To prove the proposition above one needs to consider separately the zones of low and high frequency.

1) Estimate on the zone $|\xi| \leq \delta_0$

On this region one notices $\rho(\xi) = \frac{1}{2}\log(1+|\xi|^2)(1+\log(1+|\xi|^2))$. Then, one can observe that $1 \leq 1 + \log(1+|\xi|^2) \leq 1 + \log(1+\delta_0^2)$ for $|\xi| \leq \delta_0$. Thus we get

$$\frac{1}{2}\log(1+|\xi|^2) \leq \rho(\xi) \leq \frac{1+\log(1+\delta_0^2)}{2}\log(1+\delta_0^2), \quad |\xi| \leq \delta_0.$$

Then, by applying the second estimate of Proposition 4.11, one obtains

$$\begin{aligned}
\int_{|\xi|\leq\delta_0} |\hat{u}|^2 d\xi &\leq 6\int_{|\xi|\leq\delta_0} \frac{e^{-\frac{\rho(\xi)}{2}t}}{\log(1+|\xi|^2)} |\hat{u}_1|^2 d\xi + 6\int_{|\xi|\leq\delta_0} e^{-\frac{\rho(\xi)}{2}t} |\hat{u}_0|^2 d\xi \\
&\leq 6\int_{|\xi|\leq\delta_0} \frac{e^{-\frac{\log(1+|\xi|^2)}{4}t}}{\log(1+|\xi|^2)} |\hat{u}_1|^2 d\xi + 6\int_{|\xi|\leq\delta_0} e^{-\frac{\log(1+|\xi|^2)}{4}t} |\hat{u}_0|^2 d\xi \\
&= 6\int_{|\xi|\leq\delta_0} (1+|\xi|^2)^{-\frac{t}{4}} \frac{1}{\log(1+|\xi|^2)} |\hat{u}_1|^2 d\xi \\
&+ 6\int_{|\xi|\leq\delta_0} (1+|\xi|^2)^{-\frac{t}{4}} |\hat{u}_0|^2 d\xi \\
&\leq 6\|\hat{u}_1\|_{L^\infty}^2 \int_{|\xi|\leq\delta_0} (1+|\xi|^2)^{-\frac{t}{4}} \frac{1}{\log(1+|\xi|^2)} d\xi \\
&+ 6\|\hat{u}_0\|_{L^\infty}^2 \int_{|\xi|\leq\delta_0} (1+|\xi|^2)^{-\frac{t}{4}} d\xi \\
&\leq 6\omega_n \|u_1\|_{L^1}^2 \int_0^{\delta_0} (1+r^2)^{-\frac{t}{4}} \frac{r^2}{\log(1+r^2)} r^{n-3} dr \\
&+ 6\omega_n \|u_0\|_{L^1}^2 \int_0^{\delta_0} (1+r^2)^{-\frac{t}{4}} r^{n-1} dr \\
&\leq C_{0,n}\omega_n \|u_1\|_{L^1}^2 \int_0^{\delta_0} (1+r^2)^{-\frac{t}{4}} r^{n-3} dr \\
&+ 6\omega_n \|u_0\|_{L^1}^2 \int_0^{\delta_0} (1+r^2)^{-\frac{t}{4}} r^{n-1} dr \\
&\leq C_{0,n}\omega_n \|u_1\|_{L^1}^2 \int_0^{1} (1+r^2)^{-\frac{t}{4}} r^{n-3} dr
\end{aligned}$$

$$
\begin{aligned}
&+ 6\omega_n\|u_0\|^2_{L^1}\int_0^1 (1+r^2)^{-\frac{t}{4}}r^{n-1}dr\\
&\leq C_{1,n}\|u_1\|^2_{L^1}t^{-\frac{n-2}{2}} + C_{2,n}\|u_0\|^2_{L^1}t^{-\frac{n}{2}}, \ \ t>0
\end{aligned}
$$

with some constants $C_{j,n} > 0$ $(j = 0, 1)$, where we have just used Lemma 2.28 and the fact that

$$
\lim_{\sigma\to+0}\frac{\sigma}{\log(1+\sigma)} = 1.
$$

2) Estimate on the zone $|\xi| \geq \delta_0$

In this case, one notices $\rho(\xi) = \dfrac{1}{2(1+\log(1+|\xi|^2))}$. By Proposition 4.11 and the definition of $\rho(\xi)$ we have

$$
\begin{aligned}
\int_{|\xi|\geq\delta_0}|\hat{u}|^2d\xi &\leq 6\int_{|\xi|\geq\delta_0}\frac{e^{-\frac{\rho(\xi)}{2}t}}{\log(1+|\xi|^2)}|\hat{u}_1|^2d\xi + 6\int_{|\xi|\geq\delta_0}e^{-\frac{\rho(\xi)}{2}t}|\hat{u}_0|^2d\xi\\
&= 6\int_{|\xi|\geq\delta_0}\frac{e^{-\frac{1}{4(1+\log(1+|\xi|^2))}t}}{\log(1+|\xi|^2)}|\hat{u}_1|^2d\xi + 6\int_{|\xi|\geq\delta_0}e^{-\frac{1}{4(1+\log(1+|\xi|^2))}t}|\hat{u}_0|^2d\xi\\
&\leq Ct^{-\nu}\int_{|\xi|\geq\delta_0}\frac{(1+\log(1+|\xi|^2))^{\nu}}{\log(1+|\xi|^2)}|\hat{u}_1|^2d\xi\\
&+ Ct^{-\nu'}\int_{|\xi|\geq\delta_0}(1+\log(1+|\xi|^2))^{\nu'}|\hat{u}_0|^2d\xi\\
&\leq Ct^{-\frac{n}{2}}\int_{|\xi|\geq\delta_0}\frac{(1+\log(1+|\xi|^2))^{\frac{n}{2}}}{\log(1+|\xi|^2)}|\hat{u}_1|^2d\xi\\
&+ Ct^{-\frac{n-2}{2}}\int_{|\xi|\geq\delta_0}(1+\log(1+|\xi|^2))^{\frac{n-2}{2}}|\hat{u}_0|^2d\xi,
\end{aligned}
$$

where we just applied Lemma 2.20 with $\nu = \frac{n}{2}$ and $a = -1$, and $\nu' = \frac{n-2}{2}$ and $a = -1$ to the last two integrals.

Thus we may conclude that

$$
\begin{aligned}
\int_{|\xi|\geq\delta_0}|\hat{u}|^2d\xi \ &\leq \ Ct^{-\frac{n}{2}}\int_{|\xi|\geq\delta_0}\frac{1+\log(1+|\xi|^2)}{\log(1+|\xi|^2)}(1+\log(1+|\xi|^2))^{\frac{n-2}{2}}|\hat{u}_1|^2d\xi\\
&+ \ Ct^{-\frac{n-2}{2}}\int_{|\xi|\geq\delta_0}(1+\log(1+|\xi|^2))^{\frac{n-2}{2}}|\hat{u}_0|^2d\xi\\
&\leq \ C_1t^{-\frac{n}{2}}\int_{|\xi|\geq\delta_0}(1+\log(1+|\xi|^2))^{\frac{n-2}{2}}|\hat{u}_1|^2d\xi\\
&+ \ Ct^{-\frac{n-2}{2}}\int_{|\xi|\geq\delta_0}(1+\log(1+|\xi|^2))^{\frac{n-2}{2}}|\hat{u}_0|^2d\xi\\
&\leq \ C_1t^{-\frac{n}{2}}\|u_1\|^2_{Y^{\frac{n-2}{2}}} + Ct^{-\frac{n-2}{2}}\|u_0\|^2_{Y^{\frac{n-2}{2}}},
\end{aligned}
$$

where one has just used the property

$$
\lim_{\sigma\to\infty}\frac{1+\log(1+\sigma)}{\log(1+\sigma)} = 1.
$$

By adding the two estimates for low and high frequencies and using the Plancherel Theorem, one can conclude the proof of proposition. □

4.3 ASYMPTOTIC EXPANSION

Applying the Fourier transform on the problem (4.4) one obtain the associated problem in Fourier space:

$$\begin{aligned} &(1+\log(1+|\xi|^2))\hat{u}_{tt} + \log(1+|\xi|^2)(1+\log(1+|\xi|^2))\hat{u} + \hat{u}_t = 0, \\ &\hat{u}(0,\xi) = \hat{u}_0(\xi), \\ &\hat{u}_t(0,\xi) = \hat{u}_1(\xi). \end{aligned} \tag{4.18}$$

The characteristic roots of the associated polynomial to the equation in (4.18) are given by

$$\lambda_\pm = \frac{-1 \pm \sqrt{1-4\log(1+|\xi|^2)(1+\log(1+|\xi|^2))^2}}{2(1+\log(1+|\xi|^2))}.$$

We observe that there exists a unique real number $\delta > 0$ such that

$$1-4\log(1+|\xi|^2)(1+\log(1+|\xi|^2))^2 \begin{cases} \geq 0 & \text{for} |\xi| \leq \delta, \\ < 0 & \text{for} |\xi| > \delta. \end{cases} \tag{4.19}$$

In fact, the function $f(r) = 1-4\log(1+r^2)(1+\log(1+r^2))^2$ is decreasing for $r \geq 0$, continuous and

$$f(0) = 1, f(1) = 1-4\log 2\cdot(1+\log 2)^2 < 0.$$

Therefore, by the mean value theorem there exists a unique number $\delta \in (0,1)$ that satisfies (4.19). The same theorem guarantees us the existence of $0 < \eta < \delta$ such that

$$\frac{1}{2} \leq \sqrt{1-4\log(1+|\xi|^2)(1+\log(1+|\xi|^2))^2} \leq 1, \tag{4.20}$$

whenever $|\xi| \leq \eta$.

The next lemma is very important to get sharp estimates. In the following notation $A \approx B$ means that $c_1 A \leq B \leq c_2 A$ for some positive constants c_1, c_2.

Lemma 4.13. *It holds that*

(i). $\lambda_+ \approx -\log(1+|\xi|^2)$,

(ii). $\lambda_- \approx -1$,

(iii). $\lambda_+ + \lambda_- \approx -1$,

whenever $|\xi| \leq \delta$. *And, in particular, in the case of* $|\xi| \leq \eta$, *one has*

(iv). $\lambda_+ - \lambda_- \approx 1$.

Remark 4.14. Note that the constants c_1 and c_2 appeared in Lemma 4.13 may depend on δ or η. We also note that the four items in Lemma 4.13 simultaneously hold on the zone $\{|\xi| \leq \eta\}$ because of $\eta < \delta$.

Proof. (i). For $a > 0$ and $b \geq 0$, it holds that

$$-a + \sqrt{b} = \frac{b - a^2}{a + \sqrt{b}}.$$

This identity implies that

$$\begin{aligned}-1+&\sqrt{1 - 4\log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2} = \\ &= -\frac{4\log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2}{1 + \sqrt{1 - 4\log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2}},\end{aligned}$$

since $1 - 4\log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2 \geq 0$ for $|\xi| \leq \delta$. Then,

$$\lambda_+ = -2\log(1 + |\xi|^2)\frac{1 + \log(1 + |\xi|^2)}{1 + \sqrt{1 - 4\log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2}}, \quad |\xi| \leq \delta.$$

Now, for $|\xi| \leq \delta$ we have $1 \leq 1 + \log(1 + |\xi|^2) \leq 1 + \log(1 + \delta^2) =: K_\delta$ and

$$1 \leq 1 + \sqrt{1 - 4\log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2} \leq 2.$$

Therefore we may conclude that

$$-2K_\delta \log(1 + |\xi|^2) \leq \lambda_+ \leq -\log(1 + |\xi|^2), \quad |\xi| \leq \delta.$$

This implies the desired statement of (i).

(ii). Since $0 \leq \sqrt{1 - 4\log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2} \leq 1$ in the region $|\xi| \leq \delta$,

$$\begin{aligned}\frac{-1}{1 + \log(1 + |\xi|^2)} &\leq \frac{-1 - \sqrt{1 - 4\log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2}}{2(1 + \log(1 + |\xi|^2))} \\ &\leq \frac{-1}{2(1 + \log(1 + |\xi|^2))}.\end{aligned}$$

Therefore,

$$-1 \leq \lambda_- \leq \frac{-1}{2K_\delta}.$$

(iii). To prove this item we observe that $\lambda_+ + \lambda_- = \frac{-1}{1+\log(1+|\xi|^2)}$. Hence,

$$-1 \leq \lambda_+ + \lambda_- \leq -\frac{1}{K_\delta},$$

for $|\xi| \leq \delta$. And we obtain the equivalence $\lambda_+ + \lambda_- \approx -1$.

(iv). We observe that we have chosen $\eta > 0$ in (4.20) such that

$$\frac{1}{2} \leq \sqrt{1 - 4\log(1+|\xi|^2)(1+\log(1+|\xi|^2))^2} \leq 1$$

for all $|\xi| \leq \eta$. Thus, one can deduce

$$\begin{aligned} \frac{1}{2K_\delta} &\leq \frac{1}{2(1+\log(1+|\xi|^2))} \leq \frac{\sqrt{1 - 4\log(1+|\xi|^2)(1+\log(1+|\xi|^2))^2}}{1+\log(1+|\xi|^2)} \\ &\leq \frac{1}{1+\log(1+|\xi|^2)} \leq 1. \end{aligned}$$

This estimate shows the desired equivalence $\lambda_+ - \lambda_- \approx 1$ on the region $|\xi| \leq \eta$.

□

In the next subsection to use Lemma 4.13 we work on the zone $\{|\xi| \leq \eta\}$, where η is given in (4.20).

4.3.1 Estimates on the low frequency zone $|\xi| \leq \delta$

(i) Estimates on the low frequency zone $|\xi| \leq \eta$:

We remember that η is defined in (4.20). In this case, the characteristics roots $\lambda_\pm$ are real-valued and the solution of (4.18) is explicitly given by

$$\hat{u}(t,\xi) = \frac{\lambda_- \hat{u}_0(\xi) - \hat{u}_1(\xi)}{\lambda_- - \lambda_+} e^{t\lambda_+} + \frac{\hat{u}_1(\xi) - \lambda_+ \hat{u}_0(\xi)}{\lambda_- - \lambda_+} e^{t\lambda_-}. \tag{4.21}$$

We observe that

$$\begin{aligned} \lambda_- &= -\log(1+|\xi|^2)(1+\log(1+|\xi|^2)) - (1+\log(1+|\xi|^2))\lambda_-^2, \\ \lambda_+ &= -\log(1+|\xi|^2)(1+\log(1+|\xi|^2)) - (1+\log(1+|\xi|^2))\lambda_+^2, \end{aligned}$$

for $|\xi| \leq \delta$. Therefore we can rewrite $\hat{u}(t,\xi)$ as follows

$$\hat{u}(t,\xi) = e^{-t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))} \left(H_1(t,\xi) + H_2(t,\xi)\right), \tag{4.22}$$

where

$$\begin{aligned} H_1(t,\xi) &= \frac{\lambda_- \hat{u}_0(\xi) - \hat{u}_1(\xi)}{\lambda_- - \lambda_+} e^{-t(1+\log(1+|\xi|^2))\lambda_+^2}, \\ H_2(t,\xi) &= \frac{\hat{u}_1(\xi) - \lambda_+ \hat{u}_0(\xi)}{\lambda_- - \lambda_+} e^{-t(1+\log(1+|\xi|^2))\lambda_-^2}. \end{aligned}$$

We can also use the Chill-Haraux [11] idea that has also been used in [29] to decompose $H_1(t,\xi)$ as

$$\begin{aligned} H_1(t,\xi) &= \hat{u}_0 + \hat{u}_1 + \frac{\lambda_+ - \lambda_-}{\lambda_- - \lambda_+}\hat{u}_0 - \frac{\lambda_+ - \lambda_-}{\lambda_+ - \lambda_-}\hat{u}_1 + H_1(t,\xi) \\ &= \hat{u}_0 + \hat{u}_1 - \frac{\lambda_+}{\lambda_+ - \lambda_-}\hat{u}_0 + \frac{\lambda_- \hat{u}_0 \left(e^{-t(1+\log(1+|\xi|^2))\lambda_+^2} - 1\right)}{\lambda_- - \lambda_+} \\ &+ \frac{\hat{u}_1 \left(e^{-t(1+\log(1+|\xi|^2))\lambda_+^2} - (\lambda_+ - \lambda_-)\right)}{\lambda_+ - \lambda_-}. \end{aligned}$$

By combining the last expression together with the decomposition $\hat{u}_j(\xi) = A_j(\xi) - iB_j(\xi) + P_j$ for initial data given in (2.8) we can get the following expression for $\hat{u}(t,\xi)$ which holds for $|\xi| \leq \eta$:

$$\begin{aligned}
\hat{u}(t,\xi) &= e^{-t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}\Big(A_0(\xi) - iB_0(\xi) + P_0 + A_1(\xi) - iB_1(\xi) + P_1\Big) \\
&\quad - e^{-t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}\frac{\lambda_+}{\lambda_+ - \lambda_-}\hat{u}_0 \\
&\quad + e^{-t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}\frac{\lambda_-\hat{u}_0\left(e^{-t(1+\log(1+|\xi|^2))\lambda_+^2} - 1\right)}{\lambda_- - \lambda_+} \\
&\quad + e^{-t\log(1+|\xi|^2)\left(1+\log(1+|\xi|^2)\right)}\frac{\hat{u}_1\left(e^{-t(1+\log(1+|\xi|^2))\lambda_+^2} - (\lambda_+ - \lambda_-)\right)}{\lambda_+ - \lambda_-} \\
&\quad + e^{-t\log(1+|\xi|^2)\left(1+\log(1+|\xi|^2)\right)}H_2(t,\xi).
\end{aligned} \tag{4.23}$$

Our main goal in this subsection is to introduce an asymptotic profile of the solution $\hat{u}(t,\xi)$ in the low frequency region as $t \to \infty$ in a simple form as

$$\varphi_1(t,\xi) := (P_0 + P_1)e^{-t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}. \tag{4.24}$$

For this purpose, it is necessary to find suitable estimates for the other six terms of the expression (4.23) defined by the functions

$$\begin{aligned}
F_1(t,\xi) &= e^{-t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}\left(A_0(\xi) - iB_0(\xi)\right), \\
F_2(t,\xi) &= e^{-t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}\left(A_1(\xi) - iB_1(\xi)\right), \\
F_3(t,\xi) &= -e^{-t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}\frac{\lambda_+}{\lambda_+ - \lambda_-}\hat{u}_0, \\
F_4(t,\xi) &= e^{-t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}\frac{\lambda_-\hat{u}_0\left(e^{-t(1+\log(1+|\xi|^2))\lambda_+^2} - 1\right)}{\lambda_- - \lambda_+}, \\
F_5(t,\xi) &= e^{-t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}\frac{\hat{u}_1\left(e^{-t(1+\log(1+|\xi|^2))\lambda_+^2} - (\lambda_+ - \lambda_-)\right)}{\lambda_+ - \lambda_-}, \\
F_6(t,\xi) &= e^{-t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}H_2(t,\xi).
\end{aligned}$$

Therefore, from (4.23) and (4.24), for $|\xi| \leq \eta$ we have

$$\hat{u}(t,\xi) - \varphi_1(t,\xi) = \sum_{j=1}^{6} F_j(t,\xi). \tag{4.25}$$

In order to estimate the difference given by (4.25) on the zone of low frequency $\{|\xi| \leq \eta\}$ we shall develop the next computations based on Lemmas 2.24 and 4.13. In addition, we also assume $u_0, u_1 \in L^{1,1}(\boldsymbol{R}^n)$.

Now, we first observe that

$$1 \leq 1 + \log(1+|\xi|^2) \leq 1 + \log(1+\eta^2) =: k_\eta, \quad |\xi| \leq \eta.$$

Then, for $j = 0, 1$ by using Lemma 2.24 with $\kappa = 1$ one has

$$\begin{aligned}
\int_{|\xi|\leq\eta} |F_{j+1}(t,\xi)|^2 d\xi &= \int_{|\xi|\leq\eta} e^{-2t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}|A_j(\xi) - iB_j(\xi)|^2 d\xi \\
&\leq \int_{|\xi|\leq\eta} e^{-2t\log(1+|\xi|^2)}|A_j(\xi) - iB_j(\xi)|^2 d\xi \\
&= \int_{|\xi|\leq\eta} (1+|\xi|^2)^{-2t}|A_j(\xi) - iB_j(\xi)|^2 d\xi \\
&\leq (L+M)^2\|u_j\|_{1,1}^2 \int_{|\xi|\leq\eta} (1+|\xi|^2)^{-2t}|\xi|^2 d\xi \\
&= \omega_n(L+M)^2\|u_j\|_{1,1}^2 \int_0^\eta (1+r^2)^{-2t} r^{n+1} d\xi \\
&\leq \omega_n(L+M)^2\|u_j\|_{1,1}^2 \int_0^1 (1+r^2)^{-2t} r^{n+1} d\xi \\
&\leq C\|u_j\|_{1,1}^2 t^{-\frac{n+2}{2}}
\end{aligned}$$

for $t \gg 1$ due to Lemma 2.28. Consequently, for $t \gg 1$ we have

$$\int_{|\xi|\leq\eta} |F_1(t,\xi)|^2 d\xi \leq C\|u_0\|_{1,1}^2 t^{-\frac{n+2}{2}} \quad \text{and} \quad \int_{|\xi|\leq\eta} |F_2(t,\xi)|^2 d\xi \leq C\|u_1\|_{1,1}^2 t^{-\frac{n+2}{2}}.$$

In order to get an estimate on the function $F_3(t,\xi)$, we observe that

$$\begin{aligned}
\int_{|\xi|\leq\eta} |F_3(t,\xi)|^2 d\xi &= \int_{|\xi|\leq\eta} e^{-2t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))} \left(\frac{\lambda_+}{\lambda_+ - \lambda_-}\right)^2 |\hat{u}_0|^2 d\xi \\
&\leq C \int_{|\xi|\leq\eta} e^{-2t\log(1+|\xi|^2)} \log^2(1+|\xi|^2)|\hat{u}_0|^2 d\xi,
\end{aligned}$$

because, for $|\xi| \leq \eta$, we have

$$\frac{\lambda_+}{\lambda_+ - \lambda_-} \approx -\log(1+|\xi|^2)$$

due to items (i) and (iv) of Lemma 4.13. We also observe that $\log(1+r^2) \leq r^2$ for all $r \geq 0$ and we may use this simple inequality to conclude the L^2-estimate for $F_3(t,\xi)$ as follows.

$$\begin{aligned}
\int_{|\xi|\leq\eta} |F_3(t,\xi)|^2 d\xi &\leq C\|u_0\|_1^2 \int_{|\xi|\leq\eta} (1+|\xi|^2)^{-2t}|\xi|^4 d\xi \\
&= C\omega_n\|u_0\|_1^2 \int_0^\eta (1+r^2)^{-2t} r^{n+3} dr \\
&\leq C\omega_n\|u_0\|_1^2 \int_0^1 (1+r^2)^{-t} r^{n+3} dr \\
&\leq C\omega_n\|u_0\|_1^2 t^{-\frac{n+4}{2}}, \quad t \gg 1,
\end{aligned}$$

where we also used that $|\hat{u}_0(\xi)| \leq \|u_0\|_1$ for all $\xi \in \boldsymbol{R}^n$ and Lemma 2.28.

To estimate the function $F_4(t,\xi)$ we need the elementary inequality

$$|e^{-a}-1| \le a, \quad a \ge 0. \tag{4.26}$$

We also remember that

$$\Big|\frac{\lambda_-}{\lambda_- - \lambda_+}\Big| \approx 1, \quad |\xi| \le \eta$$

from Lemma 4.13. Thus, we have

$$\begin{aligned}
\int_{|\xi|\le\eta} |F_4(t,\xi)|^2 d\xi &\le C\int_{|\xi|\le\eta} e^{-2t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}\Big(e^{-t(1+\log(1+|\xi|^2))\lambda_+^2}-1\Big)^2 |\hat{u}_0|^2 d\xi \\
&\le Ct^2\int_{|\xi|\le\eta} e^{-2t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}(1+\log(1+|\xi|^2))^2\lambda_+^4|\hat{u}_0|^2 d\xi \\
&\le Ct^2\|u_0\|_1^2\int_{|\xi|\le\eta}(1+|\xi|^2)^{-2t}\log^4(1+|\xi|^2)d\xi \\
&\le Ct^2\|u_0\|_1^2\int_{|\xi|\le\eta}(1+|\xi|^2)^{-2t}|\xi|^8 d\xi \\
&\le \omega_n Ct^2\|u_0\|_1^2 t^{-\frac{n+8}{2}} \\
&= C_n\|u_0\|_1^2 t^{-\frac{n+4}{2}}, \quad t \gg 1
\end{aligned}$$

because of $1+\log(1+|\xi|^2) \le 1+\log 2$, for $|\xi| \le \eta < 1$, where we also used the fact that $|\lambda_+| \le C\log(1+|\xi|^2)$ for $|\xi| \le \eta$. The constant $C_n > 0$ depends only on n.

In order to get estimates to the remainder function $F_5(t,\xi)$ on $|\xi| \le \eta$, we can use the inequality $(a-b)^2 \le 2a^2+2b^2$ to obtain the following estimate.

$$\begin{aligned}
&\int_{|\xi|\le\eta} |F_5(t,\xi)|^2 d\xi \\
&\quad = \int_{|\xi|\le\eta} e^{-2t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}\frac{\Big(e^{-t(1+\log(1+|\xi|^2))\lambda_+^2}-(\lambda_+-\lambda_-)\Big)^2}{(\lambda_+-\lambda_-)^2}|\hat{u}_1|^2 d\xi \\
&\quad \le 2\int_{|\xi|\le\eta} e^{-2t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}\frac{\Big(e^{-t(1+\log(1+|\xi|^2))\lambda_+^2}-1\Big)^2}{(\lambda_+-\lambda_-)^2}|\hat{u}_1|^2 d\xi \\
&\quad + 2\int_{|\xi|\le\eta} e^{-2t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}\frac{(1-(\lambda_+-\lambda_-))^2}{(\lambda_+-\lambda_-)^2}|\hat{u}_1|^2 d\xi.
\end{aligned} \tag{4.27}$$

Now, let $D = 1-4\log(1+|\xi|^2)(1+\log(1+|\xi|^2))^2$. Then we observe that $D \ge 0$ for $|\xi| \le \eta$ and

$$1-(\lambda_+-\lambda_-) = \frac{2\log(1+|\xi|^2)+\log^2(1+|\xi|^2)+4\log(1+|\xi|^2)(1+\log(1+|\xi|^2))^2}{\big(1+\log(1+|\xi|^2)\big)\big(1+\log(1+|\xi|^2)+\sqrt{D}\big)}.$$

In particular, $1-(\lambda_+-\lambda_-)$ is positive and there exists a constant $C_\eta > 0$ such that

$$|1-(\lambda_+-\lambda_-)| \le C_\eta \log(1+|\xi|^2) \tag{4.28}$$

for all $|\xi| \leq \eta$, where η is defined in (4.20). From (4.27), (4.26) and (4.28), we obtain the next estimate.

$$\begin{aligned}\int_{|\xi|\leq\eta}|F_5(t,\xi)|^2 d\xi &\leq C\int_{|\xi|\leq\eta} e^{-2t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}\frac{t^2(1+\log(1+|\xi|^2))^2\lambda_+^4}{(\lambda_+-\lambda_-)^2}|\hat{u}_1|^2 d\xi\\ &+2C_\eta\int_{|\xi|\leq\eta} e^{-2t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}\log^2(1+|\xi|^2)|\hat{u}_1|^2 d\xi\\ &\leq Ct^2\int_{|\xi|\leq\eta} e^{-t\log(1+|\xi|^2)}\log^4(1+|\xi|^2)|\hat{u}_1|^2 d\xi\\ &+2C_\eta\int_{|\xi|\leq\eta} e^{-t\log(1+|\xi|^2)}\log^2(1+|\xi|^2)|\hat{u}_1|^2 d\xi\\ &\leq Ct^2\|u_1\|_1^2\int_{|\xi|\leq 1}(1+|\xi|^2)^{-t}|\xi|^8 d\xi + 2C_\eta\|u_1\|_1^2\int_{|\xi|\leq 1}(1+|\xi|^2)^{-t}|\xi|^4 d\xi\\ &\leq C_n\|u_1\|_1^2 t^{-\frac{n+4}{2}} + 2C_{\eta,n}\|u_1\|_1^2 t^{-\frac{n+4}{2}}, \quad t\gg 1.\end{aligned}$$

Finally, by (ii) of Lemma 4.13 one has $\lambda_- \approx -1$ on the region $|\xi| \leq \eta$, so that there exists constants $c_1, c_2 > 0$ such that $c_1 \leq 2(1+\log(1+|\xi|^2))\lambda_-^2 \leq c_2$ whenever $|\xi| \leq \eta$. Therefore, it follows that

$$\begin{aligned}\int_{|\xi|\leq\eta}|F_6(t,\xi)|^2 d\xi &= \int_{|\xi|\leq\eta} e^{-2t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}H_2^2(t,\xi)d\xi\\ &\leq \int_{|\xi|\leq\eta} e^{-2t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}e^{-2t(1+\log(1+|\xi|^2))\lambda_-^2}\frac{1}{(\lambda_--\lambda_+)^2}|\hat{u}_1|^2 d\xi\\ &+\int_{|\xi|\leq\eta} e^{-2t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}e^{-2t(1+\log(1+|\xi|^2))\lambda_-^2}\left(\frac{\lambda_+}{\lambda_--\lambda_+}\right)^2|\hat{u}_0|^2 d\xi\\ &\leq Ce^{-c_1t}\int_{|\xi|\leq\eta} e^{-t\log(1+|\xi|^2)}|\hat{u}_1|^2 d\xi + Ce^{-c_1t}\int_{|\xi|\leq\eta} e^{-t\log(1+|\xi|^2)}\log^2(1+|\xi|^2)|\hat{u}_0|^2 d\xi\\ &\leq Ce^{-c_1t}\|u_1\|_1^2\int_{|\xi|\leq\eta}(1+|\xi|^2)^{-t}d\xi + Ce^{-c_1t}\|u_0\|_1^2\int_{|\xi|\leq\eta} e^{-t\log(1+|\xi|^2)}|\xi|^4 d\xi\\ &\leq C\|u_1\|_1^2 t^{-\frac{n}{2}}e^{-c_1t} + C\|u_0\|_1^2 t^{-\frac{n+4}{2}}e^{-c_1t}, \qquad t\gg 1.\end{aligned}$$

By combining the above estimates for $F_j(t,\xi)$, $j = 1,\cdots,6$, with equation (4.25), we obtain that the solution $\hat{u}(t,\xi)$ given by (4.23) has the following asymptotic property.

Proposition 4.15. *Let $n \geq 1$ and $\eta > 0$ given by (4.20). For $(u_0,u_1) \in L^{1,1}(\boldsymbol{R}^n)\times L^{1,1}(\boldsymbol{R}^n)$ the solution $\hat{u}(t,\xi)$ to problem (4.18) satisfies*

$$\int_{|\xi|\leq\eta}|\hat{u}(t,\xi)-\varphi_1(t,\xi)|^2 d\xi \leq C\Big(\|u_0\|_{1,1}^2 t^{-\frac{n+2}{2}} + \|u_1\|_{1,1}^2 t^{-\frac{n+2}{2}} + \|u_0\|_1^2 t^{-\frac{n+4}{2}}$$

$$+\|u_1\|_1^2 t^{-\frac{n+4}{2}} + \|u_1\|_1^2 t^{-\frac{n+2}{2}} + \|u_1\|_1^2 t^{-\frac{n}{2}}e^{-c_1t} + \|u_0\|_1^2 t^{-\frac{n+4}{2}}e^{-c_1t}\Big), \quad t\gg 1$$

where $\varphi_1(t,\xi)$ is defined in (4.24) and C is a positive constant that depends only on η and n.

□

(ii) Estimates on the low-middle frequency zone $\eta \leq |\xi| \leq \delta$:

To obtain sharp estimates on the low-middle frequency zone $\eta \leq |\xi| \leq \delta$ it should be noted that according to (4.19) the characteristics roots $\lambda_\pm$ are still real-valued.

We can rewrite the solution $\hat{u}(t,\xi)$, for $\eta \leq |\xi| < \delta$, as follows

$$\begin{aligned}\hat{u}(t,\xi) &= e^{-\frac{t}{2(1+\log(1+|\xi|^2))}}\left[\cosh(c(\xi)t)\hat{u}_0(\xi) + \frac{\sinh(c(\xi)t)}{c(\xi)}\hat{u}_1(\xi)\right] \\ &+ e^{-\frac{t}{2(1+\log(1+|\xi|^2))}}\frac{\sinh(c(\xi)t)}{2(1+\log(1+|\xi|^2))c(\xi)}\hat{u}_0(\xi)\end{aligned} \tag{4.29}$$

where

$$c(\xi) := \frac{\sqrt{1-4\log(1+|\xi|^2)(1+\log(1+|\xi|^2))^2}}{2(1+\log(1+|\xi|^2))} > 0, \quad |\xi| < \delta.$$

Let

$$C_\delta = \frac{1}{2(1+\log(1+\delta^2))}.$$

It is important to observe that (4.29) is not defined for $|\xi| = \delta$, because $c(\xi) = 0$ in this case. However, we note that it is a removable singularity of $\hat{u}(t,\xi)$. Moreover, for $\xi \in \boldsymbol{R}^n$ such that $|\xi| = \delta$, the eigenvalues $\lambda_\pm$ are equal and the solution formula is given by

$$\hat{u}(t,\xi) = e^{-C_\delta t}\hat{u}_0(\xi) + C_\delta t e^{-C_\delta t}\hat{u}_0(\xi) + te^{-C_\delta t}\hat{u}_1(\xi), \quad |\xi| = \delta. \tag{4.30}$$

We remember that δ is given in (4.19) and our choice for η is such that

$$\sqrt{1-4\log(1+|\xi|^2)(1+\log(1+|\xi|^2))^2} \geq \frac{1}{2}$$

when $|\xi| \leq \eta$ (see (4.20)) and this is a decreasing function on $|\xi|$. Thus, in the case for $\eta \leq |\xi| \leq \delta$, one has

$$c(\xi) \leq \frac{1}{4(1+\log(1+|\xi|^2))}.$$

Lemma 4.16. *Let $n \geq 1$ and $u_0, u_1 \in L^2(\boldsymbol{R}^n)$. Then the solution $\hat{u}(t,\xi)$ to problem (4.18) satisfies*

$$\int_{\eta\leq|\xi|\leq\delta} |\hat{u}(t,\xi)|^2 d\xi \leq 4e^{-C_\delta t}\|u_0\|_2^2 + 4Ct^2e^{-C_\delta t}\|u_0\|_2^2 + 4t^2e^{-C_\delta t}\|u_1\|_2^2 \tag{4.31}$$

for $t > 0$, where C is a positive constant that depends on η, and C_δ is defined above and it depends only on δ.

Proof. Due to the fact that $\cosh a \leq e^a$ for all $a \geq 0$ we may estimate for $t > 0$

$$\begin{aligned}e^{-\frac{t}{1+\log(1+|\xi|^2)}}\cosh^2(c(\xi)t) &\leq e^{-\frac{t}{1+\log(1+|\xi|^2)}}e^{2c(\xi)t} \leq e^{-\frac{t}{1+\log(1+|\xi|^2)}}e^{\frac{t}{2(1+\log(1+|\xi|^2))}} \\ &= e^{-\frac{t}{2(1+\log(1+|\xi|^2))}} \leq e^{-C_\delta t}, \quad \eta \leq |\xi| < \delta.\end{aligned} \tag{4.32}$$

Similarly, by using Lemma 2.21, we may obtain

$$e^{-\frac{t}{1+\log(1+|\xi|^2)}}\frac{\sinh^2(c(\xi)t)}{c(\xi)^2} \leq t^2 e^{-C_\delta t}, \quad \eta \leq |\xi| < \delta. \tag{4.33}$$

From the two solution formula (4.29), (4.30) and estimates (4.32), (4.33) combining with Young's inequality we have

$$|\hat{u}(t,\xi)|^2 \leq 4e^{-C_\delta t}|\hat{u}_0(\xi)|^2 + 4C_\eta t e^{-C_\delta t}|\hat{u}_0(\xi)|^2 + 4te^{-C_\delta t}|\hat{u}_1(\xi)|^2, \tag{4.34}$$

for $\eta \leq |\xi| \leq \delta$, where

$$C_\eta = \frac{1}{2(1+\log(1+\eta^2))}.$$

Therefore, we may obtain the desired estimate

$$\begin{aligned}\int_{\eta\leq|\xi|\leq\delta}|\hat{u}(t,\xi)|^2 d\xi \leq 4e^{-C_\delta t}\int_{\eta\leq|\xi|\leq\delta}|\hat{u}_0(\xi)|^2 d\xi + 4C_\eta^2 t^2 e^{-C_\delta t}\int_{\eta\leq|\xi|\leq\delta}|\hat{u}_0(\xi)|^2 d\xi \\ + 4t^2 e^{-C_\delta t}\int_{\eta\leq|\xi|\leq\delta}|\hat{u}_1(\xi)|^2 d\xi \\ = 4e^{-C_\delta t}\|u_0\|_2^2 + 4C_\eta^2 t^2 e^{-C_\delta t}\|u_0\|_2^2 + 4t^2 e^{-C_\delta t}\|u_1\|_2^2, \quad t \geq 1.\end{aligned}$$

□

4.3.2 Estimates on the high frequency zone $|\xi| \geq \delta$

On the zone of high frequency the characteristics roots are complex-valued (see (4.19)) and are given by

$$\lambda_\pm = -a(\xi) \pm ib(\xi),$$

where

$$a(\xi) = \frac{1}{2(1+\log(1+|\xi|^2))} \quad \text{and} \quad b(\xi) = \frac{\sqrt{4\log(1+|\xi|^2)(1+\log(1+|\xi|^2))^2 - 1}}{2(1+\log(1+|\xi|^2))}. \tag{4.35}$$

Then the solution $\hat{u}(t,\xi)$ to problem (4.18) in the high frequency zone is explicitly given by

$$\hat{u}(t,\xi) = e^{-a(\xi)t}\cos(b(\xi)t)\hat{u}_0 + \frac{a(\xi)}{b(\xi)}e^{-a(\xi)t}\sin(b(\xi)t)\hat{u}_0 + \frac{1}{b(\xi)}e^{-a(\xi)t}\sin(b(\xi)t)\hat{u}_1. \tag{4.36}$$

(i) Estimate on the high-middle frequency zone $\delta \leq |\xi| \leq \sqrt{e-1}$.

In this region, the function $a(\xi)$ is equivalent to a constant, that is $\frac{1}{4} \leq a(\xi) \leq \frac{1}{2}$.

Moreover, we can see that $\dfrac{1}{b(\xi)}$ converges to $+\infty$ when $|\xi| \to \delta^+$ according to (4.19). However, we remember that $\sin a \leq a$ for all $a \geq 0$. Thus

$$\sin(b(\xi)t) \leq b(\xi)t$$

for all $\xi \in \boldsymbol{R}^n$ and $t \geq 0$. By combining these properties together with the solution formula (4.36) one can obtain the following estimate for $t > 0$, which implies the exponential decay in such region.

$$\int_{\delta \leq |\xi| \leq \sqrt{e-1}} |\hat{u}(t,\xi)|^2 d\xi \leq e^{-\frac{t}{2}} \|u_0\|_2^2 + \frac{1}{4} t^2 e^{-\frac{t}{2}} \|u_0\|_2^2 + t^2 e^{-\frac{t}{2}} \|u_1\|_2^2. \tag{4.37}$$

(ii) Estimate on the high frequency zone $|\xi| \geq \sqrt{e-1}$

The estimates on this zone are more delicate and the derivation is one of essential contributions in this chapter. We first need to rewrite the solution formula given by (4.36) into a more suitable expression.

First we observe that for $|\xi| \geq \delta$, in particular, for $|\xi| \geq \sqrt{e-1}$, it holds that

$$b(\xi) \leq \sqrt{\log(1+|\xi|^2)}.$$

Then the mean value theorem implies, for $|\xi| \geq \sqrt{e-1}$, that

$$\cos(b(\xi)t) = \cos(\sqrt{\log(1+|\xi|^2)}t) - \sin(\theta(t,\xi)) \left[b(\xi) - \sqrt{\log(1+|\xi|^2)} \right] t,$$

with $\theta(t,\xi) = \alpha b(\xi)t + (1-\alpha)\sqrt{\log(1+|\xi|^2)}t$ for some $\alpha \in (0,1)$. Similarly,

$$\sin(b(\xi)t) = \sin(\sqrt{\log(1+|\xi|^2)}t) + \cos(\eta(t,\xi)) \left[b(\xi) - \sqrt{\log(1+|\xi|^2)} \right] t,$$

with $\eta(t,\xi) = \gamma b(\xi)t + (1-\gamma)\sqrt{\log(1+|\xi|^2)}t$ for some $\gamma \in (0,1)$.

Thus, one can rewrite $\hat{u}(t,\xi)$ as follows:

$$\begin{aligned} \hat{u}(t,\xi) = {} & e^{-a(\xi)t} \cos(\sqrt{\log(1+|\xi|^2)}t)\hat{u}_0 + te^{-a(\xi)t} \sin(\theta(\xi,t)) \left[\sqrt{\log(1+|\xi|^2)} - b(\xi) \right] \hat{u}_0 \\ & + \frac{a(\xi)}{b(\xi)} e^{-a(\xi)t} \sin(b(\xi)t)\hat{u}_0 + \frac{1}{b(\xi)} e^{-a(\xi)t} \sin(\sqrt{\log(1+|\xi|^2)}t)\hat{u}_1 \\ & + te^{-a(\xi)t} \frac{1}{b(\xi)} \cos(\eta(\xi,t)) \left[b(\xi) - \sqrt{\log(1+|\xi|^2)} \right] \hat{u}_1. \end{aligned} \tag{4.38}$$

We introduce an important function to be the asymptotic profile on the zone of high frequency for the solution $\hat{u}(t,\xi)$ given by (4.38) as follows

$$\varphi_2(t,\xi) := e^{-\frac{t}{2\log(1+|\xi|^2)}} \left(\frac{\sin(\sqrt{\log(1+|\xi|^2)}t)}{\sqrt{\log(1+|\xi|^2)}} \hat{u}_1(\xi) + \cos(\sqrt{\log(1+|\xi|^2)}t)\hat{u}_0(\xi) \right). \tag{4.39}$$

In the following part, one will prove that the function $\varphi_2(t,\xi)$ is asymptotic profile for the solution $\hat{u}(t,\xi)$ in the high frequency region $|\xi| \geq \sqrt{e-1}$. Then we obtain the following difference between the solution and the profile

$$\hat{u}(t,\xi) - \varphi_2(t,\xi) = \left(e^{-a(\xi)t} - e^{-\frac{t}{2\log(1+|\xi|^2)}} \right) \cos(\sqrt{\log(1+|\xi|^2)}t)\hat{u}_0$$

$$
\begin{aligned}
&+ te^{-a(\xi)t}\sin(\theta(\xi,t))\left[\sqrt{\log(1+|\xi|^2)}-b(\xi)\right]\hat{u}_0+\frac{a(\xi)}{b(\xi)}e^{-a(\xi)t}\sin(b(\xi)t)\hat{u}_0\\
&+ e^{-a(\xi)t}\left(\frac{1}{b(\xi)}-\frac{1}{\sqrt{\log(1+|\xi|^2)}}\right)\sin(\sqrt{\log(1+|\xi|^2)}t)\hat{u}_1\\
&+ \frac{1}{\sqrt{\log(1+|\xi|^2)}}\left(e^{-a(\xi)t}-e^{-\frac{t}{2\log(1+|\xi|^2)}}\right)\sin(\sqrt{\log(1+|\xi|^2)}t)\hat{u}_1\\
&+ te^{-a(\xi)t}\frac{1}{b(\xi)}\cos(\eta(\xi,t))\left[b(\xi)-\sqrt{\log(1+|\xi|^2)}\right]\hat{u}_1. \qquad (4.40)
\end{aligned}
$$

We consider the following six functions

$$
\begin{aligned}
G_1(t,\xi) &= \left(e^{-a(\xi)t}-e^{-\frac{t}{2\log(1+|\xi|^2)}}\right)\cos(\sqrt{\log(1+|\xi|^2)}t)\hat{u}_0,\\
G_2(t,\xi) &= te^{-a(\xi)t}\sin(\theta(\xi,t))\left[\sqrt{\log(1+|\xi|^2)}-b(\xi)\right]\hat{u}_0,\\
G_3(t,\xi) &= \frac{a(\xi)}{b(\xi)}e^{-a(\xi)t}\sin(b(\xi)t)\hat{u}_0,\\
G_4(t,\xi) &= e^{-a(\xi)t}\left(\frac{1}{b(\xi)}-\frac{1}{\sqrt{\log(1+|\xi|^2)}}\right)\sin(\sqrt{\log(1+|\xi|^2)}t)\hat{u}_1,\\
G_5(t,\xi) &= \frac{1}{\sqrt{\log(1+|\xi|^2)}}\left(e^{-a(\xi)t}-e^{-\frac{t}{2\log(1+|\xi|^2)}}\right)\sin(\sqrt{\log(1+|\xi|^2)}t)\hat{u}_1,\\
G_6(t,\xi) &= te^{-a(\xi)t}\frac{1}{b(\xi)}\cos(\eta(\xi,t))\left[b(\xi)-\sqrt{\log(1+|\xi|^2)}\right]\hat{u}_1,
\end{aligned}
$$

which are the remainder terms that appear in (4.40). From now, let us estimates these 6-remainders in the following lines. To obtain these estimates, we assume that the initial data $(u_0,u_1)\in Y^{l+1}\times Y^l$ with $l\geq 0$.

We note that on the region such that $|\xi|\geq\sqrt{e-1}$ one has $1+\log(1+|\xi|^2)\leq 2\log(1+|\xi|^2)$. Also, by Lemma 2.20 with $c=1$ and $a=-1$ one has

$$
\frac{e^{\frac{-t}{1+\log(1+|\xi|^2)}}}{(1+\log(1+|\xi|^2))^{\nu}}\leq Ct^{-\nu},\quad t>0,\ \ \xi\in\boldsymbol{R}^n, \qquad (4.41)
$$

for a fixed $\nu>0$. The above two inequalities will be used to get the next series of estimates for the functions $G_j(t,\xi)$, $j=1,\cdots,6$.

The first estimate is concerned with the function $G_1(t,\xi)$ and it can be obtained from (4.26).

$$
\begin{aligned}
\int_{|\xi|\geq\sqrt{e-1}}|G_1(t,\xi)|^2d\xi &= \int_{|\xi|\geq\sqrt{e-1}}\left(e^{-a(\xi)t}-e^{-\frac{t}{2\log(1+|\xi|^2)}}\right)^2\cos^2(\sqrt{\log(1+|\xi|^2)}t)|\hat{u}_0|^2d\xi\\
&\leq \int_{|\xi|\geq\sqrt{e-1}}\left(e^{-a(\xi)t}-e^{-\frac{t}{2\log(1+|\xi|^2)}}\right)^2|\hat{u}_0|^2d\xi\\
&= \int_{|\xi|\geq\sqrt{e-1}}e^{\frac{-t}{1+\log(1+|\xi|^2)}}\left(1-e^{\frac{-t}{2\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}}\right)^2|\hat{u}_0|^2d\xi
\end{aligned}
$$

$$\leq t^2 \int_{|\xi|\geq\sqrt{e-1}} \frac{e^{\frac{-t}{1+\log(1+|\xi|^2)}}}{4\log^2(1+|\xi|^2)(1+\log(1+|\xi|^2))^2}|\hat{u}_0|^2 d\xi$$
$$\leq t^2 \int_{|\xi|\geq\sqrt{e-1}} \frac{e^{\frac{-t}{1+\log(1+|\xi|^2)}}}{(1+\log(1+|\xi|^2))^4}|\hat{u}_0|^2 d\xi.$$

Using (4.41) and the fact that

$$\frac{e^{\frac{-t}{1+\log(1+|\xi|^2)}}}{(1+\log(1+|\xi|^2))^4} = \frac{e^{\frac{-t}{1+\log(1+|\xi|^2)}}}{(1+\log(1+|\xi|^2))^{5+l}}(1+\log(1+|\xi|^2))^{l+1}, \quad t\geq 0, \xi\in \boldsymbol{R}^n,$$

we obtain the next estimate to the function $G_1(t,\xi)$.

$$\begin{aligned}\int_{|\xi|\geq\sqrt{e-1}} |G_1(t,\xi)|^2 d\xi &\leq t^2 \int_{|\xi|\geq\sqrt{e-1}} (1+\log(1+|\xi|^2))^{l+1} \frac{e^{\frac{-t}{1+\log(1+|\xi|^2)}}}{(1+\log(1+|\xi|^2))^{5+l}}|\hat{u}_0|^2 d\xi\\ &\leq Ct^2 t^{-(l+5)} \int_{|\xi|\geq\sqrt{e-1}} (1+\log(1+|\xi|^2))^{l+1}|\hat{u}_0|^2 d\xi\\ &\leq Ct^{-(l+3)}\|u_0\|^2_{Y^{l+1}},\end{aligned}$$

for all $t>0$ and $l\geq 0$, where we have used the inequalities (4.26), (4.41) and the fact that $\log(1+|\xi|^2)\geq 1$ on the high frequency zone.

For $|\xi|\geq\delta$, we introduce the auxiliary function $R(\xi)$ defined by

$$R(\xi) = \sqrt{1-\frac{1}{4\log(1+|\xi|^2)(1+\log(1+|\xi|^2))^2}}, \tag{4.42}$$

which is well defined due to (4.19). Additionally, one notes that for $|\xi|\geq\sqrt{e-1}$ we have the following estimate

$$\begin{aligned}\left|\sqrt{\log(1+|\xi|^2)} - b(\xi)\right| &= \frac{1}{4(1+\log(1+|\xi|^2))^2\sqrt{\log(1+|\xi|^2)}\,(1+R(\xi))}\\ &\leq \frac{1}{4(1+\log(1+|\xi|^2))^2\sqrt{\log(1+|\xi|^2)}}.\end{aligned}$$

Then, for $t>0$ and $l\geq 0$, we get

$$\begin{aligned}\int_{|\xi|\geq\sqrt{e-1}} |G_2(t,\xi)|^2 d\xi &= t^2 \int_{|\xi|\geq\sqrt{e-1}} e^{-2a(\xi)t}\sin^2(\theta(\xi,t))\left[\sqrt{\log(1+|\xi|^2)}-b(\xi)\right]^2|\hat{u}_0|^2 d\xi\\ &\leq t^2 \int_{|\xi|\geq\sqrt{e-1}} \frac{e^{-\frac{t}{1+\log(1+|\xi|^2)}}}{16(1+\log(1+|\xi|^2))^4\log(1+|\xi|^2)}|\hat{u}_0|^2 d\xi\\ &\leq t^2 \int_{|\xi|\geq\sqrt{e-1}} \frac{e^{-\frac{t}{1+\log(1+|\xi|^2)}}}{8(1+\log(1+|\xi|^2))^5}|\hat{u}_0|^2 d\xi\\ &\leq t^2 \int_{|\xi|\geq\sqrt{e-1}} (1+\log(1+|\xi|^2))^{l+1}\frac{e^{-\frac{t}{1+\log(1+|\xi|^2)}}}{8(1+\log(1+|\xi|^2))^{6+l}}|\hat{u}_0|^2 d\xi\end{aligned}$$

$$
\begin{aligned}
&= Ct^2 t^{-(6+l)} \int_{|\xi|\geq\sqrt{e-1}} (1+\log(1+|\xi|^2))^{l+1}|\hat{u}_0|^2 d\xi \\
&\leq Ct^{-(l+4)}\|u_0\|^2_{Y^{l+1}},
\end{aligned}
$$

where one has just used the fact that $1+\log(1+|\xi|^2)\geq 2$ for $|\xi|\geq\sqrt{e-1}$ and (4.41).

Another important property is that $\dfrac{1}{4\log(1+|\xi|^2)(1+\log(1+|\xi|^2))^2}$ is a decreasing function of $|\xi|$, and it converges to zero as $|\xi|\to+\infty$. Hence, it follows that

$$
1 \leq \frac{1}{1-\frac{1}{4\log(1+|\xi|^2)(1+\log(1+|\xi|^2))^2}} = \frac{1}{R(\xi)^2} \leq \frac{16}{15} \tag{4.43}
$$

for $|\xi|\geq\sqrt{e-1}$.

From the above inequality one can obtain estimates of the L^2-norms of each functions $G_3(t,\cdot)$, $G_4(t,\cdot)$ and $G_5(t,\cdot)$ for $t>0$. In fact, (4.43) implies that

$$
\begin{aligned}
\int_{|\xi|\geq\sqrt{e-1}} |G_3(t,\xi)|^2 d\xi &= \int_{|\xi|\geq\sqrt{e-1}} \left(\frac{a(\xi)}{b(\xi)}\right)^2 e^{-2a(\xi)t}\sin^2(b(\xi)t)|\hat{u}_0|^2 d\xi \\
&= \int_{|\xi|\geq\sqrt{e-1}} \frac{e^{-\frac{t}{1+\log(1+|\xi|^2)}}\sin^2(b(\xi)t)|\hat{u}_0|^2}{4\log(1+|\xi|^2)(1+\log(1+|\xi|^2))^2 R(\xi)^2} d\xi \\
&\leq \frac{16}{15}\int_{|\xi|\geq\sqrt{e-1}} \frac{e^{-\frac{t}{1+\log(1+|\xi|^2)}}}{4\log(1+|\xi|^2)(1+\log(1+|\xi|^2))^2}|\hat{u}_0|^2 d\xi.
\end{aligned}
$$

Further, for $|\xi|\geq\sqrt{e-1}$, we have $1\leq\log(1+|\xi|^2)$ and then $1+\log(1+|\xi|^2)\leq 2\log(1+|\xi|^2)$. Therefore,

$$
\frac{1}{\log(1+|\xi|^2)} \leq \frac{2}{1+\log(1+|\xi|^2)}, \quad |\xi|\geq\sqrt{e-1}
$$

and we may conclude the estimate for $G_3(t,\xi)$ as follows.

$$
\begin{aligned}
\int_{|\xi|\geq\sqrt{e-1}} |G_3(t,\xi)|^2 d\xi &\leq \frac{8}{15}\int_{|\xi|\geq\sqrt{e-1}} \frac{e^{-\frac{t}{1+\log(1+|\xi|^2)}}}{(1+\log(1+|\xi|^2))^3}|\hat{u}_0|^2 d\xi \\
&= \frac{8}{15}\int_{|\xi|\geq\sqrt{e-1}} \frac{e^{-\frac{t}{1+\log(1+|\xi|^2)}}}{(1+\log(1+|\xi|^2))^{4+l}}(1+\log(1+|\xi|^2))^{l+1}|\hat{u}_0|^2 d\xi \\
&\leq Ct^{-(l+4)}\int_{|\xi|\geq\sqrt{e-1}} (1+\log(1+|\xi|^2))^{l+1}|\hat{u}_0|^2 d\xi \\
&\leq Ct^{-(4+l)}\|u_0\|^2_{Y^{l+1}}, \quad t\gg 1.
\end{aligned}
$$

To get an estimate for the L^2-norm of $G_4(t,\cdot)$ we first observe the following identity:

$$
\frac{1}{b(\xi)} - \frac{1}{\sqrt{\log(1+|\xi|^2)}} = \frac{1}{4\log^{3/2}(1+|\xi|^2)(1+\log(1+|\xi|^2))^2 R(t,\xi)\,(1+R(\xi))}
$$

for $|\xi|\geq\delta$, where $R(\xi)$ is given by (4.42).

By using the above identity, the estimate (4.43) and the fact that $1+\log(1+|\xi|^2) \leq 2\log(1+|\xi|^2)$ for $|\xi| \leq \sqrt{e-1}$, we can arrive at the L^2-estimate to the function $G_4(t,\cdot)$:

$$
\begin{aligned}
\int_{|\xi|\geq\sqrt{e-1}} |G_4(t,\xi)|^2 d\xi &= \int_{|\xi|\geq\sqrt{e-1}} e^{-2a(\xi)t}\left(\frac{1}{b(\xi)} - \frac{1}{\sqrt{\log(1+|\xi|^2)}}\right)^2 \sin^2(\sqrt{\log(1+|\xi|^2)}t)|\hat{u}_1|^2 d\xi \\
&\leq \frac{1}{15}\int_{|\xi|\geq\sqrt{e-1}} \frac{e^{-\frac{t}{(1+\log(1+|\xi|^2))}}}{\log^3(1+|\xi|^2)(1+\log(1+|\xi|^2))^4} \sin^2(\sqrt{\log(1+|\xi|^2)}t)|\hat{u}_1|^2 d\xi \\
&\leq \frac{8}{15}\int_{|\xi|\geq\sqrt{e-1}} \frac{e^{-\frac{t}{(1+\log(1+|\xi|^2))}}}{(1+\log(1+|\xi|^2))^7}|\hat{u}_1|^2 d\xi \\
&\leq Ct^{-(l+7)}\|u_1\|^2_{Y^l}.
\end{aligned}
$$

Similarly to the previous estimate for $G_1(t,\cdot)$ one obtains

$$
\begin{aligned}
&\int_{|\xi\geq\sqrt{e-1}} |G_5(t,\xi)|^2 d\xi \\
&\quad = \int_{|\xi\geq\sqrt{e-1}} \left(e^{-\frac{t}{2(1+\log(1+|\xi|^2))}} - e^{-\frac{t}{2\log(1+|\xi|^2)}}\right)^2 \frac{\sin^2(\sqrt{\log(1+|\xi|^2)}t)}{\log(1+|\xi|^2)}|\hat{u}_1|^2 d\xi \\
&\quad \leq \int_{|\xi\geq\sqrt{e-1}} \frac{e^{-\frac{t}{1+\log(1+|\xi|^2)}}}{\log(1+|\xi|^2)}\left(1 - e^{-\frac{t}{2\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}}\right)^2 |\hat{u}_1|^2 d\xi \\
&\quad \leq t^2\int_{|\xi\geq\sqrt{e-1}} \frac{e^{-\frac{t}{1+\log(1+|\xi|^2)}}}{4\log^3(1+|\xi|^2)(1+\log(1+|\xi|^2))^2}|\hat{u}_1|^2 d\xi \\
&\quad \leq 2t^2\int_{|\xi\geq\sqrt{e-1}} \frac{e^{-\frac{t}{1+\log(1+|\xi|^2)}}}{(1+\log(1+|\xi|^2))^5}|\hat{u}_1|^2 d\xi \\
&\quad \leq Ct^{-(l+3)}\|u_1\|^2_{Y^l}.
\end{aligned}
$$

Finally, we observe that

$$
\frac{b(\xi) - \sqrt{\log(1+|\xi|^2)}}{b(\xi)} = \frac{-1}{4\log(1+|\xi|^2)(1+\log(1+|\xi|^2))^2 R(\xi)(1+R(\xi))}
$$

for $|\xi| \geq \delta$, where $R(\xi)$ is given by (4.42). Thus, for $|\xi| \geq \sqrt{e-1}$ it holds that

$$
\left|\frac{b(\xi) - \sqrt{\log(1+|\xi|^2)}}{b(\xi)}\right|^2 \leq \frac{1}{15\log^2(1+|\xi|^2)(1+\log(1+|\xi|^2))^4}.
$$

Hence, from the definition of $G_6(t,\xi)$, we have

$$
\begin{aligned}
\int_{|\xi|\geq\sqrt{e-1}} |G_6(t,\xi)|^2 d\xi &\leq \frac{1}{15}t^2\int_{|\xi|\geq\sqrt{e-1}} \frac{e^{-\frac{t}{1+\log(1+|\xi|^2)}}\cos^2(\eta(\xi,t))}{\log^2(1+|\xi|^2)(1+\log(1+|\xi|^2))^4}|\hat{u}_1|^2 d\xi \\
&\leq \frac{4}{15}t^2\int_{|\xi|\geq\sqrt{e-1}} \frac{e^{-\frac{t}{1+\log(1+|\xi|^2)}}}{(1+\log(1+|\xi|^2))^6}|\hat{u}_1|^2 d\xi
\end{aligned}
$$

$$\leq Ct^{-(l+4)}\|u_1\|_{Y^l}^2, \quad t > 0.$$

The estimates for $G_j(t,\xi)$ $(j = 1, \cdots, 6)$ together with the identity (4.40) provide the following result.

Proposition 4.17. *Let $n \geq 1$, $l \geq 0$ and $(u_0, u_1) \in Y^{l+1} \times Y^l$. Then there exists a positive constant C, which is independent of t, u_0 and u_1, such that*

$$\int_{|\xi|\geq\sqrt{e-1}} |\hat{u}(t,\xi) - \varphi_2(t,\xi)|^2\, d\xi \leq C\left(\|u_0\|_{Y^{l+1}}^2 + \|u_1\|_{Y^l}^2\right) t^{-(l+3)}$$

for $t \geq 0$, where $\varphi_2(t,\xi)$ is given by (4.39).

□

4.3.3 Estimates on the whole space $\boldsymbol{R}^n$

In this subsection, we consider three special functions: $\varphi_1(t,\xi)$, $\varphi_2(t,\xi)$, which are given by (4.24) and (4.39), and

$$\varphi(\xi,t) = \varphi_1(t,\xi) + \varphi_2(t,\xi) \tag{4.44}$$

defined for $\xi \in \boldsymbol{R}^n$.

We will prove that, under certain conditions, each of them is an asymptotic profile as $t \to \infty$ of the solution $\hat{u}(t,\xi)$ in $\boldsymbol{R}^n$.

Lemma 4.18. *Let $n \geq 1$ and $(u_0, u_1) \in (L^{1,1}(\boldsymbol{R}^n) \cap Y^{l+1}) \times (L^{1,1}(\boldsymbol{R}^n) \cap Y^l)$. Then there exists a constant $C > 0$, which is independent of t, u_0, u_1 such that*

$$\int_{\boldsymbol{R}^n} |\hat{u}(t,\xi) - \varphi(t,\xi)|^2 d\xi \leq C\left(t^{-\frac{n+2}{2}} + t^{-(l+3)}\right) I_{0,l}^2$$

for $t \gg 1$, where

$$I_{0,l} := \sqrt{\|u_0\|_{1,1}^2 + \|u_1\|_{1,1}^2 + \|u_0\|_{Y^{l+1}}^2 + \|u_1\|_{Y^l}^2}. \tag{4.45}$$

Proof. On the region $|\xi| \leq \sqrt{e-1}$, the function $\log(1+|\xi|^2)$ is positive and bounded by 1, then it holds that

$$\frac{-t}{\log(1+|\xi|^2)} \leq -t,$$

for $t \geq 0$. We also have $\sin a \leq a$ for all $a \geq 0$. Having this in mind we can get, for $t \geq 0$, the estimates

$$\begin{aligned}\int_{|\xi|\leq\sqrt{e-1}} |\varphi_2(t,\xi)|^2 d\xi &\leq 2\int_{|\xi|\leq\sqrt{e-1}} e^{-\frac{t}{\log(1+|\xi|^2)}} \frac{\sin^2(\sqrt{\log(1+|\xi|^2)}t)}{\log(1+|\xi|^2)} |\hat{u}_1|^2 d\xi \\ &\quad + 2\int_{|\xi|\leq 1} e^{-\frac{t}{\log(1+|\xi|^2)}} \cos^2(\sqrt{\log(1+|\xi|^2)}t)|\hat{u}_0|^2 d\xi \\ &\leq 2t^2 e^{-t}\int_{|\xi|\leq\sqrt{e-1}} |\hat{u}_1|^2 d\xi + 2e^{-t}\int_{|\xi|\leq 1} |\hat{u}_0|^2 d\xi\end{aligned}$$

$$\leq 2t^2e^{-t}\|u_1\|_2^2 + 2e^{-t}\|u_0\|_2^2. \tag{4.46}$$

On the other hand, one knows that

$$e^{-2t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))} \leq e^{-2t\log(1+|\xi|^2)},$$

because of $1 + \log(1 + |\xi|^2) \geq 1$. Then

$$\begin{aligned}
\int_{|\xi|\geq\eta} |\varphi_1(t,\xi)|^2 d\xi &= |P_0 + P_1|^2 \int_{|\xi|\geq\eta} e^{-2t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))} d\xi \\
&\leq |P_0 + P_1|^2 \int_{|\xi|\geq\eta} (1+|\xi|^2)^{-2t} d\xi \\
&= |P_0 + P_1|^2 \int_{\eta\leq|\xi|\leq 1} (1+|\xi|^2)^{-2t} d\xi + |P_0 + P_1|^2 \int_{|\xi|\geq 1} (1+|\xi|^2)^{-2t} d\xi \\
&= \omega_n |P_0 + P_1|^2 \int_\eta^1 (1+r^2)^{-2t} r^{n-1} dr + \omega_n |P_0 + P_1|^2 \int_1^\infty (1+r^2)^{-2t} r^{n-1} dr \\
&\leq C|P_0 + P_1|^2 \left((1+\eta^2)^{-t} + \frac{2^{-t}}{t-1} \right) \\
&\leq C \left(\|u_0\|_1^2 + \|u_1\|_1^2 \right) \left((1+\eta^2)^{-t} + \frac{2^{-t}}{t-1} \right), \quad t \gg 1,
\end{aligned} \tag{4.47}$$

with a generous constant $C > 0$, due to Lemmas 2.25 and 2.29. We also note that both above estimates are of exponential type.

Under these preparations we can get the desired estimate in the statement. At first, one notices that

$$|\hat{u}(t,\xi) - \varphi(t,\xi)| = |\hat{u}(t,\xi) - \varphi_1(t,\xi) - \varphi_2(t,\xi)| \leq |\hat{u}(t,\xi) - \varphi_1(t,\xi)| + |\varphi_2(t,\xi)|.$$

From Young's inequality, it holds that

$$|\hat{u}(t,\xi) - \varphi(t,\xi)|^2 \leq 2|\hat{u}(t,\xi) - \varphi_1(t,\xi)|^2 + 2|\varphi_2(t,\xi)|^2. \tag{4.48}$$

Similarly,

$$|\hat{u}(t,\xi) - \varphi(t,\xi)|^2 \leq 2|\hat{u}(t,\xi) - \varphi_2(t,\xi)|^2 + 2|\varphi_1(t,\xi)|^2. \tag{4.49}$$

Also, one has

$$|\hat{u}(t,\xi) - \varphi(t,\xi)| \leq |\hat{u}(t,\xi)| + |\varphi_1(t,\xi)| + |\varphi_2(t,\xi)|.$$

And we obtain

$$|\hat{u}(t,\xi) - \varphi(t,\xi)|^2 \leq 2|\hat{u}(t,\xi)|^2 + 4|\varphi_1(t,\xi)|^2 + 4|\varphi_2(t,\xi)|^2. \tag{4.50}$$

Let us apply the estimates (4.48) on the region $|\xi| \leq \eta$, (4.49) on the region $|\xi| \geq \sqrt{e-1}$ and (4.50) on the middle frequency region $\eta \leq |\xi| \leq \sqrt{e-1}$, respectively. Then one can proceed the estimates as follows.

$$\int_{\boldsymbol{R}^n} |\hat{u}(t,\xi) - \varphi(t,\xi)|^2 d\xi = \int_{|\xi|\leq\eta} |\hat{u}(t,\xi) - \varphi(t,\xi)|^2 d\xi + \int_{\eta\leq|\xi|\leq\sqrt{e-1}} |\hat{u}(t,\xi) - \varphi(t,\xi)|^2 d\xi$$

$$
\begin{aligned}
&+\int_{|\xi|\geq\sqrt{e-1}}|\hat{u}(t,\xi)-\varphi(t,\xi)|^2d\xi\\
&\leq 2\int_{|\xi|\leq\eta}|\hat{u}(t,\xi)-\varphi_1(t,\xi)|^2d\xi+2\int_{|\xi|\leq\eta}|\varphi_2(t,\xi)|^2d\xi\\
&+2\int_{\eta\leq|\xi|\leq\sqrt{e-1}}|\hat{u}(t,\xi)|^2d\xi+4\int_{\eta\leq|\xi|\leq\sqrt{e-1}}|\varphi_1(t,\xi)|^2d\xi\\
&+4\int_{\eta\leq|\xi|\leq\sqrt{e-1}}|\varphi_2(t,\xi)|^2d\xi+2\int_{|\xi|\geq\sqrt{e-1}}|\hat{u}(t,\xi)-\varphi_2(t,\xi)|^2d\xi\\
&+2\int_{|\xi|\geq\sqrt{e-1}}|\varphi_1(t,\xi)|^2d\xi\\
&\leq 2\int_{|\xi|\leq\eta}|\hat{u}(t,\xi)-\varphi_1(t,\xi)|^2d\xi+2\int_{|\xi|\geq\sqrt{e-1}}|\hat{u}(t,\xi)-\varphi_2(t,\xi)|^2d\xi\\
&+4\int_{|\xi|\leq\eta}|\varphi_2(t,\xi)|^2d\xi+4\int_{|\xi|\geq\sqrt{e-1}}|\varphi_1(t,\xi)|^2d\xi\\
&+2\int_{\eta\leq|\xi|\leq\sqrt{e-1}}|\hat{u}(t,\xi)|^2d\xi+4\int_{\eta\leq|\xi|\leq\sqrt{e-1}}|\varphi_1(t,\xi)|^2d\xi\\
&+4\int_{\eta\leq|\xi|\leq\sqrt{e-1}}|\varphi_2(t,\xi)|^2d\xi\\
&=2\int_{|\xi|\leq\eta}|\hat{u}(t,\xi)-\varphi_1(t,\xi)|^2d\xi+2\int_{|\xi|\geq\sqrt{e-1}}|\hat{u}(t,\xi)-\varphi_2(t,\xi)|^2d\xi\\
&+4\int_{|\xi|\leq\sqrt{e-1}}|\varphi_2(t,\xi)|^2d\xi+4\int_{|\xi|\geq\eta}|\varphi_1(t,\xi)|^2d\xi\\
&+2\int_{\eta\leq|\xi|\leq\sqrt{e-1}}|\hat{u}(t,\xi)|^2d\xi. \qquad (4.51)
\end{aligned}
$$

Propositions 4.15 and 4.17 tell us that

$$
\begin{aligned}
&\int_{|\xi|\leq\eta}|\hat{u}(t,\xi)-\varphi_1(t,\xi)|^2d\xi\leq CI_{0,l}^2t^{-\frac{n+2}{2}},\\
&\int_{|\xi|\geq\sqrt{e-1}}|\hat{u}(t,\xi)-\varphi_2(t,\xi)|^2d\xi\leq CI_{0,l}^2t^{-(l+3)}
\end{aligned}
$$

for $t\gg 1$. According to (4.46) and (4.47), the L^2-norm for $\varphi_1(t,\cdot)$ on the zone $|\xi|\geq\eta$ and for $\varphi_2(t,\cdot)$ on the zone $|\xi|\leq\sqrt{e-1}$ decay with exponential rate. Furthermore, from Lemma 4.16 and estimate (4.37) the L^2-norm of $\hat{u}(t,\cdot)$ on the middle frequency zone $\eta\leq|\xi|\leq\sqrt{e-1}$ also has exponential decay. By combining (4.51) with the above informations we conclude the desired estimate

$$
\int_{\boldsymbol{R}^n}|\hat{u}(t,\xi)-\varphi(t,\xi)|^2d\xi\leq\tilde{C}I_{0,l}^2(t^{-\frac{n+2}{2}}+t^{-(l+3)})\quad t\gg 1.
$$

□

Lemma 4.19. *Let $u_0,u_1\in L^1(\boldsymbol{R}^n)$ and the function $\varphi_1(t,\xi)$ is defined in (4.24). Then there exists positive constants C_1,C_2, depending only on dimension n, such that*

$$
C_1|P_0+P_1|^2t^{-\frac{n}{2}}\leq\int_{\boldsymbol{R}^n}|\varphi_1(t,\xi)|^2d\xi\leq C_2(\|u_0\|_1^2+\|u_1\|_1^2)t^{-\frac{n}{2}} \qquad (4.52)
$$

for $t \gg 1$.

Proof. The function $\varphi_1(t,\xi)$ satisfies

$$|\varphi_1(t,\xi)| \leq |P_0 + P_1|(1+|\xi|^2)^{-t}$$

for $\xi \in \boldsymbol{R}^n$, since $1+\log(1+|\xi|^2) \geq 1$. By using Lemmas 2.28 and 2.25, we immediately concluded that

$$\begin{aligned}
\int_{\boldsymbol{R}^n} |\varphi_1(t,\xi)|^2 d\xi &\leq |P_0+P_1|^2 \int_{\boldsymbol{R}^n_\xi} (1+|\xi|^2)^{-2t} d\xi \\
&= \omega_n |P_0+P_1|^2 \int_0^1 (1+r^2)^{-2t} r^{n-1} dr + \omega_n |P_0+P_1|^2 \int_1^\infty (1+r^2)^{-2t} r^{n-1} dr \\
&\leq C\omega_n |P_0+P_1|^2 \left(t^{-\frac{n}{2}} + \frac{2^{-t}}{t-1} \right) \\
&\leq C\omega_n (\|u_0\|_1^2 + \|u_1\|_1^2) t^{-\frac{n}{2}}
\end{aligned}$$

for $t \gg 1$.

On the other hand, for $|\xi| \leq \eta$, we have $1+\log(1+|\xi|^2) \leq 1+\log(1+|\eta|^2) = k_\eta$. Thus,

$$|\varphi_1(t,\xi)| \geq |P_0+P_1|(1+|\xi|^2)^{-k_\eta t}$$

for $|\xi| \leq \eta$. First, we choose $t_0 > 0$ such that, for all $t > t_0$ it holds that $t^{-\frac{1}{2}} \leq \eta$, and

$$\frac{1}{e^{4k_\eta}} \leq \left(1+\frac{1}{t}\right)^{-2k_\eta t} \leq 1.$$

Such t_0 exists, because one has

$$\lim_{t\to\infty} \left(1+\frac{1}{t}\right)^{-2k_\eta t} = \frac{1}{e^{2k_\eta}}.$$

For this choice, we can compute as follows:

$$\begin{aligned}
\int_{\boldsymbol{R}^n} |\varphi_1(t,\xi)|^2 d\xi \geq \int_{|\xi|\leq\eta} |\varphi_1(t,\xi)|^2 d\xi &\geq \omega_n |P_0+P_1|^2 \int_0^\eta (1+r^2)^{-2k_\eta t} r^{n-1} dr \\
&\geq \omega_n |P_0+P_1|^2 \int_0^{t^{-\frac{1}{2}}} (1+r^2)^{-2k_\eta t} r^{n-1} dr \\
&\geq \omega_n |P_0+P_1|^2 \left(1+\frac{1}{t}\right)^{-2k_\eta t} \int_0^{t^{-\frac{1}{2}}} r^{n-1} dr \\
&= \frac{\omega_n}{n} |P_0+P_1|^2 \left(1+\frac{1}{t}\right)^{-2k_\eta t} t^{-\frac{n}{2}} \\
&\geq \frac{\omega_n e^{-4k_\eta}}{n} |P_0+P_1|^2 t^{-\frac{n}{2}}
\end{aligned}$$

for $t > t_0$. □

Lemma 4.20. *Let $n \geq 1$, $l \geq 0$ and $(u_0, u_1) \in Y^{l+1} \times Y^l$. Then there exists a constant $C > 0$, which is independent of u_0, u_1 and t, such that*

$$\int_{\boldsymbol{R}^n} |\varphi_2(t,\xi)|^2 d\xi \leq C I_{0,l}^2 t^{-(l+1)}$$

for all $t > 0$, where $I_{0,l}$ is given in (4.45).

Proof. By definition of $\varphi_2(t,\xi)$ in (4.39), we have

$$|\varphi_2(t,\xi)| \leq e^{-\frac{t}{2\log(1+|\xi|^2)}} \frac{|\sin(\sqrt{\log(1+|\xi|^2)}t)|}{\sqrt{\log(1+|\xi|^2)}} |\hat{u}_1| + e^{-\frac{t}{2\log(1+|\xi|^2)}} |\cos(\sqrt{\log(1+|\xi|^2)}t)||\hat{u}_0|.$$

Hence, the Young's inequality enable us to get

$$|\varphi_2(t,\xi)|^2 \leq 2e^{-\frac{t}{\log(1+|\xi|^2)}} \frac{\sin^2(\sqrt{\log(1+|\xi|^2)}t)}{\log(1+|\xi|^2)} |\hat{u}_1|^2 + 2e^{-\frac{t}{\log(1+|\xi|^2)}} \cos^2(\sqrt{\log(1+|\xi|^2)}t)|\hat{u}_0|^2. \tag{4.53}$$

It follows from (4.46), we get

$$\int_{|\xi|\leq\sqrt{e-1}} |\varphi_2(t,\xi)|^2 d\xi \leq 2t^2 e^{-t} \|u_1\|_2^2 + 2e^{-t} \|u_0\|_2^2, \quad t > 0. \tag{4.54}$$

On the high frequency zone $|\xi| \geq \sqrt{e-1}$ it holds that

$$\frac{1}{1+\log(1+|\xi|^2)} \leq \frac{1}{\log(1+|\xi|^2)} \leq \frac{2}{1+\log(1+|\xi|^2)}. \tag{4.55}$$

By using the inequality (4.55) and the estimates (4.41) and (4.53), one can obtain

$$\begin{aligned}
\int_{|\xi|\geq\sqrt{e-1}} |\varphi_2(t,\xi)|^2 d\xi &\leq 2\int_{|\xi|\geq\sqrt{e-1}} e^{-\frac{t}{\log(1+|\xi|^2)}} \frac{\sin^2(\sqrt{\log(1+|\xi|^2)}t)}{\log(1+|\xi|^2)} |\hat{u}_1|^2 d\xi \\
&\quad + 2\int_{|\xi|\geq\sqrt{e-1}} e^{-\frac{t}{\log(1+|\xi|^2)}} \cos^2(\sqrt{\log(1+|\xi|^2)}t)|\hat{u}_0|^2 d\xi \\
&\leq 4\int_{|\xi|\geq\sqrt{e-1}} e^{-\frac{t}{1+\log(1+|\xi|^2)}} \frac{1}{1+\log(1+|\xi|^2)} |\hat{u}_1|^2 d\xi \\
&\quad + 2\int_{|\xi|\geq\sqrt{e-1}} e^{-\frac{t}{1+\log(1+|\xi|^2)}} |\hat{u}_0|^2 d\xi \\
&= 4\int_{|\xi|\geq\sqrt{e-1}} \frac{e^{-\frac{t}{1+\log(1+|\xi|^2)}}}{(1+\log(1+|\xi|^2))^{l+1}} (1+\log(1+|\xi|^2))^l |\hat{u}_1|^2 d\xi \\
&\quad + 2\int_{|\xi|\geq\sqrt{e-1}} \frac{e^{-\frac{t}{1+\log(1+|\xi|^2)}}}{(1+\log(1+|\xi|^2))^{l+1}} (1+\log(1+|\xi|^2))^{l+1} |\hat{u}_0|^2 d\xi \\
&\leq 4Ct^{-(l+1)} \|u_1\|_{Y^l}^2 + 2Ct^{-(l+1)} \|u_0\|_{Y^{l+1}}^2, \quad t > 0.
\end{aligned} \tag{4.56}$$

By combining estimates (4.54) and (4.56) one can conclude

$$\int_{\boldsymbol{R}^n} |\varphi_2(t,\xi)|^2 d\xi \leq Ct^{-(l+1)} (\|u_1\|_{Y^l}^2 + \|u_0\|_{Y^{l+1}}^2), \quad t \gg 1$$

for some generous constant $C > 0$, independent of u_0, u_1 and t. □

4.3.4 The asymptotic profile formulas

In this subsection we compile the results obtained in previous subsections. We observe that each of the functions $\varphi_1(t,\xi)$, $\varphi_2(t,\xi)$ and $\varphi(t,\xi)$ given by (4.24), (4.39) and (4.44) is asymptotic profile to the solution of the problem (4.1)–(4.2) in the Fourier space depending on the regularity of the initial data.

First, from Lemma 4.18 we have the estimate

$$\int_{\boldsymbol{R}^n} |\hat{u}(t,\xi)-\varphi(t,\xi)|^2 d\xi \leq C I_{0,l}^2 \left(t^{-\frac{n+2}{2}} + t^{-(l+3)}\right) =: P_n(t), \tag{4.57}$$

where $I_{0,l}$ is defined by (4.45).

Since we can write as $\hat{u}(t,\xi)-\varphi_1(t,\xi) = \hat{u}(t,\xi)-\varphi(t,\xi)+\varphi_2(t,\xi)$, and $\hat{u}(t,\xi)-\varphi_2(t,\xi) = \hat{u}(t,\xi)-\varphi(t,\xi)+\varphi_1(t,\xi)$, from Lemmas 4.20 and 4.19 one has

$$\begin{aligned}\int_{\boldsymbol{R}^n} |\hat{u}(t,\xi)-\varphi_1(t,\xi)|^2 d\xi &\leq 2\int_{\boldsymbol{R}^n} |\hat{u}(t,\xi)-\varphi(t,\xi)|^2 d\xi + 2\int_{\boldsymbol{R}^n} |\varphi_2(t,\xi)|^2 d\xi \\ &\leq 2C I_{0,l}^2 \left(t^{-\frac{n+2}{2}} + t^{-(l+3)} + t^{-(l+1)}\right) \\ &\leq 4C I_{0,l}^2 \left(t^{-\frac{n+2}{2}} + t^{-(l+1)}\right) =: M_n(t), \end{aligned} \tag{4.58}$$

and

$$\begin{aligned}\int_{\boldsymbol{R}^n} |\hat{u}(t,\xi)-\varphi_2(t,\xi)|^2 d\xi &\leq 2\int_{\boldsymbol{R}^n} |\hat{u}(t,\xi)-\varphi(t,\xi)|^2 d\xi + 2\int_{\boldsymbol{R}^n} |\varphi_1(t,\xi)|^2 d\xi \\ &\leq 2C I_{0,l}^2 \left(t^{-\frac{n+2}{2}} + t^{-(l+3)} + t^{-\frac{n}{2}}\right) \\ &\leq 4C I_{0,l}^2 \left(t^{-(l+3)} + t^{-\frac{n}{2}}\right) =: Q_n(t). \end{aligned} \tag{4.59}$$

By an asymptotic profile we mean the term of $\hat{u}(t,\xi)$ that decays with the slowest time rate. According to Lemmas 4.19 and 4.20 we know that

$$\begin{aligned}&\int_{\boldsymbol{R}^n} |\varphi_1(t,\xi)|^2 \leq C I_{0,l}^2 t^{-\frac{n}{2}}, \quad \int_{\boldsymbol{R}^n} |\varphi_2(t,\xi)|^2 \leq C I_{0,l}^2 t^{-(l+1)}, \\ &\int_{\boldsymbol{R}^n} |\varphi(t,\xi)|^2 \leq C I_{0,l}^2 (t^{-\frac{n}{2}} + t^{-(l+1)}).\end{aligned}$$

Therefore, the asymptotic profile of the solution $\hat{u}(t,\xi)$ as $t \to +\infty$ is

(i). $\varphi_1(t,\xi)$ if $\frac{n}{2} < l+1$ (compare with (4.58)),

(ii). $\varphi(t,\xi)$ if $\frac{n}{2} = l+1$ (compare with (4.57)),

(iii). $\varphi_2(t,\xi)$ if $\frac{n}{2} > l+1$ (compare with (4.59)).

Next, we state three results on the asymptotic profile. To prove them it is still necessary to discuss the decay rate of $P_n(t), M_n(t)$ and $Q_n(t)$ related to the differences between the solution $\hat{u}(t,\xi)$ and the suitable asymptotic profiles. For this purpose we introduce a value $l^*(n)$ on the regularity $l \geq 1$ of the initial data such that

$$l^*(n) := \frac{n}{2} - 1.$$

This value expresses a kind of critical number on the regularity $l \geq 1$, which divides the property of the solution $u(t,x)$ into three types: one is diffusive-like (Theorem 4.21), the other is wave-like (Theorem 4.22) and the remaining is both of them (Theorem 4.23).

In the following results we require $l \geq 1$ since it is necessary for the existence and uniqueness of the solution (see Theorem 4.6). We also remember that the constant $I_{0,l}$ depends on initial data $(u_0,u_1) \in (L^{1,1}(\boldsymbol{R}^n) \cap Y^{l+1}) \times (L^{1,1}(\boldsymbol{R}^n) \cap Y^l)$ and it is given by (4.45).

Theorem 4.21. *Let $n \geq 1$ and $l \geq 1$. If $(u_0,u_1) \in (L^{1,1}(\boldsymbol{R}^n) \cap Y^{l+1}) \times (L^{1,1}(\boldsymbol{R}^n) \cap Y^l)$, then there exists a constant $C > 0$, which is independent of t, u_0, u_1 such that*

$$\|u(t,\cdot) - \mathcal{F}^{-1}(\varphi_1(t,\xi))(\cdot)\|_2$$

$$\leq \begin{cases} CI_{0,l}^2 t^{-\frac{n+2}{4}} & \text{if } l \geq 1 \text{ and } n \leq 2;\ \text{if } n \geq 3 \text{ and } l \geq n/2, \\ CI_{0,l}^2 t^{-\frac{l+1}{2}} & \text{if } n \geq 4 \text{ and } n/2 - 1 < l \leq n/2;\ \text{if } n = 3 \text{ and } 1 \leq l \leq 3/2, \end{cases}$$

for $t \gg 1$, where

$$\varphi_1(t,\xi) := e^{-t\log(1+|\xi|^2)(1+\log(1+|\xi|^2))}(P_0 + P_1).$$

Proof. In discussions below, we justify that the asymptotic profile in the Fourier space is $\varphi_1(t,\xi)$, if $l^*(n) < l$. We analyse the decay rate for the difference between the solution and its asymptotic profile, that is, for the term $M_n(t)$ given by (4.58).
First, we consider $l = 1$. In this case, $\varphi_1(t,\xi)$ is asymptotic profile for $n < 4$.
• If $n \leq 2$, then $\frac{n+2}{2} \leq 2 = l+1$, and so $M_n(t) \leq 8CI_{0,l}^2 t^{-\frac{n+2}{2}}$.
• If $n = 3$, then $2 = l+1 < \frac{n+2}{2} = \frac{5}{2}$. Thus $M_n(t) \leq 8CI_{0,l}^2 t^{-(l+1)} = 8CI_{0,l}^2 t^{-2}$.
Now let us consider the case $l > 1$.
• In this case, the rate $t^{-(l+1)}$ is better than t^{-2}. Therefore, if $n \leq 2$, we have $M_n(t) \leq 8CI_{0,l}^2 t^{-\frac{n+2}{2}}$.
• If $n > 2$ and $\frac{n}{2} \leq l$, then $l > 1$ and $M_n(t) \leq 8CI_{0,l}^2 t^{-\frac{n+2}{2}}$.
• If $n \geq 4$ and $\frac{n}{2} - 1 < l \leq \frac{n}{2}$, we obtain $l > 1$ and $M_n(t) \leq 8CI_{0,l}^2 t^{-(l+1)}$.
• For $n = 3$, we need $1 < l \leq \frac{3}{2}$, in order that $M_n(t) \leq 8CI_{0,l}^2 t^{-(l+1)}$.
These observations together with the Plancherel Theorem imply the desired statement of Theorem 4.21. □

Theorem 4.22. *Let $n \geq 5$ and $l \geq 1$. If $(u_0,u_1) \in (L^{1,1}(\boldsymbol{R}^n) \cap Y^{l+1}) \times (L^{1,1}(\boldsymbol{R}^n) \cap Y^l)$, then there exists a constant $C > 0$, which is independent of t, u_0, u_1 such that*

$$\|u(t,\cdot) - \mathcal{F}^{-1}(\varphi_2(t,\xi))(\cdot)\|_2$$

$$\leq \begin{cases} CI_{0,l}^2 t^{-\frac{n}{4}} & \textit{if } 1 \leq l < n/2-1 \textit{ and } 5 \leq n \leq 8; \textit{ if } n > 8 \textit{ and } n/2-3 < l < n/2-1, \\ CI_{0,l}^2 t^{-\frac{l+3}{2}} & \textit{if } n > 8 \textit{ and } 1 \leq l \leq n/2-3, \end{cases}$$

for $t \gg 1$, *where*

$$\varphi_2(t,\xi) := e^{-\frac{t}{2\log(1+|\xi|^2)}} \frac{\sin(\sqrt{\log(1+|\xi|^2)}t)}{\sqrt{\log(1+|\xi|^2)}} \hat{u}_1(\xi) + e^{-\frac{t}{2\log(1+|\xi|^2)}} \cos(\sqrt{\log(1+|\xi|^2)}t)\hat{u}_0(\xi).$$

Proof. Similarly to the proof of the previous theorem, we observe that $\varphi_2(t,\xi)$ is asymptotic profile (in the Fourier space) when $l^*(n) > l$.
If $l = 1$, $\varphi_2(t,\xi)$ is asymptotic profile for $n > 4$.
• If $4 < n \leq 8$, then $\frac{n}{2} \leq 4 = l+3$. So $Q_n(t) \leq 8CI_{0,l}^2 t^{-\frac{n}{2}}$.
• For $n > 8$, we have $\frac{n}{2} > 4 = l+3$ and $Q_n(t) \leq 8CI_{0,l}^2 t^{-(l+3)} = 8CI_{0,l}^2 t^{-4}$.
By assuming $l > 1$, it is necessary that $n > 2l+2 > 4$.
• If $4 < n \leq 8$, then $\frac{n}{2} \leq 4 < l+3$. Therefore, $Q_n(t) \leq 8CI_{0,l}^2 t^{-\frac{n}{2}}$.
• For $n > 8$ and $\frac{n}{2} - 3 < l < \frac{n}{2} - 1$, we have $l > 1$ and $Q_n(t) \leq 8CI_{0,l}^2 t^{-\frac{n}{2}}$.
• If $n > 8$ and $1 < l \leq \frac{n}{2} - 3$, we obtain $Q_n(t) \leq 8CI_{0,l}^2 t^{-(l+3)}$.
This analysis and the Plancherel Theorem prove the result. □

Theorem 4.23. *Let* $n \geq 4$ *and* $l = \frac{n}{2} - 1$. *If* $(u_0, u_1) \in (L^{1,1}(\boldsymbol{R}^n) \cap Y^{l+1}) \times (L^{1,1}(\boldsymbol{R}^n) \cap Y^l)$, *then there exists a constant* $C > 0$, *which is independent of* t, u_0, u_1 *such that*

$$\|u(t,\cdot) - \mathcal{F}^{-1}(\varphi(t,\xi))(\cdot)\|_2 \leq CI_{0,l}^2 t^{-\frac{n+2}{4}}$$

for $t \gg 1$, *where*

$$\varphi(t,\xi) := \varphi_1(t,\xi) + \varphi_2(t,\xi).$$

Proof. In this case, we have $l^*(n) = l$. This condition implies that $\frac{n+2}{2} = l+2 < l+3$. Then we have $P_n(t) \leq 2CI_{0,l}^2 t^{-\frac{n+2}{2}}$. Due to $l \geq 1$, this estimate holds only for $n \geq 4$. The result follows based on this analysis and from the Plancherel Theorem. □

4.4 OPTIMAL DECAY RATES OF THE SOLUTION

From the discussions to get Theorems 4.21, 4.22 and 4.23, we still can get crucial results regarding decay rates of the solution $u(t,x)$ to problem (4.1)–(4.2). Moreover, it is also possible to prove the optimality of this decay rates.

We have already used the decomposition such as

$$\hat{u}(t,\xi) = \hat{u}(t,\xi) - \varphi(t,\xi) + \varphi_1(t,\xi) + \varphi_2(t,\xi),$$

where $\varphi(t,\xi) = \varphi_1(t,\xi) + \varphi_2(t,\xi)$ with $\varphi_1(t,\xi)$ and $\varphi_2(t,\xi)$ are given by (4.24), (4.39). Since u_0 and u_1 have the required regularity in Lemmas 4.18, 4.19 and 4.20, one can get

$$\int_{\boldsymbol{R}^n} |\hat{u}(t,\xi)|^2 dx \leq 4\int_{\boldsymbol{R}^n} |\hat{u}(t,\xi) - \varphi(t,\xi)|^2 d\xi + 4\int_{\boldsymbol{R}^n} |\varphi_1(t,\xi)|^2 d\xi + 4\int_{\boldsymbol{R}^n} |\varphi_2(t,\xi)|^2 d\xi$$

$$\leq 4CI_{0,l}^2(t^{-\frac{n+2}{2}} + t^{-(l+3)} + t^{-\frac{n}{2}} + t^{-(l+1)}) \quad (t \gg 1)$$
$$\leq KI_{0,l}^2(t^{-\frac{n}{2}} + t^{-(l+1)}) =: R_n(t) \tag{4.60}$$

with some constant $K > 0$.

In the same way as we did in the previous subsection, we compare $\frac{n}{2}$ and $(l+1)$ in order to obtain the decay rate of the solution.

(i). If $l \geq 1$ and $n \leq 3$, then $\frac{n}{2} < 2 \leq l+1$. So $R_n(t) \leq 2KI_{0,l}^2 t^{-\frac{n}{2}}$.

(ii). If $n > 4$ and $l > \frac{n}{2} - 1$, we have $R_n(t) \leq 2KI_{0,l}^2 t^{-\frac{n}{2}}$.

(iii). If $n \geq 4$ and $1 \leq l \leq \frac{n}{2} - 1$, then $l+1 \leq \frac{n}{2}$. Thus $R_n(t) \leq 2KI_{0,l}^2 t^{-(l+1)}$.

The last item (iii) combined with the expression (4.60) and Plancherel Theorem completes the proof of the following Theorem 4.24.

Theorem 4.24. *Let $n \geq 4$ and $1 \leq l \leq \frac{n}{2} - 1$. If $(u_0, u_1) \in (L^{1,1}(\boldsymbol{R}^n) \cap Y^{l+1}) \times (L^{1,1}(\boldsymbol{R}^n) \cap Y^l)$, then the solution $u(t,x)$ to problem (4.1)–(4.2) satisfies*

$$\|u(t,\cdot)\|_2 \leq CI_{0,l} t^{-\frac{l+1}{2}}$$

for $t \gg 1$, where C is a positive constant which depends only on n.

□

Remark 4.25. The decay rate obtained in Theorem 4.24 seems exactly optimal, however, one cannot obtain the lower bound of time-decay rate. This is still open.

Items (i) and (ii) give us conditions for the decay rate of the solution to be better than $t^{-\frac{n}{2}}$. Furthermore, we may prove that this rate is optimal under these same conditions. Recalling the fact that the condition $(u_0, u_1) \in Y^2 \times Y^1$ is necessary for the existence and uniqueness of the solution $u(t,x)$ to problem (4.1)–(4.2) according to Theorem 4.6, we state Theorems 4.26 and 4.27.

Theorem 4.26. *Let $1 \leq n \leq 3$. If $(u_0, u_1) \in (L^{1,1}(\boldsymbol{R}^n) \cap Y^2) \times (L^{1,1}(\boldsymbol{R}^n) \cap Y^1)$, then there exists constants $C_1, C_2 > 0$ independent of t such that*

$$C_1|P_0 + P_1| t^{-\frac{n}{4}} \leq \|u(t,\cdot)\|_2 \leq C_2 I_{0,1} t^{-\frac{n}{4}}$$

for all $t \gg 1$ provided that $P_1 + P_0 \neq 0$.

Theorem 4.27. *Let $n \geq 4$ and $\varepsilon > 0$. If $(u_0, u_1) \in (L^{1,1}(\boldsymbol{R}^n) \cap Y^{\frac{n}{2}+\varepsilon}) \times (L^{1,1}(\boldsymbol{R}^n) \cap Y^{\frac{n-2}{2}+\varepsilon})$, then there exists constants $C_1, C_2 > 0$ independent of t such that*

$$C_1|P_0 + P_1| t^{-\frac{n}{4}} \leq \|u(t,\cdot)\|_2 \leq C_2 I_{0,\frac{n}{2}+\varepsilon-1} t^{-\frac{n}{4}}$$

for all $t \gg 1$, provided that $P_1 + P_0 \neq 0$.

Proof of Theorems 4.26 and 4.27. We first observe the case of $(u_0, u_1) \in Y^{l+1} \times Y^l$ with $l > \dfrac{n-2}{2}$. On the one hand,

$$\begin{aligned}|\hat{u}(t,\xi)|^2 &\leq (|\hat{u}(t,\xi) - \varphi(t,\xi)| + |\varphi(t,\xi)|)^2 \\ &\leq 2|\hat{u}(t,\xi) - \varphi(t,\xi)|^2 + 2|\varphi(t,\xi)|^2 \\ &\leq 4|\hat{u}(t,\xi) - \varphi(t,\xi)|^2 + 4|\varphi_1(t,\xi)|^2 + 4|\varphi_2(t,\xi)|^2.\end{aligned}$$

So,

$$\begin{aligned}\int_{\boldsymbol{R}^n} |\hat{u}(t,\xi)|^2 d\xi &\leq 4\int_{\boldsymbol{R}^n} |\hat{u}(t,\xi) - \varphi(t,\xi)|^2 d\xi + 4\int_{\boldsymbol{R}^n} |\varphi_1(t,\xi)|^2 d\xi + 4\int_{\boldsymbol{R}^n} |\varphi_2(t,\xi)|^2 d\xi \\ &\leq CI_{0,l}^2 \left(t^{-\frac{n+2}{2}} + t^{-\frac{n}{2}} + t^{-(l+1)}\right) \\ &\leq CI_{0,l}^2 t^{-\frac{n}{2}},\end{aligned}$$

since $l + 1 > \dfrac{n}{2}$, due to Lemmas 4.19, 4.20 and 4.18. Thus, the upper bound estimates in Theorems 4.26 and 4.27 can be proved by choosing $l = 1$ and $l = \dfrac{n-2}{2} + \varepsilon$, respectively.

On the other hand, since one has $|\varphi(t,\xi)| \leq |\varphi(t,\xi) - \hat{u}(t,\xi)| + |\hat{u}(t,\xi)|$ and $|\varphi_1(t,\xi)| \leq |\varphi_1(t,\xi) + \varphi_2(t,\xi)| + |\varphi_2(t,\xi)|$, by using Young's inequality, we obtain

$$\begin{aligned}|\hat{u}(t,\xi)|^2 &\geq \frac{1}{2}|\varphi(t,\xi)|^2 - |\varphi(t,\xi) - \hat{u}(t,\xi)|^2 \\ &\geq \frac{1}{4}|\varphi_1(t,\xi)|^2 - \frac{1}{2}|\varphi_2(t,\xi)|^2 - |\varphi(t,\xi) - \hat{u}(t,\xi)|^2.\end{aligned}$$

From the estimates just obtained in Lemmas 4.19, 4.20 and 4.18, one can obtain the following expression:

$$\begin{aligned}\int_{\boldsymbol{R}^n} |\hat{u}(t,\xi)|^2 d\xi &\geq \frac{1}{4}\int_{\boldsymbol{R}^n} |\varphi_1(t,\xi)|^2 d\xi - \frac{1}{2}\int_{\boldsymbol{R}^n} |\varphi_2(t,\xi)|^2 d\xi - \int_{\boldsymbol{R}^n} |\hat{u}(t,\xi) - \varphi(t,\xi)|^2 d\xi \\ &\geq C_1|P_0 + P_1|^2 t^{-\frac{n}{2}} - CI_{0,l}^2 t^{-(l+1)} - CI_{0,l}^2 t^{-\frac{n+2}{2}} - CI_{0,l}^2 t^{-(l+3)} \\ &= t^{-\frac{n}{2}}\left(C_1|P_0 + P_1|^2 - CI_0^2 t^{-\frac{2l-n+2}{2}} - CI_0^2 t^{-1} - CI_0^2 t^{-\frac{2l-n+6}{2}}\right). \qquad (4.61)\end{aligned}$$

If $\frac{n}{2} < l + 1$, then $\frac{2l-n+2}{2} > 0$ and $\frac{2l-n+6}{2} > 0$, because of $l + 1 < l + 3$. Hence, one has

$$\lim_{t\to\infty}\left(CI_0^2 t^{-\frac{2l-n+6}{2}} + CI_0^2 t^{-1} + CI_0^2 t^{-\frac{2l-n+2}{2}}\right) = 0,$$

so that there exists $t_1 \gg 1$ such that

$$CI_0^2 t^{-\frac{2l-n+6}{2}} + CI_0^2 t^{-1} + CI_0^2 t^{-\frac{2l-n+2}{2}} \leq \frac{C_1}{2}|P_0 + P_1|^2$$

for all $t \geq t_1$ in the case of $|P_1 + P_0| \neq 0$. That is, for $t \geq t_1$ it holds that

$$C_1|P_0 + P_1|^2 - CI_0^2 t^{-\frac{2l-n+6}{2}} - CI_0^2 t^{-1} - CI_0^2 t^{-\frac{2l-n+2}{2}} \geq \frac{C_1}{2}|P_0 + P_1|^2.$$

Therefore, one can arrive at the crucial estimate

$$\int_{\boldsymbol{R}^n_\xi} |\hat{u}(t,\xi)|^2 d\xi \geq \frac{C_1}{2}|P_0 + P_1|^2 t^{-\frac{n}{2}} \tag{4.62}$$

for $t \geq t_1$ because of (4.61). By choosing $l = 1$ in Theorem 4.26, and $l = \dfrac{n-2}{2} + \varepsilon$ in Theorem 4.27, one can get the desired estimates. □

Remark 4.28. It should be noted that the problem associated to equation

$$u_{tt} + Lu_{tt} + Lu + L^2 u + L^\theta u_t = 0,$$

with $0 < \theta \leq 1/2$ can still be studied for both operators $L = -\Delta$ and the logarithmic-Laplacian operator. As mentioned in introduction chapter, in [22] Horbach-Ikehata-Charão derived optimal asymptotic properties for $\theta > 1/2$ and the optimality of the intermediate case $0 < \theta \leq 1/2$ it seems to be open.

5 THE WAVE EQUATION WITH LOGARITHMIC TYPE DAMPING DEPENDING ON SMALL PARAMETER

In this chapter we study the Cauchy problem associated to the wave equation with a damping term of logarithmic type depending on small parameter $0 < \theta < \frac{1}{2}$ as follows

$$u_{tt} - \Delta u + L_\theta u_t = 0, \quad (t, x) \in (0, \infty) \times \boldsymbol{R}^n, \tag{5.1}$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \boldsymbol{R}^n. \tag{5.2}$$

The operator L_θ is given by (1.4)–(1.5) and $L_\theta u_t$ is the dissipative term of the system.

This research is a counter part of that was initiated by Charão-D'Abbicco-Ikehata considered in [4] for the large parameter case $\theta > \frac{1}{2}$. The case $\theta = \frac{1}{2}$ seems to be open.

By a similar argument used in Section 3.1 to prove existence and uniqueness of solution to the problem (3.1)–(3.2), one can prove that the problem (5.1)-(5.2) has a unique weak solution

$$u \in C([0, \infty); H^1(\boldsymbol{R}^n)) \cap C^1([0, \infty); L^2(\boldsymbol{R}^n))$$

for each $(u_0, u_1) \in H^1(\boldsymbol{R}^n) \times L^2(\boldsymbol{R}^n)$.

We notice that we also may apply the multipliers method already used in Sections 3.2 and 4.2 to obtain estimates to the solution and to the total energy of the system:

$$E_u(t) := \frac{1}{2} \left(\|u_t(t, \cdot)\|_{L^2}^2 + \|\nabla u(t, \cdot)\|_{L^2}^2 \right).$$

However, it is expected that this method will not produce optimal estimates, due to $\log(1 + |\xi|^{2\theta}) \approx |\xi|^{2\theta}$ in the low frequency region $|\xi| \leq 1$ and Ikehata-Natsume have studied the problem associated to the symbol $|\xi|^{2\theta}$ in [27]. Therefore, the multiplier method by Charão-da Luz-Ikehata as [7] should be more suitable to this problem resulting in the so-called almost optimal decay estimates.

We derived a *double diffusion-like* asymptotic profile as $t \to \infty$ and optimal estimates in time of solutions as $t \to \infty$ in L^2-sense. An important discovery in this research is that in the case when $n = 1$, we present a threshold $\theta^* = \frac{1}{4}$ of the parameter $\theta \in (0, \frac{1}{2})$ such that the solution of the Cauchy problem decays with some optimal rate for $\theta \in (0, \theta^*)$ as $t \to \infty$, while the L^2-norm of the corresponding solution never decays for $\theta \in [\theta^*, \frac{1}{2})$ and it blows up in infinite time. The case $\theta \in (0, \theta^*)$ indicates an usual diffusion phenomenon, while when $\theta \in [\theta^*, \frac{1}{2})$ the double diffusion phenomenon is crucial to estimate the solution in L^2-sense. Such a double diffusion in the one dimensional case is a quite novel phenomenon discovered through our new model produced by logarithmic damping with a small parameter θ. It might be prepared in the usual structural damping case such as $(-\Delta)^\theta u_t$ with $\theta \in (0, 1/2)$, however it seems that nobody ever pointed out even in the case of structural damping.

The results obtained in this chapter was published in *Journal of Differential Equations* (see [40]).

In Section 5.1 we find an asymptotic profile of solutions in the L^2-framework to the problem (5.1)-(5.2) in the case when (ideally speaking) $0 < \theta < \dfrac{1}{2}$. After that, in Section 5.2 we use such asymptotic profile to investigate the optimal decay rates, depending on the dimension n and the parameter θ, of solutions to problem (5.1)-(5.2). Without loss of generality we can assume that the initial amplitude $u_0 = 0$ when one concentrates only on capturing the leading term as time goes to infinity.

5.1 ASYMPTOTIC EXPANSION

The purpose of this section is to find an asymptotic profile to the solution of the problem (5.1)–(5.2) for $0 < \theta < \frac{1}{2}$. Through a leading term, we can find optimal decay and/or growth rates.

The associated problem to (5.1)-(5.2) in Fourier space is

$$\hat{u}_{tt} + \log(1 + |\xi|^{2\theta})\hat{u}_t + |\xi|^2\hat{u} = 0, \quad t > 0, \quad \xi \in \boldsymbol{R}^n, \tag{5.3}$$

$$\hat{u}(0, \xi) = 0, \quad \hat{u}_t(0, \xi) = \hat{u}_1(\xi), \quad \xi \in \boldsymbol{R}^n, \tag{5.4}$$

where the associated characteristic polynomial is

$$\lambda^2 + \log(1 + |\xi|^{2\theta})\lambda + |\xi|^2 = 0.$$

The characteristics roots are expressed as

$$\lambda_\pm = \frac{-\log(1 + |\xi|^{2\theta}) \pm \sqrt{\log^2(1 + |\xi|^{2\theta}) - 4|\xi|^2}}{2}, \quad \xi \in \boldsymbol{R}^n. \tag{5.5}$$

Lemma 5.1. *There exists $\delta = \delta(\theta)$, $0 < \delta < 1$ such that*

$$\log^2(1 + |\xi|^{2\theta}) - 4|\xi|^2 \geq 0 \textit{ for } |\xi| \leq \delta, \tag{5.6}$$

$$\log^2(1 + |\xi|^{2\theta}) - 4|\xi|^2 < 0 \textit{ for } |\xi| > \delta. \tag{5.7}$$

Proof. Working with $r = |\xi|$, we first observe that $\log(1 + r^2) < 2r$ for all $r > 0$. Also, $r^{2\theta} \leq r^2$ for $r \geq 1$, since $\theta \leq 1$. Therefore, in the case $\theta < \dfrac{1}{2}$, one has

$$\log(1 + r^{2\theta}) \leq \log(1 + r^2) < 2r \tag{5.8}$$

for all $r \geq 1$. Thus we may conclude that the function $f(r) := \log(1+r^{2\theta}) - 2r$ is negative for all $r \geq 1$. However, the similar phenomena does not happen near the origin. In fact, we first notice that

$$\lim_{r \to +0} \frac{\log(1 + r^{2\theta})}{r} = \infty,$$

for $\theta \in (0, \frac{1}{2})$. Therefore, there exists $r_0 = r_0(\theta) < 1$ such that

$$\frac{\log(1 + r^{2\theta})}{r} > 2$$

for all $r \in \boldsymbol{R}^n$ satisfying $0 < r < r_0$. Then, $f(r) = \log(1 + r^{2\theta}) - 2r \geq 0$ for $0 \leq r < r_0$. Furthermore, for $r \geq 0$ one can get

$$f''(r) = \frac{2\theta r^{2\theta-2}\left[(2\theta-1)(1+r^{2\theta}) - 2\theta r^{2\theta}\right]}{(1+r^{2\theta})^2} = \frac{2\theta r^{2\theta-2}\left[2\theta - 1 - r^{2\theta}\right]}{(1+r^{2\theta})^2}.$$

Since $0 < \theta < \frac{1}{2}$, the function $f : [0, \infty) \to \boldsymbol{R}$ satisfies $f''(r) < 0$. Due to $f(0) = 0$ and (5.8) one can conclude that there exists a unique number $\delta = \delta(\theta)$, $0 < \delta < 1$, such that $f(r) \geq 0$ for all $0 \leq r \leq \delta$ and $f(r) \leq 0$ for all $r \geq \delta$. Finally, one can write

$$\log^2(1 + |\xi|^{2\theta}) - 4|\xi|^2 = f(|\xi|)\left(\log(1 + |\xi|^{2\theta}) + 2|\xi|\right).$$

Therefore, using the properties of the function $f(r) = f(|\xi|)$ one can obtain the desired statement. □

By Lemma 5.1, we see that that the characteristics roots (5.5) are real-valued for $|\xi| \leq \delta$ and complex-valued for $|\xi| > \delta$. This is a crucial different point from that observed in the case of $\theta \geq 1/2$.

5.1.1 Estimates on the region $|\xi| \leq \delta$

First part of this section we analyze the behavior of the characteristics roots near the origin $\xi = 0$. To do that we need some remarks and lemmas.

Remark 5.2. For $q \geq 0$ it is easy to check the inequality $\frac{1}{2}r^q \leq \log(1 + r^q) \leq r^q$ for $r \in [0, 1]$. In particular, for $0 < \theta < \frac{1}{2}$ we have

$$\frac{1}{2}|\xi|^{2\theta} \leq \log(1 + |\xi|^{2\theta}) \leq \frac{3}{2}|\xi|^{2\theta}, \tag{5.9}$$
$$\frac{1}{2}|\xi|^{2} \leq \log(1 + |\xi|^{2}) \leq \frac{3}{2}|\xi|^{2}, \tag{5.10}$$
$$\frac{1}{2}|\xi|^{2-2\theta} \leq \log(1 + |\xi|^{2-2\theta}) \leq \frac{3}{2}|\xi|^{2-2\theta} \tag{5.11}$$

for $|\xi| \leq 1$.

We note that for $0 \leq \theta < 1/2$ it holds that

$$\lim_{r \to +0} \frac{r^{4-4\theta}}{r^2} = 0.$$

Thus, there exists $\delta_1 = \delta_1(\theta) > 0$ that satisfies

$$\frac{|\xi|^{4-4\theta}}{|\xi|^2} \leq \frac{1}{25} \tag{5.12}$$

whenever $0 < |\xi| \leq \delta_1$. Moreover, we can prove that $\delta_1 < \delta$. In fact, from (5.12) one has

$$25|\xi|^2 \leq |\xi|^{4\theta}$$

for $0 \leq |\xi| \leq \delta_1$. In this region it also holds $|\xi|^{4\theta} \leq 4\log^2(1+|\xi|^{2\theta})$, due to (5.9). Thus

$$\log^2(1+|\xi|^{2\theta}) \geq \frac{25}{4}|\xi|^2 \geq \frac{16}{3}|\xi|^2 \geq 4|\xi|^2 \tag{5.13}$$

for $|\xi| \leq \delta_1$. Comparing (5.13) with (5.6)–(5.7), we may conclude that $\delta_1 < \delta$. From (5.13) we also obtain

$$\log^2(1+|\xi|^{2\theta}) \geq \frac{16}{3}|\xi|^2$$

whenever $|\xi| \leq \delta_1$.

Now we define a new number:

$$\eta := \sup\{\alpha > 0; \frac{|\xi|^{4-4\theta}}{|\xi|^2} \leq \frac{1}{25} \text{ for } 0 < |\xi| \leq \alpha\}. \tag{5.14}$$

We note that η is positive and is well defined, because the set

$$\{\alpha > 0; \frac{|\xi|^{4-4\theta}}{|\xi|^2} \leq \frac{1}{25} \text{ for } |\xi| \leq \alpha\}$$

is not empty (δ_1 is a member of this set) and is bounded from above. In fact, for example, δ is an upper bound for this set and $\eta < \delta$ with δ defined in Lemma 5.1. In particular, the following two properties are true for $|\xi| \leq \eta$:

$$\frac{3}{4}\log^2(1+|\xi|^{2\theta}) \geq 4|\xi|^2, \tag{5.15}$$

$$25|\xi|^{4-4\theta} \leq |\xi|^2. \tag{5.16}$$

Lemma 5.3. *Let η be the number defined by (5.14). Then, for $|\xi| \leq \eta$ it holds that*

(i). $\lambda_+ - \lambda_- \approx \log(1+|\xi|^{2\theta})$;

(ii). $\lambda_+ \approx -\log(1+|\xi|^{2-2\theta}) \approx -|\xi|^{2-2\theta}$;

(iii). $\lambda_- \approx -\log(1+|\xi|^{2\theta})$.

Proof. (i) The upper estimate is simple because for $|\xi| \leq \eta < \delta$ it holds that

$$\lambda_+ - \lambda_- = \sqrt{\log^2(1+|\xi|^{2\theta}) - 4|\xi|^2} \leq \sqrt{\log^2(1+|\xi|^{2\theta})} = \log(1+|\xi|^{2\theta}).$$

On the other hand, by (5.15) we have

$$\frac{1}{4}\log^2(1+|\xi|^{2\theta}) \leq \log^2(1+|\xi|^{2\theta}) - 4|\xi|^2, \quad |\xi| \leq \eta.$$

For this reason, in the zone $|\xi| \leq \eta$ it holds that

$$\frac{1}{2}\log(1+|\xi|^{2\theta}) \leq \sqrt{\log^2(1+|\xi|^{2\theta}) - 4|\xi|^2}.$$

(ii). The inequality (5.16) provides us

$$0 \geq 25|\xi|^{4-4\theta} - 5|\xi|^2 + 4|\xi|^2 = 25|\xi|^{4-4\theta} - 5|\xi|^{2\theta}|\xi|^{2-2\theta} + 4|\xi|^2. \tag{5.17}$$

The lower inequality in (5.9) implies that $-10\log(1+|\xi|^{2\theta}) \leq -5|\xi|^{2\theta}$ for $|\xi| \leq 1$ and, in particular, for $|\xi| \leq \eta$. By combining this fact with (5.17), we obtain

$$25|\xi|^{4-4\theta} - 10\log(1+|\xi|^{2\theta})|\xi|^{2-2\theta} + 4|\xi|^2 \leq 0, \quad |\xi| \leq \eta.$$

Adding $\log^2(1+|\xi|^{2\theta})$ on both sides we may obtain

$$\begin{aligned}\Big(\log(1+|\xi|^{2\theta}) - 5|\xi|^{2-2\theta}\Big)^2 &= \log^2(1+|\xi|^{2\theta}) - 10\log(1+|\xi|^{2\theta})|\xi|^{2-2\theta} + 25|\xi|^{4-4\theta} \\ &\leq \log^2(1+|\xi|^{2\theta}) - 4|\xi|^2.\end{aligned}$$

Hence, for $|\xi| \leq \eta$, $\log(1+|\xi|^{2\theta}) - 5|\xi|^{2-2\theta} \leq \sqrt{\log^2(1+|\xi|^{2\theta}) - 4|\xi|^2}$ and

$$-\frac{5}{2}|\xi|^{2-2\theta} \leq \frac{-\log(1+|\xi|^{2\theta}) + \sqrt{\log^2(1+|\xi|^{2\theta}) - 4|\xi|^2}}{2} = \lambda_+.$$

Furthermore, we also concludes that

$$-5\log(1+|\xi|^{2-2\theta}) \leq -\frac{5}{2}|\xi|^{2-2\theta} \leq \lambda_+ \tag{5.18}$$

on the zone $|\xi| \leq \eta$, due to (5.11).

In order to prove the upper estimate part of (ii) we first observe that

$$0 \leq |\xi|^2 + |\xi|^{4-4\theta} = 4|\xi|^2 + |\xi|^{4-4\theta} - 3|\xi|^{2\theta}|\xi|^{2-2\theta}.$$

In the zone $|\xi| \leq \eta$ it holds that $-3|\xi|^{2\theta} \leq -2\log(1+|\xi|^{2\theta})$ by (5.9), which implies that

$$-3|\xi|^{2\theta}|\xi|^{2-2\theta} \leq -2\log(1+|\xi|^{2\theta})|\xi|^{2-2\theta}.$$

By using the inequality just above we may obtain that

$$0 \leq 4|\xi|^2 + |\xi|^{4-4\theta} - 2\log(1+|\xi|^{2\theta})|\xi|^{2-2\theta}. \tag{5.19}$$

We add $\log^2(1+|\xi|^{2\theta})$ in both side of (5.19) in order to get the following estimate:

$$\begin{aligned}\log^2(1+|\xi|^{2\theta}) - 4|\xi|^2 &\leq \log^2(1+|\xi|^{2\theta}) - 2\log(1+|\xi|^{2\theta})|\xi|^{2-2\theta} + |\xi|^{4-4\theta} \\ &= \Big(\log(1+|\xi|^{2\theta}) - |\xi|^{2-2\theta}\Big)^2.\end{aligned}$$

This implies

$$\lambda_+ = \frac{-\log(1+|\xi|^{2\theta}) + \sqrt{\log^2(1+|\xi|^{2\theta}) - 4|\xi|^2}}{2} \leq -\frac{1}{2}|\xi|^{2-2\theta}. \tag{5.20}$$

When one derives (5.20), one must check the fact that $\log(1+|\xi|^{2\theta}) - |\xi|^{2-2\theta} \geq 0$ on $|\xi| \leq \eta$. Indeed, this can be easily observed by a combination of (5.15) and (5.16).

Now, by combining inequalities (5.20) and (5.11) one obtain

$$\lambda_+ \leq -\frac{1}{2}|\xi|^{2-2\theta} \leq -\frac{1}{3}\log(1+|\xi|^{2-2\theta})$$

because of $|\xi| \leq \eta$. The inequalities just above and (5.18) imply the desired statement of item (ii).

(iii). In the course of the proof of item (i) in the region $|\xi| \leq \eta$, we also have

$$-\log(1+|\xi|^{2\theta}) \leq -\sqrt{\log^2(1+|\xi|^{2\theta}) - 4|\xi|^2} \leq -\frac{1}{2}\log(1+|\xi|^{2\theta}).$$

Therefore, one can easily conclude that

$$-\log(1+|\xi|^{2\theta}) \leq \frac{-\log(1+|\xi|^{2\theta}) - \sqrt{\log^2(1+|\xi|^{2\theta}) - 4|\xi|^2}}{2} \leq -\frac{3}{4}\log(1+|\xi|^{2\theta}),$$

for $|\xi| \leq \eta$. This implies the desired statement of item (iii). □

5.1.1.1 Estimates on the low-frequency zone $|\xi| \leq \eta^3$

We first remember the number $\eta \in (0,\delta)$ defined in (5.9)-(5.16). Also, since $0 < \eta < 1$, we have $\eta^3 < \eta$. In the zone of low frequency $|\xi| \leq \eta^3 < \eta$, the characteristics roots $\lambda_\pm$ are real and the solution of (5.3)-(5.4) is explicitly given by

$$\hat{u}(t,\xi) = \frac{e^{t\lambda_+} - e^{t\lambda_-}}{\lambda_+ - \lambda_-}\hat{u}_1(\xi). \tag{5.21}$$

The purpose in this section is to get an asymptotic profile to the solution $\hat{u}(t,\xi)$ and, in order to do that, we need to obtain useful estimates. For this reason, we define a function $g : [0,\delta] \to \boldsymbol{R}$ inspired by an idea from [18], as follows. A discovery of this function $g(s)$ is one of decisive points in our proof.

$$g(s) := \begin{cases} 1 + \sqrt{1 - \frac{4s^6}{\log^2(1+s^{6\theta})}} & \text{if } 0 < s \leq \delta \\ 2 & \text{if } s = 0. \end{cases} \tag{5.22}$$

Note that for $0 < \theta < 1/2$,

$$\lim_{s\to 0^+} \frac{s^6}{\log^2(1+s^{6\theta})} = 0.$$

Remark 5.4. Let $t > 0$ and $\xi \in \boldsymbol{R}^n$, $0 < |\xi| \leq \eta$, be fixed. We recall that $\eta < \delta < 1$. Let us consider the function $h(s)$ defined on $[0,\eta]$ as follows:

$$h(s) := e^{-\frac{t\log(1+|\xi|^{2\theta})}{2}g(s)}.$$

We see that $h(s)$ is differentiable on $(0,\eta)$. Then, it should be noted that one can apply the mean value theorem in the interval $[0,s]$ for each $s \in (0,\eta]$ to get

$$\begin{aligned}\frac{h(s)-h(0)}{s} &= \frac{e^{-\frac{t\log(1+|\xi|^{2\theta})}{2}g(s)} - e^{-\frac{t\log(1+|\xi|^{2\theta})}{2}g(0)}}{s} \\ &= -\frac{t\log(1+|\xi|^{2\theta})}{2}e^{-\frac{t\log(1+|\xi|^{2\theta})}{2}g(\alpha s)}g'(\alpha s)\end{aligned} \tag{5.23}$$

with some $\alpha = \alpha(s,t,|\xi|) \in (0,1)$.

We observe that on the low frequency zone $0 \le |\xi| \le \eta^3$ it holds that

$$\lambda_- = -\frac{\log(1+|\xi|^{2\theta})}{2} g(\sqrt[3]{|\xi|}).$$

By applying (5.23) for $t > 0$ and $s = \sqrt[3]{|\xi|}$, $0 < |\xi| \le \eta^3$, we have

$$e^{t\lambda_-} = e^{-t\log(1+|\xi|^{2\theta})} - \frac{t}{2}\log(1+|\xi|^{2\theta})\sqrt[3]{|\xi|} e^{-\frac{t\log(1+|\xi|^{2\theta})}{2} g(\alpha\sqrt[3]{|\xi|})} g'(\alpha\sqrt[3]{|\xi|}) \tag{5.24}$$

with $\alpha := \alpha(s,t,|\xi|) = \alpha(t,|\xi|) \in (0,1)$.

From the Chill-Haraux [11] idea, we also observe that

$$\lambda_+ = -\frac{\lambda_+^2 + |\xi|^2}{\log(1+|\xi|^{2\theta})},$$

so that one has

$$e^{t\lambda_+} = e^{-\frac{|\xi|^2}{\log(1+|\xi|^{2\theta})}t} e^{-\frac{\lambda_+^2}{\log(1+|\xi|^{2\theta})}t}. \tag{5.25}$$

On the other hand, because of (5.5) we see that

$$\frac{1}{\lambda_+ - \lambda_-} = \frac{1}{\log(1+|\xi|^{2\theta})} + R(|\xi|)$$

where

$$R(r) = \frac{4r^2}{\log^3(1+r^{2\theta})\sqrt{1-\frac{4r^2}{\log^2(1+r^{2\theta})}}\left(1+\sqrt{1-\frac{4r^2}{\log^2(1+r^{2\theta})}}\right)}. \tag{5.26}$$

Now we assume that the initial data $u_1 \in L^1(\boldsymbol{R}^n)$ to use the decomposition

$$\hat{u}_1(\xi) = A_{u_1}(\xi) - iB_{u_1}(\xi) + P_{u_1} =: A_1(\xi) - iB_1(\xi) + P_1$$

as in (2.8). By combining (5.24), (5.25) and (5.26) with this decomposition of initial data, we can write the solution of (5.3)-(5.4) given by (5.21), for $|\xi| \le \eta^3$, as follows.

$$\begin{aligned}
\hat{u}(t,\xi) = {} & \frac{e^{-\frac{|\xi|^2}{\log(1+|\xi|^{2\theta})}t}}{\log(1+|\xi|^{2\theta})} P_1 - \frac{e^{-t\log(1+|\xi|^{2\theta})}}{\log(1+|\xi|^{2\theta})} P_1 + R(|\xi|) e^{-\frac{|\xi|^2}{\log(1+|\xi|^{2\theta})}t} \hat{u}_1(\xi) \\
& + \frac{e^{-\frac{|\xi|^2}{\log(1+|\xi|^{2\theta})}t}}{\log(1+|\xi|^{2\theta})} (A_1(\xi) - iB_1(\xi)) + e^{-\frac{|\xi|^2}{\log(1+|\xi|^{2\theta})}t} \frac{e^{-\frac{\lambda_+^2}{\log(1+|\xi|^{2\theta})}t} - 1}{\lambda_+ - \lambda_-} \hat{u}_1(\xi) \\
& - \frac{e^{-t\log(1+|\xi|^{2\theta})}}{\log(1+|\xi|^{2\theta})} (A_1(\xi) - iB_1(\xi)) - R(|\xi|) e^{-t\log(1+|\xi|^{2\theta})} \hat{u}_1(\xi) \\
& + t\frac{\log(1+|\xi|^{2\theta})\sqrt[3]{|\xi|}}{2(\lambda_+ - \lambda_-)} e^{-\frac{t\log(1+|\xi|^{2\theta})}{2} g\left(\alpha\sqrt[3]{|\xi|}\right)} g'\left(\alpha\sqrt[3]{|\xi|}\right) \hat{u}_1(\xi).
\end{aligned} \tag{5.27}$$

We introduce an asymptotic profile as $t \to +\infty$ of the solution $\hat{u}(t,\xi)$ in the low frequency region $|\xi| \leq \eta^3 < \eta$ in the simple form

$$\varphi(t,\xi) := \frac{e^{-\frac{|\xi|^2}{\log(1+|\xi|^{2\theta})}t}}{\log(1+|\xi|^{2\theta})}P_1 - \frac{e^{-\log(1+|\xi|^{2\theta})t}}{\log(1+|\xi|^{2\theta})}P_1. \tag{5.28}$$

Thus, we need to prove that

$$\|\hat{u}(t,\cdot) - \varphi(t,\cdot)\| \to 0, \quad t \to \infty$$

with a better decay rate than the components in the right hand side of (5.28). To prove this result, we consider the following six remainder functions.

$$\begin{aligned}
F_1(t,\xi) &= R(|\xi|)e^{-\frac{|\xi|^2}{\log(1+|\xi|^{2\theta})}t}\hat{u}_1(\xi),\\
F_2(t,\xi) &= -R(|\xi|)e^{-t\log(1+|\xi|^{2\theta})}\hat{u}_1(\xi),\\
F_3(t,\xi) &= \frac{e^{-\frac{|\xi|^2}{\log(1+|\xi|^{2\theta})}t}}{\log(1+|\xi|^{2\theta})}(A_1(\xi) - iB_1(\xi)),\\
F_4(t,\xi) &= e^{-\frac{|\xi|^2}{\log(1+|\xi|^{2\theta})}t}\frac{e^{-\frac{\lambda_+^2}{\log(1+|\xi|^{2\theta})}t} - 1}{\lambda_+ - \lambda_-}\hat{u}_1(\xi),\\
F_5(t,\xi) &= -\frac{e^{-t\log(1+|\xi|^{2\theta})}}{\log(1+|\xi|^{2\theta})}(A_1(\xi) - iB_1(\xi)),\\
F_6(t,\xi) &= t\frac{\log(1+|\xi|^{2\theta})\sqrt[3]{|\xi|}}{2(\lambda_+ - \lambda_-)}e^{-\frac{t\log(1+|\xi|^{2\theta})}{2}g\left(\alpha\sqrt[3]{|\xi|}\right)}g'\left(\alpha\sqrt[3]{|\xi|}\right)\hat{u}_1(\xi).
\end{aligned}$$

From (5.27) and (5.28), for $|\xi| \leq \eta$, we have

$$\hat{u}(t,\xi) - \varphi(t,\xi) = \sum_{j=1}^{6} F_j(t,\xi).$$

In order to obtain decay rates in time to these functions we assume the additional condition on the initial data such that

$$u_1 \in L^{1,2\theta}(\boldsymbol{R}^n), \;\; 0 < \theta < 1/2.$$

To begin with, we estimate the function $F_3(t,\xi)$. Indeed, by using the estimates in (5.11), Lemma 2.24 with $\kappa := \theta \in (0,1/2)$ (this is our crucial idea) and the inequalities (5.9) and (5.11) one can estimate

$$\begin{aligned}
\int_{|\xi|\leq\eta^3<1}|F_3(t,\xi)|^2d\xi &= \int_{|\xi|\leq\eta^3}\frac{e^{-\frac{2|\xi|^2}{\log(1+|\xi|^{2\theta})}t}}{\log^2(1+|\xi|^{2\theta})}|A_1(\xi) - iB_1(\xi)|^2d\xi\\
&\leq \int_{|\xi|\leq\eta^3}\frac{e^{-\frac{2t}{3}\log(1+|\xi|^{2-2\theta})}}{\log^2(1+|\xi|^{2\theta})}|A_1(\xi) - iB_1(\xi)|^2d\xi
\end{aligned}$$

$$
\begin{aligned}
&\leq 2(M+K)^2\|u_1\|^2_{L^{1,2\theta}}\int_{|\xi|\leq\eta^3}\frac{e^{-\frac{2t}{3}\log(1+|\xi|^{2-2\theta})}}{\log^2(1+|\xi|^{2\theta})}|\xi|^{4\theta}d\xi\\
&\leq C(M+K)^2\|u_1\|^2_{L^{1,2\theta}}\int_{|\xi|\leq\eta^3}(1+|\xi|^{2-2\theta})^{-\frac{2t}{3}}d\xi\\
&= C(M+K)^2\omega_n\|u_1\|^2_{L^{1,2\theta}}\int_0^{\eta^3}(1+r^{2-2\theta})^{-\frac{2t}{3}}r^{n-1}dr\\
&\leq C\|u_1\|^2_{L^{1,2\theta}}t^{-\frac{n}{2(1-\theta)}}, \quad t\gg 1. \qquad (5.29)
\end{aligned}
$$

In the third line of the sequence of estimates (5.29), we observe that

$$
\lim_{\sigma\to+0}\frac{\sigma}{\log(1+\sigma)}=1,
$$

which justifies the constant $C>0$ in the subsequent line. The last inequality is due to Lemma 2.30.

Similarly, we can also estimate

$$
\begin{aligned}
\int_{|\xi|\leq\eta^3}|F_5(t,\xi)|^2d\xi &= \int_{|\xi|\leq\eta^3}\frac{e^{-2\log(1+|\xi|^{2\theta})t}}{\log^2(1+|\xi|^{2\theta})}|A_1(\xi)-iB_1(\xi)|^2d\xi\\
&= \int_{|\xi|\leq\eta^3}\frac{(1+|\xi|^{2\theta})^{-2t}}{\log^2(1+|\xi|^{2\theta})}|A_1(\xi)-iB_1(\xi)|^2d\xi\\
&\leq C(K+M)^2\|u_1\|^2_{L^{1,2\theta}}\int_{|\xi|\leq\eta^3}(1+|\xi|^{2\theta})^{-2t}d\xi\\
&= C\omega_n(K+M)^2\|u_1\|^2_{L^{1,2\theta}}\int_0^{\eta^3}(1+r^{2\theta})^{-2t}r^{n-1}dr\\
&\leq \frac{C}{\theta}\|u_1\|^2_{L^{1,2\theta}}t^{-\frac{n}{2\theta}}, \quad t\gg 1, \qquad (5.30)
\end{aligned}
$$

where the last inequality is due to Lemma 2.32.

On the next estimates to the functions $F_j(t,\xi)$ we also rely on Lemma 2.30 or Lemma 2.32.

In order to estimate $F_4(t,\xi)$ we use the fact that $|e^{-a}-1|\leq a$ for all $a\geq 0$. Then, Lemma 5.3 and inequality (5.11) imply the existence of a constant $C>0$ such that

$$
\begin{aligned}
\int_{|\xi|\leq\eta^3}|F_4(t,\xi)|^2d\xi &= \int_{|\xi|\leq\eta^3}\left(\frac{e^{\frac{-\lambda_+^2}{\log(1+|\xi|^{2\theta})}t}-1}{\lambda_+-\lambda_-}\right)^2 e^{-\frac{2|\xi|^2}{\log(1+|\xi|^{2\theta})}t}|\hat{u}_1(\xi)|^2d\xi\\
&\leq t^2\|u_1\|_1^2\int_{|\xi|\leq\eta^3}\frac{\lambda_+^4}{\log^2(1+|\xi|^{2\theta})}\frac{e^{-\frac{2|\xi|^2}{\log(1+|\xi|^{2\theta})}t}}{(\lambda_+-\lambda_-)^2}d\xi\\
&\leq Ct^2\|u_1\|_1^2\int_{|\xi|\leq\eta^3}\frac{|\xi|^{8-8\theta}}{\log^4(1+|\xi|^{2\theta})}e^{-\frac{2t}{3}\log(1+|\xi|^{2-2\theta})}d\xi\\
&\leq Ct^2\|u_1\|_1^2\int_{|\xi|\leq\eta^3}(1+|\xi|^{2-2\theta})^{-\frac{2t}{3}}|\xi|^{8-16\theta}d\xi
\end{aligned}
$$

$$
\begin{aligned}
&= C\omega_n t^2\|u_1\|_1^2 \int_0^{\eta^3} (1+r^{2-2\theta})^{-\frac{2t}{3}} r^{7-16\theta+n} dr \\
&\leq C\|u_1\|_1^2 t^2 t^{-\frac{8-16\theta+n}{2(1-\theta)}} \\
&= C\|u_1\|_1^2 t^{-\frac{4-12\theta+n}{2(1-\theta)}}, \quad t \gg 1. \qquad (5.31)
\end{aligned}
$$

Remark 5.5. Note that in the above estimate (5.31) to apply Lemma 2.30 it is necessary to check $7-16\theta+n > -1$, but this holds for $0 \leq \theta < 1/2$. Moreover, according to our computations above, we have to prove that all L^2-norm of the six functions $F_1(t,\xi), \cdots, F_6(t,\xi)$ decay to zero in time. However, to get such decay estimates in (5.31), we need additional restriction such that $0 \leq \theta < \frac{5}{12} < \frac{1}{2}$ in the case $n = 1$. For $n \geq 2$ this restriction is not necessary, because $t^{-\frac{4-12\theta+n}{2(1-\theta)}} \to 0$ when $t \to \infty$ for any $\theta \in (0, \frac{1}{2})$.

Now we want to obtain an estimate for $F_1(t,\cdot)$ on the region $|\xi| \leq \eta^3$. Initially, from (5.9) we may see that

$$
\begin{aligned}
\int_{|\xi|\leq\eta^3} |F_1(t,\xi)|^2 d\xi &= \int_{|\xi|\leq\eta^3} e^{-\frac{2|\xi|^2}{\log(1+|\xi|^{2\theta})}t} |R(\xi)|^2 |\hat{u}_1(\xi)|^2 d\xi \\
&\approx \int_{|\xi|\leq\eta^3} e^{-\log(1+|\xi|^{2-2\theta})t} |R(\xi)|^2 |\hat{u}_1(\xi)|^2 d\xi \\
&\leq \|u_1\|_1^2 \int_{|\xi|\leq\eta^3} e^{-\log(1+|\xi|^{2-2\theta})t} |R(\xi)|^2 d\xi. \qquad (5.32)
\end{aligned}
$$

Here, the function $R(r)$ is bounded on the low frequency zone for $0 < \theta \leq \frac{1}{3}$, because of

$$
\lim_{r\to+0} R(r) = \begin{cases} 0 & \text{for } \ 0 < \theta < \frac{1}{3}, \\ 4 & \text{for } \ \theta = \frac{1}{3}. \end{cases} \qquad (5.33)
$$

Therefore, for $0 < \theta \leq \frac{1}{3}$ and $n \geq 1$, from (5.32) and (5.33), we may conclude the existence of a positive constant C such that

$$
\begin{aligned}
\int_{|\xi|\leq\eta^3} |F_1(t,\xi)|^2 d\xi &\leq C\|u_1\|_1^2 \int_{|\xi|\leq\eta^3} e^{-\log(1+|\xi|^{2-2\theta})t} d\xi \\
&= C\|u_1\|_1^2 \omega_n \int_0^{\eta^3} (1+r^{2-2\theta})^{-t} r^{n-1} dr \\
&\sim \|u_1\|_1^2 t^{-\frac{n}{2(1-\theta)}}, \quad t \gg 1. \qquad (5.34)
\end{aligned}
$$

Furthermore, in the case of $0 \leq \theta \leq \frac{5}{12}$ we also notice that the function $R(r)\sqrt{r}$ is bounded in the region $|\xi| \leq \eta^3$, because

$$
\lim_{r\to 0} \sqrt{r} R(r) = \begin{cases} 0 & \text{for } 0 < \theta < \frac{5}{12}, \\ 4 & \text{for } \ \theta = \frac{5}{12}. \end{cases}
$$

Thus, if $n \geq 2$, we can get other estimate to $F_1(t,\cdot)$ for $0 \leq \theta \leq \frac{5}{12}$, from (5.32), as follows.

$$
\int_{|\xi|\leq\eta^3} |F_1(t,\xi)|^2 d\xi \leq \omega_n \|u_1\|_1^2 \int_0^{\eta^3} (1+r^{2-2\theta})^{-t} |R(r)|^2 r^{n-1} dr
$$

$$\leq C\|u_1\|_1^2 \int_0^{\eta^3} (1+r^{2-2\theta})^{-t} r^{n-2} dr$$
$$\leq C\|u_1\|_1^2 t^{-\frac{n-1}{2(1-\theta)}}, \quad t \gg 1. \tag{5.35}$$

In Section 5.2, we prove that the asymptotic profile decays with the rate $t^{-\frac{n-4\theta}{2(1-\theta)}}$ for $n \geq 2$ (see Lemma 5.14). For this reason, we only consider the estimate (5.34), which holds for $n \geq 2$, when $\theta > \frac{1}{4}$.

Similarly to the way used to obtain estimates for $F_1(t,\cdot)$ one can arrive at the following estimates for $F_2(t,\cdot)$:

$$\int_{|\xi|\leq\eta^3} |F_2(t,\xi)|^2 d\xi \leq \begin{cases} \frac{C}{\theta}\|u_1\|_1^2 t^{-\frac{n}{2\theta}} & \text{for } n \geq 1 \text{ and } 0 < \theta \leq \frac{1}{3}, \quad t \gg 1, \\ \frac{C}{\theta}\|u_1\|_1^2 t^{-\frac{n-1}{2\theta}} & \text{for } n \geq 2 \text{ and } \frac{1}{4} < \theta \leq \frac{5}{12}, \quad t \gg 1. \end{cases} \tag{5.36}$$

Let us estimate the L^2-norm of $F_6(t,\xi)$ at the final stage in this subsection. To do that we need to analyse the function $g(s)$ given by (5.22). Note that it is easy to see that

$$1 \leq g(s) \leq 2 \tag{5.37}$$

for $s \leq \delta$. Its derivative is given by

$$g'(s) = \frac{1}{2\sqrt{1-\frac{4s^6}{\log^2(1+s^{6\theta})}}} \left(\frac{48\theta s^{6\theta+5}}{(1+s^{6\theta})\log^3(1+s^{6\theta})} - \frac{24s^5}{\log^2(1+s^{6\theta})} \right).$$

Then, for $\theta \in [0, \frac{5}{12}]$, the function $g'(s)$ is bounded on the interval $0 < s \leq \eta$. In fact, the limits

$$\lim_{s\to+0} \frac{s^{6\theta+5}}{(1+s^{6\theta})\log^3(1+s^{6\theta})} \qquad \text{and} \qquad \lim_{s\to+0} \frac{s^5}{\log^2(1+s^{6\theta})}$$

are finite because of $0 \leq \theta \leq \frac{5}{12}$. It should be mentioned that the same does note happen on the zone $\eta < s < \delta$ because

$$\lim_{s\to\delta-0} \left(\sqrt{1-\frac{4s^6}{\log^2(1+s^{6\theta})}} \right)^{-1} = +\infty$$

(see (5.6)–(5.7)). Recall again that for $\theta \in (0, 1/2)$

$$\lim_{s\to+0} \frac{s^6}{\log^2(1+s^{6\theta})} = 0.$$

By summarizing above facts, there exists a constant $K > 0$ depending on $\theta \in [0, \frac{5}{12}]$ and $\eta > 0$ such that for all $s \in [0, \eta]$ it holds that

$$|g'(s)| \leq K.$$

In particular, for $|\xi| \in [0, \eta^3]$, we have $\sqrt[3]{|\xi|} \in [0, \eta]$ and $\alpha(t, \xi)\sqrt[3]{|\xi|} \in [0, \eta]$. Thus

$$|g'(\alpha\sqrt[3]{|\xi|})| \le K, \quad |\xi| \le \eta^3. \tag{5.38}$$

From (5.37) and (5.38), for $0 < \theta \le \frac{5}{12}$ and $n \ge 1$ we can estimate the L^2-norm of $F_6(t, \cdot)$ as follows:

$$\begin{aligned}
&\int_{|\xi|\le\eta^3} |F_6(t, \xi)|^2 d\xi = \\
&= \frac{1}{4}t^2 \int_{|\xi|\le\eta^3} e^{-2\frac{t\log(1+|\xi|^{2\theta})}{2} g\left(\alpha\sqrt[3]{|\xi|}\right)} \frac{\log^2(1+|\xi|^{2\theta})|\xi|^{\frac{2}{3}}}{(\lambda_+ - \lambda_-)^2} |g'\left(\alpha\sqrt[3]{|\xi|}\right)|^2 |\hat{u}_1(\xi)|^2 d\xi \\
&\le Ct^2\|u_1\|_1^2 \int_{|\xi|\le\eta^3} e^{-t\log(1+|\xi|^{2\theta})}|\xi|^{\frac{2}{3}} d\xi \\
&= C\omega_n t^2\|u_1\|_1^2 \int_0^{\eta^3} (1 + r^{2\theta})^{-t} r^{n-\frac{1}{3}} dr \\
&\sim \frac{1}{\theta} t^2\|u_1\|_1^2 t^{-\frac{n+\frac{2}{3}}{2\theta}} \\
&= \frac{1}{\theta}\|u_1\|_1^2 t^{-\frac{n-4\theta+\frac{2}{3}}{2\theta}}, \quad t \gg 1.
\end{aligned} \tag{5.39}$$

As a result one can conclude the following Propositions. In that case, it is essential whether the factor $1/\theta$ can be included or not in the final estimates as the coefficient.

Proposition 5.6. *Let $n = 1$, $0 < \theta \le \frac{1}{3}$, and $\varphi(t, \xi)$ be given by (5.28). If $u_1 \in L^{1,2\theta}(\boldsymbol{R})$, then*

$$\int_{|\xi|\le\eta^3} |\hat{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi \le \begin{cases} C(\|u_1\|_1^2 + \|u_1\|_{L^{1,2\theta}}^2)\left(t^{-\frac{1}{2(1-\theta)}} + \frac{1}{\theta}t^{-\frac{1}{2\theta}}\right), & \text{if } 0 < \theta \le \frac{1}{6}, \\ C(\|u_1\|_1^2 + \|u_1\|_{L^{1,2\theta}}^2)\left(t^{-\frac{1}{2(1-\theta)}} + \frac{1}{\theta}t^{-\frac{\frac{5}{3}-4\theta}{2\theta}}\right), & \text{if } \frac{1}{6} < \theta \le \frac{1}{3} \end{cases},$$

for $t \gg 1$.

Proof. The proof is obtained by choosing the slowest estimates as $t \to \infty$ among (5.29), (5.30), (5.31), (5.34), (5.36) and (5.39). Note that the case $1/6 < \theta \le 1/3$ is coming from the relation such that $\frac{\frac{5}{3}-4\theta}{2\theta} \le \frac{n}{2\theta}$ with $n = 1$. □

Proposition 5.7. *Let $n \ge 2$, $0 < \theta \le \frac{5}{12}$ and $\varphi(t, \xi)$ be given by (5.28). If $u_1 \in L^{1,2\theta}(\boldsymbol{R}^n)$, then*

$$\int_{|\xi|\le\eta^3} |\hat{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi \le \begin{cases} C(\|u_1\|_1^2 + \|u_1\|_{L^{1,2\theta}}^2)\left(t^{-\frac{n}{2(1-\theta)}} + \frac{1}{\theta}t^{-\frac{n}{2\theta}}\right), & \text{if } 0 < \theta \le \frac{1}{6}, \\ C(\|u_1\|_1^2 + \|u_1\|_{L^{1,2\theta}}^2)\left(t^{-\frac{n}{2(1-\theta)}} + \frac{1}{\theta}t^{-\frac{n-4\theta+\frac{2}{3}}{2\theta}}\right), & \text{if } \frac{1}{6} < \theta \le \frac{1}{3}, \\ C(\|u_1\|_1^2 + \|u_1\|_{L^{1,2\theta}}^2)\left(t^{-\frac{n-1}{2(1-\theta)}} + \frac{1}{\theta}t^{-\frac{n-1}{2\theta}}\right), & \text{if } \frac{1}{3} < \theta \le \frac{5}{12} \end{cases}$$

for $t \gg 1$.

Proof. We may conclude this result by comparing the estimates (5.29), (5.30), (5.31), (5.34), (5.35), (5.36) and (5.39). Note that the case $1/6 < \theta \leq 1/3$ is coming from the relation such that $\frac{n-4\theta+\frac{2}{3}}{2\theta} \leq \frac{n}{2\theta}$ with $n \geq 2$. ☐

5.1.1.2 Estimates on the middle-frequency zone $\eta^3 \leq |\xi| \leq \delta$

We call the zone $\eta^3 \leq |\xi| \leq \delta$ the middle frequency. On this zone the characteristics roots given by (5.5) are real and therefore the solution of (5.3)-(5.4) is given by

$$\hat{u}(t,\xi) = e^{-t\frac{\log(1+|\xi|^{2\theta})}{2}} \frac{\sinh(C(\xi)t)}{2C(\xi)} \hat{u}_1(\xi),$$

where

$$C(\xi) = \frac{\sqrt{\log^2(1+|\xi|^{2\theta}) - 4|\xi|^2}}{2}.$$

We remember that η is defined in (5.14). Since the function $|\xi| \mapsto \frac{|\xi|^{4-4\theta}}{|\xi|^2}$ is increasing for $0 < \theta < \frac{1}{2}$, we may observe that

$$\eta^3 = \sup\{\alpha > 0; \frac{|\xi|^{4-4\theta}}{|\xi|^2} \leq \frac{1}{25^3} \text{ for } 0 < |\xi| \leq \alpha\}. \tag{5.40}$$

Lemma 5.8. *There exists $\beta = \beta(\theta)$, $0 < \beta \leq \eta^3$, such that*

$$\frac{2}{25^3} \log^2(1+|\xi|^{2\theta}) \geq 4|\xi|^2 \textit{ for } |\xi| \leq \beta$$
$$\frac{2}{25^3} \log^2(1+|\xi|^{2\theta}) \leq 4|\xi|^2 \textit{ for } |\xi| \geq \beta.$$

Proof. The argument used to prove the existence of δ as in Lemma 5.1 can be also used to prove the existence of $\beta = \beta(\theta) \in (0,1)$, which satisfies both conclusions of this lemma. So, it suffices to check that $\beta \leq \eta^3$.

From Remark 5.2, we know that $\log^2(1+|\xi|^{2\theta}) \leq |\xi|^{4\theta}$, for $|\xi| \leq 1$. Thus, if $|\xi| \leq \beta$, we have

$$\frac{2}{25^3}|\xi|^{4\theta} \geq \frac{2}{25^3} \log^2(1+|\xi|^{2\theta}) \geq 4|\xi|^2.$$

This implies

$$2 \times 25^3 |\xi|^{4-4\theta} \leq |\xi|^2, \qquad |\xi| \leq \beta.$$

and the condition $\frac{|\xi|^{4-4\theta}}{|\xi|^2} \leq \frac{1}{25^3}$ is satisfied for $|\xi| \leq \beta$. Therefore, one has $\beta \leq \eta^3$ from (5.40). ☐

In other words, Lemma 5.8 tells us that

$$\log^2(1+|\xi|^{2\theta}) - 4|\xi|^2 < \frac{25^3 - 2}{25^3} \log^2(1+|\xi|^{2\theta}) \quad \text{for} \quad |\xi| \geq \beta$$

and in particular, the definition of $C(\xi)$ implies that

$$0 < 2C(\xi) = \sqrt{\log^2(1+|\xi|^{2\theta}) - 4|\xi|^2} < \frac{\sqrt{25^3-2}}{\sqrt{25^3}} \log(1+|\xi|^{2\theta}) \quad \text{for} \quad \beta \leq |\xi| < \delta.$$

Therefore, if $\eta^3 \leq |\xi| < \delta$ one has

$$-\log(1+|\xi|^{2\theta}) + 2C(\xi) < \Big(\frac{\sqrt{25^3-2}}{\sqrt{25^3}} - 1\Big) \log(1+|\xi|^{2\theta}) = -c\log(1+|\xi|^{2\theta}) \qquad (5.41)$$

with $0 < c < 1$ a constant, due to the fact that $\beta \leq \eta^3$.

Now, from Lemma 2.21 and inequality (5.41) we can prove the exponential decay for the L^2-norm of $\hat{u}(t,\cdot)$ on the middle frequency zone as follows:

$$\begin{aligned}
\int_{\eta^3\leq|\xi|<\delta} |\hat{u}(t,\xi)|^2 d\xi &= \int_{\eta^3\leq|\xi|<\delta} e^{-t\log(1+|\xi|^{2\theta})} \frac{\sinh^2(C(\xi)t)}{4(C(\xi))^2} |\hat{u}_1(\xi)|^2 d\xi \\
&\leq \frac{K^2}{4} t^2 \int_{\eta^3\leq|\xi|\leq\delta} e^{-t\log(1+|\xi|^{2\theta})+2C(\xi)t} |\hat{u}_1(\xi)|^2 d\xi \\
&\leq K^2 t^2 \int_{\eta^3\leq|\xi|\leq\delta} e^{-ct\log(1+|\xi|^{2\theta})} |\hat{u}_1(\xi)|^2 d\xi \quad (c>0) \\
&= K^2 t^2 \int_{\eta^3\leq|\xi|\leq\delta} (1+|\xi|^{2\theta})^{-ct} |\hat{u}_1(\xi)|^2 d\xi \\
&\leq K^2 \omega_n t^2 \|u_1\|_1^2 \int_{\eta^3}^{\delta} (1+r^{2\theta})^{-ct} r^{n-1} dr \\
&\leq C t^2 (1+\eta^{6\theta})^{-ct} \|u_1\|_1^2, \qquad t \gg 1, \qquad (5.42)
\end{aligned}$$

with C a positive constant depending on the space dimension n and $c > 0$ a constant given in (5.41).

5.1.2 Estimates on the high-frequency zone $|\xi| \geq \delta$

On the high frequency zone $|\xi| > \delta$ the characteristics roots are complex and the solution of (5.3)-(5.4) is given by

$$\hat{u}(t,\xi) = \frac{1}{b(\xi)} e^{-a(\xi)t} \sin(b(\xi)t) \hat{u}_1(\xi)$$

where

$$a(\xi) = \frac{\log(1+|\xi|^{2\theta})}{2}, \quad b(\xi) = \frac{\sqrt{4|\xi|^2 - \log^2(1+|\xi|^{2\theta})}}{2}.$$

We know that $|\sin a| \leq a$ for all $a \geq 0$. Then $|\frac{\sin(b(\xi)t)}{b(\xi)}| \leq t$ for all $t \geq 0$ and so one has

$$\int_{|\xi|>\delta} |\hat{u}(t,\xi)|^2 d\xi = \int_{|\xi|>\delta} (1+|\xi|^{2\theta})^{-t} \frac{\sin^2(b(\xi)t)}{b(\xi)^2} |\hat{u}_1(\xi)|^2 d\xi$$

$$
\begin{aligned}
&\leq t^2\|u_1\|_1^2 \int_{|\xi|\geq\delta} (1+|\xi|^{2\theta})^{-t} d\xi \\
&= \omega_n t^2 \|u_1\|_1^2 \int_\delta^1 (1+r^{2\theta})^{-t} r^{n-1} dr + \omega_n t^2 \|u_1\|_1^2 \int_1^\infty (1+r^{2\theta})^{-t} r^{n-1} dr \\
&\sim \|u_1\|_1^2 t^2 \Big((1+\delta^{2\theta})^{-t} + \frac{2^{-t}}{t-1} \Big), \qquad t \gg 1. \qquad (5.43)
\end{aligned}
$$

The last inequality is obtained by using Lemma (2.33).

5.1.3 The asymptotic profile

From the estimates obtained in Propositions 5.6, 5.7 and inequalities (5.42) and (5.43), we can conclude that the the inverse transform of the function $\varphi(t,\cdot)$ given in (5.28) is the asymptotic profile as $t \to \infty$ to the solution $u(t,x)$ of the problem (5.1)-(5.2). Such a result is stated in the next theorem.

Theorem 5.9. *(i). Let $n=1$, $0<\theta\leq \frac{1}{3}$, and $u_1 \in L^{1,2\theta}(\boldsymbol{R}) \cap L^2(\boldsymbol{R})$. Then it holds that*

$$
\begin{aligned}
&\|u(t,\cdot) - \mathcal{F}^{-1}(\varphi(t,\xi))(\cdot)\|_{L^2} \\
&\leq \begin{cases} C(\|u_1\|_1 + \|u_1\|_{L^{1,2\theta}}) \left(t^{-\frac{1}{4(1-\theta)}} + \frac{1}{\sqrt{\theta}} t^{-\frac{1}{4\theta}} \right), & \text{if } 0<\theta\leq\frac{1}{6}, \\ C(\|u_1\|_1 + \|u_1\|_{L^{1,2\theta}}) \left(t^{-\frac{1}{4(1-\theta)}} + \frac{1}{\sqrt{\theta}} t^{-\frac{\frac{5}{3}-4\theta}{4\theta}} \right), & \text{if } \frac{1}{6}<\theta\leq\frac{1}{3} \end{cases}
\end{aligned}
$$

for $t \gg 1$, where $u(t,x)$ is a unique solution to problem (5.1)-(5.2) with $u_0 = 0$.

(ii). Let $n\geq 2$, $0<\theta\leq\frac{5}{12}$, and $u_1 \in L^{1,2\theta}(\boldsymbol{R}^n) \cap L^2(\boldsymbol{R}^n)$. Then it holds that

$$
\begin{aligned}
&\|u(t,\cdot) - \mathcal{F}^{-1}(\varphi(t,\xi))(\cdot)\|_{L^2} \\
&\leq \begin{cases} C(\|u_1\|_1 + \|u_1\|_{L^{1,2\theta}}) \left(t^{-\frac{n}{4(1-\theta)}} + \frac{1}{\sqrt{\theta}} t^{-\frac{n}{4\theta}} \right), & \text{if } 0<\theta\leq\frac{1}{6}, \\ C(\|u_1\|_1 + \|u_1\|_{L^{1,2\theta}}) \left(t^{-\frac{n}{4(1-\theta)}} + \frac{1}{\sqrt{\theta}} t^{-\frac{n-4\theta+\frac{2}{3}}{4\theta}} \right), & \text{if } \frac{1}{6}<\theta\leq\frac{1}{3}, \\ C(\|u_1\|_1 + \|u_1\|_{L^{1,2\theta}}) \left(t^{-\frac{n-1}{4(1-\theta)}} + \frac{1}{\sqrt{\theta}} t^{-\frac{n-1}{4\theta}} \right), & \text{if } \frac{1}{3}<\theta\leq\frac{5}{12} \end{cases}
\end{aligned}
$$

for $t \gg 1$, where $u(t,x)$ is a unique solution to problem (5.1)-(5.2) with $u_0 = 0$.

Proof. We first note that $\log(1+|\xi|^{2\theta}) \leq |\xi|^{2\theta}$ for all $\xi \in \boldsymbol{R}^n$, which implies

$$
\frac{|\xi|^{2\theta}}{\log(1+|\xi|^{2\theta})} \geq 1.
$$

Then, one can get the next estimate for $t \gg 1$ on the zone of high frequency $|\xi| \geq \eta^3$ as follows:

$$
\int_{|\xi|\geq\eta^3} |\varphi(t,\xi)|^2 d\xi \leq 2P_1^2 \int_{|\xi|\geq\eta^3} \frac{e^{-\frac{2|\xi|^2}{\log(1+|\xi|^{2\theta})}t}}{\log^2(1+|\xi|^{2\theta})} d\xi + 2P_1^2 \int_{|\xi|\geq\eta^3} \frac{e^{-2\log(1+|\xi|^{2\theta})t}}{\log^2(1+|\xi|^{2\theta})} d\xi
$$

$$\begin{aligned}
&\approx P_1^2 \int_{|\xi|\geq\eta^3} \frac{e^{-\frac{2|\xi|^{2\theta}|\xi|^{2-2\theta}}{\log(1+|\xi|^{2\theta})}t}}{\log^2(1+|\xi|^{2\theta})} d\xi + P_1^2 \int_{|\xi|\geq\eta^3} \frac{(1+|\xi|^{2\theta})^{-t}}{\log^2(1+|\xi|^{2\theta})} d\xi \\
&\leq P_1^2 \int_{|\xi|\geq\eta^3} \frac{e^{-2t|\xi|^{2-2\theta}}}{\log^2(1+|\xi|^{2\theta})} d\xi + P_1^2 \int_{|\xi|\geq\eta^3} \frac{(1+|\xi|^{2\theta})^{-t}}{\log^2(1+|\xi|^{2\theta})} d\xi \\
&\leq \frac{P_1^2}{\log^2(1+\eta^{6\theta})} \int_{|\xi|\geq\eta^3} e^{-2t|\xi|^{2-2\theta}} d\xi + \frac{P_1^2\omega_n}{\log^2(1+\eta^{6\theta})} \int_{\eta^3}^1 (1+r^{2\theta})^{-t} r^{n-1} dr \\
&+ \frac{P_1^2\omega_n}{(\log 2)^2} \int_1^\infty (1+r^{2\theta})^{-t} r^{n-1} dr \\
&\leq \frac{P_1^2}{\log^2(1+\eta^{6\theta})} e^{-t\eta^{6-6\theta}} \int_{|\xi|\geq\eta^3} e^{-|\xi|^{2-2\theta}} d\xi + \frac{P_1^2\omega_n}{\log^2(1+\eta^{6\theta})} \int_{\eta^3}^1 (1+r^{2\theta})^{-t} r^{n-1} dr \\
&+ \frac{P_1^2\omega_n}{(\log 2)^2} \int_1^\infty (1+r^{2\theta})^{-t} r^{n-1} dr \\
&\leq C P_1^2 \Big(e^{-t\eta^{6-6\theta}} + (1+\eta^{6\theta})^{-t} + \frac{2^{-t}}{t-1} \Big), \quad t \gg 1.
\end{aligned} \tag{5.44}$$

Now, it follows from the Plancherel Theorem that

$$\int_{\boldsymbol{R}^n} |u(t,x) - \mathcal{F}^{-1}(\varphi(t,\xi))(x)|^2 dx = \int_{\boldsymbol{R}^n} |\hat{u}(t,\xi) - \varphi(t,\xi)|^2 d\xi$$

for $t \geq 0$. Furthermore, one has

$$\begin{aligned}
\int_{\boldsymbol{R}^n} |\hat{u}(t,\xi) - \varphi(t,\xi)|^2 d\xi \leq & \int_{|\xi|\leq\eta^3} |\hat{u}(t,\xi) - \varphi(t,\xi)|^2 d\xi + \int_{|\xi|\geq\eta^3} |\hat{u}(t,\xi)|^2 d\xi \\
& + \int_{|\xi|\geq\eta^3} |\varphi(t,\xi)|^2 d\xi
\end{aligned} \tag{5.45}$$

for $t > 0$.

From (5.42) and (5.43), we know that the L^2-estimates on the zone $|\xi| \geq \eta^3$ to $\hat{u}(t,\xi)$ are of exponential type. The estimate to $\varphi(t,\xi)$ on $|\xi| \geq \eta^3$ obtained in (5.44) is also faster than those obtained in Propositions 5.6 and 5.7. The result of Theorem 5.9 follows by combining Propositions 5.6 and 5.7, with inequalities (5.42), (5.43), (5.44) and (5.45). □

Remark 5.10. In the results of Theorem 5.9, one can notice the coefficient $1/\sqrt{\theta}$ in front of each final estimates. By observing this coefficient, one may conclude that we have captured the unique nature for the log-damping (or fractional damping) with parameter $\theta > 0$. This property can be found by searching the leading term more precisely than previous researches .

Remark 5.11. It follows from Theorem 5.9 that $\hat{u}(t,\xi) \sim P_1(\varphi_1(t,\xi) - \varphi_2(t,\xi))$ in $L^2(\boldsymbol{R}^n_\xi)$ as $t \to \infty$. It is important to notice that $\varphi_1(t,\xi)$ and $\varphi_2(t,\xi)$ are exact solutions of the first order in time equations in the Fourier space, respectively:

$$-\Delta v + L_\theta v_t = 0,$$

and

$$L_\theta v + v_t = 0.$$

In some sense, the solution to problem (5.1)-(5.2) with small parameters $\theta \in (0, 1/2)$ has a double diffusion phenomenon. This kind of important double diffusion phenomenon has been first discovered by D'Abbicco-Ebert [12] to the equation

$$u_{tt} - \Delta u + (-\Delta)^\theta u_t = 0 \tag{5.46}$$

with $\theta \in (0, 1/2)$. Theorem 5.9 corresponds to that of [12, Theorem 2]. We find that (5.1)-(5.2) has a similar property to it. While, in the case when $n \geq 2$ and $\theta \in (0, 1/2)$ an asymptotic profile of the solution to (5.46) is captured as

$$\frac{e^{-t|\xi|^{2(1-\theta)}}}{|\xi|^{2\theta}} \tag{5.47}$$

in [30, Theorem 1.5]. In some sense, (5.47) is similar to $\varphi_1(t, \xi)$ because of $\log(1+r^{2\theta}) \sim r^{2\theta}$ for small $r > 0$.

Remark 5.12. A restriction $\theta \in (0, \dfrac{1}{3}]$ or $\theta \in (0, \dfrac{5}{12}]$ is just a technical condition, however, in the course of proof of Theorem 5.9 one has frequently used the following fact

$$\lim_{r \to +0} \frac{\log(1 + r^{2\theta})}{r} = \infty. \tag{5.48}$$

(5.48) is also true in a more wider range $\theta \in (0, \frac{1}{2})$. So, reconsidering, the case of $\theta \in (\frac{1}{3}, \frac{1}{2})$ for $n = 1$ or $\theta \in (\frac{5}{12}, \frac{1}{2})$ for $n \geq 2$ is still open.

Remark 5.13. The condition $u_1 \in L^2(\boldsymbol{R}^n)$ in Theorem 5.9 is used to make sure the unique existence of the mild solution $u(t, x)$. However, it does not affect directly on the L^2-estimate of the solution, even in the high-frequency estimates although the estimate (5.44) in the high frequency zone can be easily estimated in terms of $||u_1||$ instead of $||u_1||_1$.

5.2 OPTIMALITY OF THE DECAY RATES

Our goal in this section is to prove two theorems about asymptotic behavior of the solution of the problem (5.1)-(5.2). Assuming certain conditions under the dimension n and parameter θ, namely, if $n = 1$ and $0 < \theta < \frac{1}{4}$ or if $n \geq 2$ and $0 < \theta \leq \frac{5}{12}$, we obtain the optimal decay rate to the solution (see Theorem 5.17). On the other hand, when $n = 1$ and $\frac{1}{4} \leq \theta \leq \frac{1}{3}$ we show that the solution to the problem (5.1)-(5.2) blows-up as $t \to \infty$ (see Theorem 5.21).

In order to prove such results, we prepare some lemmas. From (5.28), we have

$$\varphi(t, \xi) = \varphi_1(t, \xi) - \varphi_2(t, \xi), \quad t \geq 0, \quad \xi \in \boldsymbol{R}^n, \tag{5.49}$$

where

$$\varphi_1(t,\xi) := \frac{e^{-\frac{|\xi|^2}{\log(1+|\xi|^{2\theta})}t}}{\log(1+|\xi|^{2\theta})}P_1, \quad \varphi_2(t,\xi) := \frac{e^{-\log(1+|\xi|^{2\theta})t}}{\log(1+|\xi|^{2\theta})}P_1.$$

Lemma 5.14. *Let $n=1$ with $0<\theta<\frac{1}{4}$ and $n\geq 2$ with $0<\theta\leq\frac{5}{12}$. If $u_1\in L^1(\boldsymbol{R}^n)$, then*

$$C_1P_1^2t^{-\frac{n-4\theta}{2(1-\theta)}} \leq \int_{\boldsymbol{R}^n}|\varphi(t,\xi)|^2d\xi \leq C_2P_1^2\left(t^{-\frac{n-4\theta}{2(1-\theta)}}+\frac{1}{\theta}t^{-\frac{n-4\theta}{2\theta}}\right), \quad t\gg 1,$$

where the constants C_1, C_2 depend only on θ and n.

Proof. First we note that

$$\begin{aligned}\int_{\boldsymbol{R}^n}|\varphi(t,\xi)|^2d\xi &\leq 2\int_{\boldsymbol{R}^n}|\varphi_1(t,\xi)|^2d\xi + 2\int_{\boldsymbol{R}^n}|\varphi_2(t,\xi)|^2d\xi \\ &= 2\int_{|\xi|\leq\eta}|\varphi_1(t,\xi)|^2d\xi + 2\int_{|\xi|\geq\eta}|\varphi_1(t,\xi)|^2d\xi \\ &+ 2\int_{|\xi|\leq\eta}|\varphi_2(t,\xi)|^2d\xi + 2\int_{|\xi|\geq\eta}|\varphi_2(t,\xi)|^2d\xi, \quad t>0. \end{aligned} \tag{5.50}$$

By using the equivalences obtained in Remark 5.2, we have

$$\begin{aligned}\int_{|\xi|\leq\eta}|\varphi_1(t,\xi)|^2d\xi &\approx P_1^2\int_{|\xi|\leq\eta}\frac{e^{-t\log(1+|\xi|^{2-2\theta})}}{\log^2(1+|\xi|^{2\theta})}d\xi \leq P_1^2\int_{|\xi|\leq\eta}\frac{(1+|\xi|^{2-2\theta})^{-t}}{\log^2(1+|\xi|^{2\theta})}d\xi \\ &= \omega_nP_1^2\int_0^\eta\frac{(1+r^{2-2\theta})^{-t}}{\log^2(1+r^{2\theta})}r^{n-1}dr \\ &= \omega_nP_1^2\int_0^\eta\frac{(1+r^{2-2\theta})^{-t}}{\log^2(1+r^{2\theta})}r^{n-1-4\theta}r^{4\theta}dr \\ &\leq 4\omega_nP_1^2\int_0^\eta\frac{(1+r^{2-2\theta})^{-t}}{r^{4\theta}}r^{n-1-4\theta}r^{4\theta}dr \\ &= 4\omega_nP_1^2\int_0^\eta(1+r^{2-2\theta})^{-t}r^{n-1-4\theta}dr \\ &\leq CP_1^2t^{-\frac{n-4\theta}{2(1-\theta)}}, \quad t\gg 1. \end{aligned} \tag{5.51}$$

The last decay estimate is obtained from Lemma 2.30 since $n-4\theta>0$. In the same way, by using Lemma 2.32 for $n-4\theta>0$, we have the next estimate.

$$\begin{aligned}\int_{|\xi|\leq\eta}|\varphi_2(t,\xi)|^2d\xi &= P_1^2\int_{|\xi|\leq\eta}\frac{e^{-2t\log(1+|\xi|^{2\theta})}}{\log^2(1+|\xi|^{2\theta})}d\xi \leq \omega_nP_1^2\int_0^\eta\frac{(1+r^{2\theta})^{-t}}{\log^2(1+r^{2\theta})}r^{n-1}dr \\ &\leq 4\omega_nP_1^2\int_0^\eta(1+r^{2\theta})^{-t}r^{n-1-4\theta}dr \leq C\frac{1}{\theta}P_1^2t^{-\frac{n-4\theta}{2\theta}}, \ t\gg 1. \end{aligned} \tag{5.52}$$

Further, from (5.44), the L^2-estimate to $\varphi(t,\xi)$ on the zone $|\xi|\geq\eta$ is of exponential type, because $|\xi|\geq\eta$ implies that $|\xi|\geq\eta^3$. Therefore, there exists a constant $C>0$ such that

$$\int_{\boldsymbol{R}^n}|\varphi(t,\xi)|^2d\xi \leq CP_1^2\left(t^{-\frac{n-4\theta}{2(1-\theta)}}+\frac{1}{\theta}t^{-\frac{n-4\theta}{2\theta}}\right), \quad t\gg 1,$$

due to (5.50), (5.51) and (5.52).

In order to prove the estimate from bellow, from Remark 5.2 we have

$$\begin{aligned}
\int_{\boldsymbol{R}^n} |\varphi_1(t,\xi)|^2 d\xi \geq \int_{|\xi|\leq\eta} |\varphi_1(t,\xi)|^2 d\xi \approx P_1^2 \int_{|\xi|\leq\eta} \frac{e^{-t\log(1+|\xi|^{2-2\theta})}}{\log^2(1+|\xi|^{2\theta})} d\xi \\
= \omega_n P_1^2 \int_0^\eta \frac{e^{-t\log(1+r^{2-2\theta})}}{\log^2(1+r^{2\theta})} r^{n-1} dr \\
\geq C\omega_n P_1^2 \int_0^\eta \frac{e^{-t\log(1+r^{2-2\theta})}}{r^{4\theta}} r^{n-1} dr \\
= C\omega_n P_1^2 \int_0^\eta (1+r^{2-2\theta})^{-t} r^{n-1-4\theta} dr \\
\geq C P_1^2 t^{-\frac{n-4\theta}{2(1-\theta)}}, \qquad (5.53)
\end{aligned}$$

because of $n-4\theta>0$, due to Remark 2.31, where $C>0$ is a generous constant. We also notice that $|\varphi_1(t,\xi)| \leq |\varphi(t,\xi)| + |\varphi_2(t,\xi)|$ and, from Young's inequality, $|\varphi_1(t,\xi)|^2 \leq 2|\varphi(t,\xi)|^2 + 2|\varphi_2(t,\xi)|^2$. Thus,

$$|\varphi(t,\xi)|^2 \geq \frac{1}{2}|\varphi_1(t,\xi)|^2 - |\varphi_2(t,\xi)|^2, t\geq 0, \xi\in\boldsymbol{R}^n.$$

Then, from (5.53) and (5.52), we have

$$\begin{aligned}
\int_{|\xi|\leq\eta} |\varphi(t,\xi)|^2 d\xi \geq \frac{1}{2}\int_{|\xi|\leq\eta} |\varphi_1(t,\xi)|^2 d\xi - \int_{|\xi|\leq\eta} |\varphi_2(t,\xi)|^2 d\xi \\
\geq K_1 P_1^2 t^{-\frac{n-4\theta}{2(1-\theta)}} - K_2\frac{1}{\theta} P_1^2 t^{-\frac{n-4\theta}{2\theta}} \\
= P_1^2 t^{-\frac{n-4\theta}{2(1-\theta)}} \left(K_1 - K_2\frac{1}{\theta} t^{-\frac{8\theta^2-2\theta n+n-4\theta}{2\theta(1-\theta)}}\right). \qquad (5.54)
\end{aligned}$$

Since $0<\theta<\frac{1}{2}$ and $n-4\theta>0$, one can conclude that $8\theta^2-2\theta n+n-4\theta>0$. Therefore, it follows from (5.54) that

$$\int_{\boldsymbol{R}^n} |\varphi(t,\xi)|^2 d\xi \geq \int_{|\xi|\leq\eta} |\varphi(t,\xi)|^2 d\xi \geq \frac{K_1}{2} P_1^2 t^{-\frac{n-4\theta}{2(1-\theta)}}, \quad t\gg 1.$$

These arguments imply the desired estimate for $\varphi(t,\xi)$. □

The above arguments do not hold for $n=1$ and $\frac{1}{4}\leq\theta\leq\frac{1}{3}$, because the integrals

$$\int_0^\eta \frac{(1+r^{2\theta})^{-t}}{\log^2(1+r^{2\theta})} r^{n-1} dr, \quad \int_0^\eta \frac{(1+r^{2-2\theta})^{-t}}{\log^2(1+r^{2\theta})} r^{n-1} dr$$

are divergent for all $t>0$. For this reason, we need to estimate the L^2-norm of the function $\varphi(t,\xi)$ itself:

$$\varphi(t,\xi) = \frac{e^{-\frac{|\xi|^2}{\log(1+|\xi|^{2\theta})}t}}{\log(1+|\xi|^{2\theta})} P_1 - \frac{e^{-\log(1+|\xi|^{2\theta})t}}{\log(1+|\xi|^{2\theta})} P_1.$$

Lemma 5.15. *Let $n = 1$ and $\theta > \frac{1}{4}$. If $u_1 \in L^1(\boldsymbol{R})$, there exist constants $C_1, C_2 > 0$ such that*

$$C_1 P_1^2 t^{\frac{4\theta-1}{2\theta}} \leq \int_{\boldsymbol{R}} |\varphi(t,\xi)|^2 d\xi \leq C_2 \frac{1}{4\theta - 1} P_1^2 t^{\frac{4\theta-1}{2\theta}}, \quad t \gg 1.$$

Proof. We first note that from (5.44) the L^2-norm of $\varphi(t,\xi)$ decays exponentially on the high frequency region $|\xi| \geq \eta > \eta^3$. So, in this proof it suffices to consider the integral only in the low frequency zone $0 < |\xi| \leq \eta$.

Now, we notice that

$$-\log(1 + r^{2\theta}) = \frac{r^2 - \log^2(1 + r^{2\theta})}{\log(1 + r^{2\theta})} - \frac{r^2}{\log(1 + r^{2\theta})},$$

so that one has

$$\begin{aligned}
\log(1 + |\xi|^{2\theta})\varphi(t,\xi) &= P_1 \left(e^{-\frac{|\xi|^2}{\log(1+|\xi|^{2\theta})}t} - e^{-\log(1+|\xi|^{2\theta})t} \right) \\
&= P_1 \left(e^{-\frac{|\xi|^2}{\log(1+|\xi|^{2\theta})}t} - e^{t\frac{|\xi|^2 - \log^2(1+|\xi|^{2\theta})}{\log(1+|\xi|^{2\theta})} - t\frac{|\xi|^2}{\log(1+|\xi|^{2\theta})}} \right) \\
&= P_1 e^{-\frac{|\xi|^2}{\log(1+|\xi|^{2\theta})}t} \left(1 - e^{-t\frac{\log^2(1+|\xi|^{2\theta}) - |\xi|^2}{\log(1+|\xi|^{2\theta})}} \right). \qquad (5.55)
\end{aligned}$$

Due to the fact that for $0 \leq r \leq 1$ we have $\frac{1}{2} r^{2\theta} \leq \log(1 + r^{2\theta}) \leq r^{2\theta}$, thus one has

$$r^{2\theta} \frac{(1 - 4r^{2-4\theta})}{4} \leq \frac{\log^2(1 + r^{2\theta}) - r^2}{\log(1 + r^{2\theta})} \leq 2r^{2\theta}(1 - r^{2-4\theta}). \qquad (5.56)$$

Moreover, since $\theta < \frac{1}{2}$ we have $2 - 4\theta > 0$. Therefore, there exists $\beta = \beta(\theta) > 0$, with $\beta \leq \eta$ such that

$$1 - 4r^{2-4\theta} \geq \frac{1}{2},$$

for $0 \leq r \leq \beta$. Thus,

$$\frac{1}{2} \leq 1 - 4r^{2-4\theta} \leq 1 - r^{2-4\theta} \leq 1 \qquad (5.57)$$

for $0 \leq r \leq \beta$. From (5.56) and (5.57) one can get

$$\frac{1}{8} r^{2\theta} \leq \frac{\log^2(1 + r^{2\theta}) - r^2}{\log(1 + r^{2\theta})} \leq 2r^{2\theta},$$

for $0 < r \leq \beta$. This implies

$$1 - e^{-\frac{1}{8}tr^{2\theta}} \leq 1 - e^{-t\frac{\log^2(1+r^{2\theta}) - r^2}{\log(1+r^{2\theta})}} \leq 1 - e^{-2tr^{2\theta}}$$

and

$$\frac{1 - e^{-\frac{1}{8}tr^{2\theta}}}{r^{2\theta}} \leq \frac{1 - e^{-t\frac{\log^2(1+r^{2\theta}) - r^2}{\log(1+r^{2\theta})}}}{\log(1 + r^{2\theta})} \leq 2\frac{1 - e^{-2tr^{2\theta}}}{r^{2\theta}}, \qquad (5.58)$$

for $0 < r \leq \beta$. Since

$$\lim_{\sigma\to 0}\frac{1-e^{-\sigma}}{\sigma}=1,$$

there exists $\alpha > 0$ such that $\alpha \leq \beta$ and

$$\frac{1}{2}\leq\frac{1-e^{-\sigma}}{\sigma}\leq\frac{3}{2}, \tag{5.59}$$

for all $0 < \sigma \leq \alpha$. Based on these preparations let us prove the desired estimate for $\varphi(t,\xi)$.

(1) The lower estimate of lemma:

For $0 < r \leq \left(\frac{8\alpha}{t}\right)^{\frac{1}{2\theta}}$ it holds that $0 < \sigma = \frac{1}{8}tr^{2\theta} \leq \alpha$. Applying estimate (5.59) we get

$$\frac{1}{2}\leq\frac{1-e^{-\frac{1}{8}tr^{2\theta}}}{\frac{1}{8}tr^{2\theta}}\leq\frac{3}{2}. \tag{5.60}$$

From (5.58) and (5.60), for $0 < r \leq \left(\frac{8\alpha}{t}\right)^{\frac{1}{2\theta}}$, it holds that

$$\frac{1-e^{-t\frac{\log^2(1+r^{2\theta})-r^2}{\log(1+r^{2\theta})}}}{\log(1+r^{2\theta})}\geq\frac{t}{16}. \tag{5.61}$$

Let $t_0 > 0$ be such that $\left(\frac{8\alpha}{t_0}\right)^{\frac{1}{2\theta}} \leq \alpha$, and consider $t \geq t_0$. By combining (5.49) with (5.55) and (5.61), since $\alpha \leq \beta \leq \eta$, we obtain

$$\begin{aligned}\int_{|\xi|\leq\eta}|\varphi(t,\xi)|^2d\xi &= P_1^2\int_{|\xi|\leq\eta}\left(\frac{e^{-\frac{|\xi|^2}{\log(1+|\xi|^{2\theta})}t}-e^{-\log(1+|\xi|^{2\theta})t}}{\log(1+|\xi|^{2\theta})}\right)^2d\xi\\ &= \omega_1P_1^2\int_0^\eta e^{-\frac{2r^2}{\log(1+r^{2\theta})}t}\left(\frac{1-e^{-t\frac{\log^2(1+r^{2\theta})-r^2}{\log(1+r^{2\theta})}}}{\log(1+r^{2\theta})}\right)^2dr\\ &\geq \frac{\omega_1}{16^2}P_1^2t^2\int_0^{\left(\frac{8\alpha}{t}\right)^{\frac{1}{2\theta}}}e^{-\frac{2r^2}{\log(1+r^{2\theta})}t}dr,\quad t\geq t_0. \end{aligned} \tag{5.62}$$

We also notice that

$$2\log(1+r^{2-2\theta})\leq 2r^{2-2\theta}\leq\frac{2r^2}{\log(1+r^{2\theta})}\leq 4r^{2-2\theta}\leq 8\log(1+r^{2-2\theta})\quad 0<r\leq 1. \tag{5.63}$$

Thus, from (5.62) and (5.63) one has

$$\begin{aligned}\int_{|\xi|\leq\eta}|\varphi(t,\xi)|^2d\xi &\geq \frac{\omega_1}{16^2}P_1^2t^2\int_0^{\left(\frac{8\alpha}{t}\right)^{\frac{1}{2\theta}}}e^{-8t\log(1+r^{2-2\theta})}dr\\ &= \frac{\omega_1}{16^2}P_1^2t^2\int_0^{\left(\frac{8\alpha}{t}\right)^{\frac{1}{2\theta}}}(1+r^{2-2\theta})^{-8t}dr\end{aligned}$$

$$\geq \frac{\omega_1}{16^2}P_1^2t^2\left(1+\left(\frac{8\alpha}{t}\right)^{\frac{2-2\theta}{2\theta}}\right)^{-8t}\int_0^{\left(\frac{8\alpha}{t}\right)^{\frac{1}{2\theta}}}dr, \quad t\geq t_0. \tag{5.64}$$

Now we observe that $1<\frac{2-2\theta}{2\theta}<3$ for $\frac{1}{4}<\theta<\frac{1}{2}$. Then there exists $T\geq t_0$ such that

$$\frac{1}{2}\leq\left(1+\left(\frac{8\alpha}{t}\right)^{\frac{2-2\theta}{2\theta}}\right)^{-t}\leq\frac{3}{2} \tag{5.65}$$

for all $t\geq T$, because of the fact that

$$\lim_{t\to+\infty}\left(1+\frac{1}{t^q}\right)^{-t}=1$$

provided that $q>1$. By combining estimates (5.64) and (5.65) one can arrive at the desired estimate from below such that

$$\begin{aligned}\int_{|\xi|\leq\eta}|\varphi(t,\xi)|^2d\xi &\geq \frac{\omega_n}{16^2}P_1^2t^2\left(1+\left(\frac{8\alpha}{t}\right)^{\frac{2-2\theta}{2\theta}}\right)^{-t}\int_0^{\left(\frac{8\alpha}{t}\right)^{\frac{1}{2\theta}}}dr\\ &\geq \frac{\omega_n}{2\times16^2}P_1^2t^2\int_0^{\left(\frac{8\alpha}{t}\right)^{\frac{1}{2\theta}}}dr\\ &= \frac{\omega_n}{2\times16^2}P_1t^2\left(\frac{8\alpha}{t}\right)^{\frac{1}{2\theta}}\\ &= CP_1^2t^2t^{-\frac{1}{2\theta}}\\ &= CP_1^2t^{\frac{4\theta-1}{2\theta}}, \quad t\geq T\end{aligned}$$

with some constant $C=C_\theta>0$.

(2) The upper estimate of lemma:
From (5.55), we have

$$\begin{aligned}\int_{|\xi|\leq\eta}|\varphi(t,\xi)|^2d\xi &= \omega_1P_1^2\int_0^\eta e^{-\frac{2r^2}{\log(1+r^{2\theta})}t}\left(\frac{1-e^{-t\frac{\log^2(1+r^{2\theta})-r^2}{\log(1+r^{2\theta})}}}{\log(1+r^{2\theta})}\right)^2dr\\ &= A_1(t,\theta)+A_2(t,\theta),\end{aligned}$$

where

$$A_1(t,\theta):=\omega_1P_1^2\int_0^{\left(\frac{\alpha}{2t}\right)^{\frac{1}{2\theta}}}e^{-\frac{2r^2}{\log(1+r^{2\theta})}t}\left(\frac{1-e^{-t\frac{\log^2(1+r^{2\theta})-r^2}{\log(1+r^{2\theta})}}}{\log(1+r^{2\theta})}\right)^2dr, \tag{5.66}$$

$$A_2(t,\theta):=\omega_1P_1^2\int_{\left(\frac{\alpha}{2t}\right)^{\frac{1}{2\theta}}}^{\eta}e^{-\frac{2r^2}{\log(1+r^{2\theta})}t}\left(\frac{1-e^{-t\frac{\log^2(1+r^{2\theta})-r^2}{\log(1+r^{2\theta})}}}{\log(1+r^{2\theta})}\right)^2dr, \tag{5.67}$$

which holds for $t \geq t_0$.

Now, for $0 < r \leq \left(\frac{\alpha}{2t}\right)^{\frac{1}{2\theta}}$ by using inequality (5.59) we have that

$$\frac{1-e^{-2tr^{2\theta}}}{r^{2\theta}} \leq 3t. \tag{5.68}$$

Thus for $0 < r \leq \left(\frac{\alpha}{2t}\right)^{\frac{1}{2\theta}}$, by combining (5.58) with (5.68) it holds that

$$\frac{1-e^{-t\frac{\log^2(1+r^{2\theta})-r^2}{\log(1+r^{2\theta})}}}{\log(1+r^{2\theta})} \leq 6t, \quad t \geq t_0. \tag{5.69}$$

The definition of $A_1(t,\theta)$ and the inequality (5.69) imply that

$$\begin{aligned}
A_1(t,\theta) &= \omega_1 P_1^2 \int_0^{\left(\frac{\alpha}{2t}\right)^{\frac{1}{2\theta}}} e^{-\frac{2r^2}{\log(1+r^{2\theta})}t} \left(\frac{1-e^{-t\frac{\log^2(1+r^{2\theta})-r^2}{\log(1+r^{2\theta})}}}{\log(1+r^{2\theta})} \right)^2 dr \\
&\leq 36t^2\omega_1 P_1^2 \int_0^{\left(\frac{\alpha}{2t}\right)^{\frac{1}{2\theta}}} e^{-\frac{2r^2}{\log(1+r^{2\theta})}t} dr \leq 36t^2\omega_1 P_1^2 \int_0^{\left(\frac{\alpha}{2t}\right)^{\frac{1}{2\theta}}} dr \\
&= 36t^2\omega_1 P_1^2 \left(\frac{\alpha}{2t}\right)^{\frac{1}{2\theta}} = CP_1^2 t^2 t^{-\frac{1}{2\theta}} \\
&= CP_1^2 t^{\frac{4\theta-1}{2\theta}}, \quad t \gg 1
\end{aligned} \tag{5.70}$$

with some generous constant $C > 0$. Note that the estimate given by (5.70) is also holds for $\theta = \frac{1}{4}$.

In order to estimate $A_2(t,\theta)$, we use (5.58) for $\theta > 1/4$ to get the following estimate

$$\begin{aligned}
A_2(t,\theta) &= \omega_1 P_1^2 \int_{\left(\frac{\alpha}{2t}\right)^{\frac{1}{2\theta}}}^{\eta} e^{-\frac{2r^2}{\log(1+r^{2\theta})}t} \left(\frac{1-e^{-t\frac{\log^2(1+r^{2\theta})-r^2}{\log(1+r^{2\theta})}}}{\log(1+r^{2\theta})} \right)^2 dr \\
&\leq 4\omega_1 P_1^2 \int_{\left(\frac{\alpha}{2t}\right)^{\frac{1}{2\theta}}}^{\eta} e^{-\frac{2r^2}{\log(1+r^{2\theta})}t} \left(\frac{1-e^{-2tr^{2\theta}}}{r^{2\theta}} \right)^2 dr \\
&\leq 4\omega_1 P_1^2 \int_{\left(\frac{\alpha}{2t}\right)^{\frac{1}{2\theta}}}^{\eta} \frac{1}{r^{4\theta}} dr \\
&= 4\omega_1 P_1^2 \frac{1}{1-4\theta} \left(\eta^{1-4\theta} - \left(\frac{\alpha}{2t}\right)^{\frac{1-4\theta}{2\theta}} \right) \\
&\leq CP_1^2 \frac{1}{4\theta-1} t^{\frac{4\theta-1}{2\theta}}, \quad t \geq t_0
\end{aligned} \tag{5.71}$$

with $C = 4\omega_1 \left(\frac{\alpha}{2}\right)^{\frac{1-4\theta}{2\theta}}$. It is important to emphasize that the above estimate holds only for $\theta \neq \frac{1}{4}$ and we have just used it for $1 - 4\theta < 0$ to obtain (5.71). The estimates for A_1 and A_2 prove the desired estimate from below of lemma. □

As a special case one can introduce the following log-order blowup result for the case of $\theta = \frac{1}{4}$.

Lemma 5.16. *Let $n = 1$ and $\theta = \frac{1}{4}$. For $u_1 \in L^1(\boldsymbol{R})$ the following optimal estimate holds.*

$$C_1 P_1^2 \log t \le \int_{\boldsymbol{R}} |\varphi(t,\xi)|^2 d\xi \le C_2 P_1^2 \log t, \quad t \gg 1,$$

with some constants $C_1, C_2 > 0$.

Proof. We consider the functions $A_1(t, \frac{1}{4})$ and $A_2(t, \frac{1}{4})$ given by (5.66) and (5.67) with $\theta = \frac{1}{4}$. The estimate (5.70) also holds for $\theta = \frac{1}{4}$ and it tells us the fact that

$$A_1(t, \frac{1}{4}) \le C P_1^2, \quad t \gg 1. \tag{5.72}$$

While, by definition (5.67) and (5.58) we have

$$\begin{aligned}
A_2(t, \frac{1}{4}) &= \omega_1 P_1^2 \int_{\left(\frac{\alpha}{2t}\right)^2}^{\eta} e^{-\frac{2r^2}{\log(1+\sqrt{r})} t} \left(\frac{1 - e^{-t \frac{\log^2(1+\sqrt{r}) - r^2}{\log(1+\sqrt{r})}}}{\log(1+\sqrt{r})} \right)^2 dr \\
&\le 4\omega_1 P_1^2 \int_{\left(\frac{\alpha}{2t}\right)^2}^{\eta} e^{-\frac{2r^2}{\log(1+\sqrt{r})} t} \left(\frac{1 - e^{-2t\sqrt{r}}}{\sqrt{r}} \right)^2 dr \\
&\le 4\omega_1 P_1^2 \int_{\left(\frac{\alpha}{2t}\right)^2}^{\eta} \frac{1}{r} dr \\
&= 4\omega_1 P_1^2 \left(\log \eta - \log \left(\frac{\alpha}{2t} \right)^2 \right) \\
&= 4\omega_1 P_1^2 \Big(\log \eta - 2 \log \alpha + 2 \log 2 + 2 \log t \Big) \\
&\le C P_1^2 \log t, \quad t \gg 1.
\end{aligned} \tag{5.73}$$

The estimates (5.72) and (5.73) allow us to conclude the upper estimate

$$\int_{\boldsymbol{R}} |\varphi(t,\xi)|^2 d\xi \le C_2 \log t, \quad t \gg 1, \tag{5.74}$$

with some constant $C_2 > 0$.

On the other hand, by (5.49) one can get

$$|\varphi(t,\xi)|^2 \ge \frac{1}{2} |\varphi_1(t,\xi)|^2 - |\varphi_2(t,\xi)|^2, \quad t > 0, \ \xi \in \boldsymbol{R}.$$

Thus, for $t > 0$,

$$\begin{aligned}
\int_{\boldsymbol{R}} |\varphi(t,\xi)|^2 d\xi &\ge \int_{t^{-1}}^{t^{-\frac{2}{3}}} |\varphi(t,\xi)|^2 d\xi \ge \frac{1}{2} \int_{t^{-1}}^{t^{-\frac{2}{3}}} |\varphi_1(t,\xi)|^2 d\xi - \int_{t^{-1}}^{t^{-\frac{2}{3}}} |\varphi_2(t,\xi)|^2 d\xi \\
&= P_1^2 \left(\frac{1}{2} K_1(t) - K_2(t) \right),
\end{aligned} \tag{5.75}$$

where

$$K_1(t) := \int_{t^{-1}}^{t^{-\frac{2}{3}}} \frac{e^{-\frac{2|\xi|^2}{\log(1+\sqrt{|\xi|})}t}}{\log^2(1+\sqrt{|\xi|})} d\xi,$$

$$K_2(t) := \int_{t^{-1}}^{t^{-\frac{2}{3}}} \frac{e^{-2\log(1+\sqrt{|\xi|})t}}{\log^2(1+\sqrt{|\xi|})} d\xi.$$

We remember that

$$\frac{1}{2}\sqrt{|\xi|} \le \log(1+\sqrt{|\xi|}) \le \sqrt{|\xi|}, \quad |\xi| \le 1. \tag{5.76}$$

Thus one has

$$\begin{aligned} K_1(t) &= \int_{t^{-1}}^{t^{-\frac{2}{3}}} \frac{e^{-\frac{2|\xi|^2}{\log(1+\sqrt{|\xi|})}t}}{\log^2(1+\sqrt{|\xi|})} d\xi \ge \int_{t^{-1}}^{t^{-\frac{2}{3}}} \frac{e^{-4|\xi|^{\frac{3}{2}}t}}{|\xi|} d\xi \\ &= \omega_1 \int_{t^{-1}}^{t^{-\frac{2}{3}}} \frac{e^{-4r^{\frac{3}{2}}t}}{r} dr \ge \omega_1 e^{-4} \int_{t^{-1}}^{t^{-\frac{2}{3}}} \frac{1}{r} dr \\ &= \omega_1 e^{-4}\big(-\frac{2}{3}\log t + \log t\big) \\ &= \omega_1 \frac{e^{-4}}{3}\log t, \quad t \ge 1. \end{aligned} \tag{5.77}$$

Similarly, in the case when large $t > 1$ such that $t^{-\frac{2}{3}} < 1$ it follows from (5.76) that

$$\begin{aligned} K_2(t) &= \int_{t^{-1}}^{t^{-\frac{2}{3}}} \frac{e^{-2\log(1+\sqrt{|\xi|})t}}{\log^2(1+\sqrt{|\xi|})} d\xi \le 4\int_{t^{-1}}^{t^{-\frac{2}{3}}} \frac{e^{-\sqrt{|\xi|}t}}{|\xi|} d\xi \\ &= 4\omega_1 \int_{t^{-1}}^{t^{-\frac{2}{3}}} \frac{e^{-\sqrt{r}t}}{r} dr \le 4\omega_1 e^{-\sqrt{t}} \int_{t^{-1}}^{t^{-\frac{2}{3}}} \frac{1}{r} dr \\ &\le \frac{4\omega_1}{3} e^{-\sqrt{t}} \log t, \quad t \gg 1. \end{aligned} \tag{5.78}$$

Therefore, from (5.75), (5.77) and (5.78) one has

$$\begin{aligned} \int_{\boldsymbol{R}} |\varphi(t,\xi)|^2 d\xi &\ge P_1^2 \left(\frac{1}{2}K_1(t) - K_2(t)\right) \\ &\ge P_1^2\omega_1 \left(\frac{e^{-4}}{6}\log t - \frac{4}{3}e^{-\sqrt{t}}\log t\right) \\ &= P_1^2\omega_1 \log t \left(\frac{e^{-4}}{6} - \frac{4}{3}e^{-\sqrt{t}}\right), \quad t \gg 1 \end{aligned}$$

which implies the desired estimate from below to the case $\theta = 1/4$ and $n = 1$:

$$\int_{\boldsymbol{R}} |\varphi(t,\xi)|^2 d\xi \ge C P_1^2 \log t, \quad t \gg 1 \tag{5.79}$$

with some constant $C > 0$. The estimates (5.74) and (5.79) complete the proof of lemma. □

As an application of Theorem 5.9 one can derive the following sharp decay estimates, which imply the optimal decay rates of the L^2-norm of the solution to the problem (5.1)-(5.2).

Theorem 5.17. *Let $n = 1$ with $0 < \theta < \frac{1}{4}$ and $n \geq 2$ with $0 < \theta \leq \frac{5}{12}$. For $u_1 \in L^{1,2\theta}(\boldsymbol{R}^n) \cap L^2(\boldsymbol{R}^n)$, it holds that*

$$K_1|P_1|t^{-\frac{n-4\theta}{4(1-\theta)}} \leq \|u(t,\cdot)\| \leq K_2(|P_1| + \|u_1\|_{L^{1,2\theta}}) \left(t^{-\frac{n-4\theta}{4(1-\theta)}} + \frac{1}{\sqrt{\theta}} t^{-\frac{n-4\theta}{4\theta}} \right), \quad t \gg 1$$

with some constant $K_1, K_2 > 0$ depending only on n and θ, where $u(t,x)$ is a unique solution to problem (5.1)-(5.2) with $u_0 = 0$.

Proof. One first observes that

$$\int_{\boldsymbol{R}^n} |\hat{u}(t,\xi)|^2 d\xi \leq 2 \int_{\boldsymbol{R}^n} |\hat{u}(t,\xi) - \varphi(t,\xi)|^2 d\xi + 2 \int_{\boldsymbol{R}^n} |\varphi(t,\xi)|^2 d\xi. \tag{5.80}$$

By combining Lemma 5.14 and Theorem 5.9 with (5.80), we have

$$\int_{\boldsymbol{R}^n} |\hat{u}(t,\xi)|^2 d\xi \leq C(P_1^2 + \|u_1\|^2_{L^{1,2\theta}}) \left(t^{-\frac{n-4\theta}{2(1-\theta)}} + \frac{1}{\theta} t^{-\frac{n-4\theta}{2\theta}} \right), \quad t \gg 1. \tag{5.81}$$

We can also observe that for $0 < \theta < 1/2$ it holds that $2\theta \leq 2 - 2\theta$. Therefore $\frac{n-4\theta}{2-2\theta} \leq \frac{n-4\theta}{2\theta}$. Thus the decay rate $t^{-\frac{n-4\theta}{2\theta}}$ is faster than $t^{-\frac{n-4\theta}{2-2\theta}}$. It results the following upper bound to the L^2-norm of the Fourier transformed solution $\hat{u}(t,\cdot)$ such that

$$\int_{\boldsymbol{R}^n} |\hat{u}(t,\xi)|^2 d\xi \leq C(P_1^2 + \|u_1\|^2_{L^{1,2\theta}})(1 + \frac{1}{\theta}) t^{-\frac{n-4\theta}{2(1-\theta)}}, \quad t \gg 1.$$

By the Plancherel Theorem and from (5.81) the upper bound estimate of the statement of Theorem 5.17 follows with a generous constant $C > 0$.

In order to obtain the lower bound, we observe that

$$|\varphi(t,\xi)| \leq |\hat{u}(t,\xi) - \varphi(t,\xi)| + |\hat{u}(t,\xi)|.$$

By Young's inequality, we may obtain

$$|\varphi(t,\xi)|^2 \leq 2|\hat{u}(t,\xi) - \varphi(t,\xi)|^2 + 2|\hat{u}(t,\xi)|^2.$$

Therefore,

$$|\hat{u}(t,\xi)|^2 \geq \frac{1}{2}|\varphi(t,\xi)|^2 - |\hat{u}(t,\xi) - \varphi(t,\xi)|^2, \quad t \geq 0, \quad \xi \in \boldsymbol{R}^n.$$

Thus,

$$\int_{\boldsymbol{R}^n} |\hat{u}(t,\xi)|^2 d\xi \geq \frac{1}{2} \int_{\boldsymbol{R}^n} |\varphi(t,\xi)|^2 d\xi - \int_{\boldsymbol{R}^n} |\hat{u}(t,\xi) - \varphi(t,\xi)|^2 d\xi. \tag{5.82}$$

First, we consider the case $n \geq 1$ and $0 < \theta \leq \frac{1}{6}$. By combining (5.82) with the lower estimate of Lemma 5.14 and estimate of Theorem 5.9, we obtain

$$\begin{aligned} \|\hat{u}(t,\cdot)\|^2 &\geq \frac{C_1}{2} P_1^2 t^{-\frac{n-4\theta}{2(1-\theta)}} - C_2(\|u_1\|_1^2 + \|u_1\|_{L^{1,2\theta}}^2)\left(t^{-\frac{n}{2(1-\theta)}} + \frac{1}{\theta} t^{-\frac{n}{2\theta}}\right) \\ &= t^{-\frac{n-4\theta}{2(1-\theta)}} \left(\frac{C_1}{2} P_1^2 - C_2(\|u_1\|_1^2 + \|u_1\|_{L^{1,2\theta}}^2)\left(t^{-\frac{4\theta}{2(1-\theta)}} + \frac{1}{\theta} t^{-\frac{n+4\theta^2-2n\theta}{2\theta(1-\theta)}}\right)\right), \end{aligned} \tag{5.83}$$

for $t \gg 1$ and positive constants C_1, C_2. But for $0 < \theta \leq \frac{1}{6}$ we notice that $n+4\theta^2-2n\theta > 0$, so that one can get

$$\lim_{t\to\infty} \left(\frac{C_1}{2} P_1^2 - C_2(\|u_1\|_1^2 + \|u_1\|_{L^{1,2\theta}}^2)\left(t^{-\frac{4\theta}{2(1-\theta)}} + \frac{1}{\theta} t^{-\frac{n+4\theta^2-2n\theta}{2\theta(1-\theta)}}\right)\right) = \frac{C_1}{2} P_1^2.$$

Therefore, there exists $t_1 > 0$ such that

$$\frac{C_1}{4} P_1^2 \leq \frac{C_1}{2} P_1^2 - C_2(\|u_1\|_1^2 + \|u_1\|_{L^{1,2\theta}}^2)\left(t^{-\frac{4\theta}{2(1-\theta)}} + \frac{1}{\theta} t^{-\frac{n+4\theta^2-2n\theta}{2\theta(1-\theta)}}\right) \leq C_1 P_1^2, \quad t \gg t_1.$$

From (5.83) it follows that

$$\|\hat{u}(t,\cdot)\|^2 \geq \frac{C_1}{4} P_1^2 t^{-\frac{n-4\theta}{2(1-\theta)}}, \tag{5.84}$$

for $t \gg 1$.

The estimate (5.84) implies the desired estimate for lower bound in t in the case when $n \geq 1$ and $0 < \theta \leq \frac{1}{6}$.

Analogously, we may obtain the results for $n = 1$ with $\frac{1}{6} < \theta < \frac{1}{4}$, and for $n \geq 2$ with $\frac{1}{6} < \theta \leq \frac{5}{12}$ based on the results of Theorem 5.9 for these values of θ and Lemma 5.14.

This completes the proof of Theorem 5.17. □

Remark 5.18. A similar L^p-L^q type "decay" estimates only from above has been already studied precisely in [13] and [14, Corollary 2.2] to the solution of the equation (5.46) for $n = 1$ and $0 < \theta < 1/4$, or $n \geq 2$ and $0 < \theta < 1/2$. The lower bound itself in Theorem 5.17 seems new.

Remark 5.19. As a result of Theorem 5.17, one can observe that $\|u(t,\cdot)\| \sim t^{-\frac{n-4\theta}{4(1-\theta)}}$ $(t \to \infty)$. Thus, as for an ultimate situation when $\theta \to 0^+$ formally, the optimal decay order will approach $t^{-\frac{n}{4}}$, which is the Gauss kernel. This is quite natural because in the case when $\theta = 0$, the equation corresponds to the frequently studied damped wave equation. In this sense, all results in this chapter reflect a diffusive aspect of the equation (5.1) with small θ. This property is quite different from those studied in [4] for large $\theta \geq \frac{1}{2}$. In [4], a wave like property is captured.

Remark 5.20. The optimal decay order $t^{-\frac{n-4\theta}{4(1-\theta)}}$ obtained in Theorem 5.17 has a close relation to that studied in [14], [7] and [30, (1.13) with $l = k = 0$] for the equation (5.46). In particular, the (almost) optimal decay rate of the "energy" and L^2-norm of the solutions are studied by developing a new energy method in the Fourier space in [7]. So, the structure of the equation (5.1) is quite similar to (5.46) with $\theta \in (0, 1/2)$.

Contrary to the decay results as in Theorem 5.17, one can observe the following surprising property, which shows infinite time blowup results of the solution to problem (5.1)–(5.2) in the one dimensional case. We believe this is the first discovery in the damped wave equation community. In [15] and [13], when they apply the decay estimates of the solution for the equation (5.46) to the nonlinear problems, they necessarily avoid to treat the case of $n = 1$ with and $1/4 \le \theta < 1/2$. The following crucial result makes their mechanism clear because of $\log(1 + r^{2\theta}) \sim r^{2\theta}$ for small $r > 0$.

Theorem 5.21. *Let $n = 1$ with $\frac{1}{4} \le \theta \le \frac{1}{3}$. For $u_1 \in L^{1,2\theta}(\boldsymbol{R}) \cap L^2(\boldsymbol{R})$, there exists positive constants K_1, K_2, which depend only on θ, such that*

$$K_1|P_1|t^{\frac{4\theta-1}{4\theta}} \le \|u(t,\cdot)\| \le K_2\big(\frac{1}{\sqrt{4\theta-1}}|P_1| + \|u_1\|_{L^{1,2\theta}}\big)t^{\frac{4\theta-1}{4\theta}}, \quad t \gg 1 \tag{5.85}$$

for $\frac{1}{4} < \theta \le \frac{1}{3}$ and

$$K_1|P_1|\sqrt{\log t} \le \|u(t,\cdot)\| \le K_2\big(|P_1| + \|u_1\|_{L^{1,2\theta}}\big)\sqrt{\log t}, \quad t \gg 1 \tag{5.86}$$

for the case $\theta = \frac{1}{4}$.

Proof of Theorem 5.21. The proof of Theorem 5.21 can be obtained in the same way as in Theorem 5.17, but using Lemmas 5.15 and 5.16 instead of Lemma 5.14 and observing that the estimates to $\|\varphi(t,\cdot)\|^2$ in Lemmas 5.15 and 5.16 are also worse than the estimates to $\|\hat{u}(t,\cdot) - \varphi(t,\cdot)\|^2$ in Theorem 5.9. □

Remark 5.22. We find the number $\theta^* = \dfrac{1}{4}$ as a critical value in the one dimensional case because θ^* divides the structure of the corresponding solution $u(t,\cdot)$ into two parts: one is decay property for $0 < \theta < \theta^*$, while the other is the infinite time blow-up results in the case of $\theta^* \le \theta < \frac{1}{2}$. Moreover, we note that in Theorem (5.21) there is not a contradiction between the estimate (5.85) when $\theta \to (1/4)^+$ and the estimate (5.86) for $\theta = 1/4$ because of the singularity $\sqrt{\frac{1}{4\theta-1}}$ at $\theta = 1/4$.

Remark 5.23. The statement of Theorems 5.17 and 5.21 present the restrictions $\theta \le \frac{1}{3}$ for $n = 1$ and $\theta \le \frac{5}{12}$ for $n \ge 2$. By combining the solution formula (5.21), with Lemma 5.3 and the asymptotic estimates in Lemmas 2.30, 2.32, 5.15 and 5.16, the missing estimates to the solution for $\theta < \frac{1}{2}$ can be derived and are of the order $t^{\frac{4\theta-1}{4\theta}}$ for $n = 1$ and $t^{-\frac{n-4\theta}{4(1-\theta)}}$ for $n = 2$. However, the optimality in these cases is still open, but we believe that they are optimal.

Remark 5.24. We emphasize that the problem (5.1)–(5.2) with $\theta = 1/2$ seems to be open until now. It may be in some sense another critical value to that problem and it will be important to study such case.

6 FINAL REMARKS

In this work, we study three new models of evolution equation based on the logarithmic-Laplacian operators L_θ. In each one of the three problems we proved decay and/or blow-up in infinite time estimates to solutions as in the wave problems with the usual Laplacian operator. This property of operator L to produce the same estimates as the Laplacian operator, even though it is a weaker operator, indicates that it is a more efficient than the Laplacian operator.

We emphasize that we also prove new blow-up estimates to the solution of problem

$$u_{tt} - \Delta u + L_\theta u_t = 0 \tag{6.1}$$

when $n = 1$ with $\frac{1}{4} \leq \theta < \frac{1}{2}$. These optimal blow-up estimates to the solutions of wave problems with usual dissipation $(-\Delta)^\theta u_t$, $\frac{1}{4} \leq \theta < \frac{1}{2}$, in one dimension seem have not yet been discovered in works by other authors. Since $\log(1 + |\xi|^{2\theta}) \approx |\xi|^{2\theta}$ for $|\xi| \leq 1$, the same optimal estimates to the solution of usual wave problems can be obtained.

There are some open problems that can still be studied. The problem associated to equation

$$u_{tt} + Lu_{tt} + Lu + L^2u + L^\theta u_t = 0,$$

with $0 < \theta \leq 1/2$ also presents the regularity loss property. One can to investigate asymptotic profile to the solution and using it to derived optimal decay rates for problems with $0 < \theta \leq 1/2$.

We wish to study the problem (6.1) with $\theta = \frac{1}{2}$, due to it is another critical value which divides the asymptotic profile into diffusion-like and wave-like. One can also investigate asymptotic profiles to solutions of the problems (6.1) for values of θ that were not covered in Theorem 5.9.

In addition, semilinear problems associated with the models presented in this work can be studied. In particular, we want to investigate the so-called critical exponent to the problem

$$u_{tt} - \Delta u + L_\theta u_t = |u|^p, \quad 0 \leq \theta \leq 1,$$

which has not yet been discovered for the usual wave equation with dissipation $-\Delta u_t$.

Other problems involving coupled systems as, for example, thermo-elastic systems, Maxwell system and the linearized compressible Navier-Stokes system may be studied under effects of the L_θ operator.

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